Minimum dilatation problem and quasi-periodicity conjecture

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Overview of Talk

Minimum dilatation problem and quasi-periodicity conjecture

I. Pseudo-Anosov mapping classes and their dilatations

II. Minimum dilatation problem

III. Two models for small dilatation mapping classes.

I. Pseudo-Anosov mapping classes and their dilatations

Background and Notation

Objects of study:

$S = S_{g,n}$ a compact oriented surface with $\chi_{\text{top}}(S) < 0$

$\phi : S \to S$, *mapping class*, an isotopy class of orientation preserving homeomorphisms, (S, ϕ)

$\text{Mod}(S)$ the *mapping class group*

Goal:

Describe the dynamics of minimum and "small" dilatation pseudo-Anosov mapping classes.

Some references:

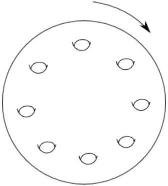
Work of Thurston [F-L-P], A Primer on Mapping Class Groups [F-M]

Classification of mapping classes

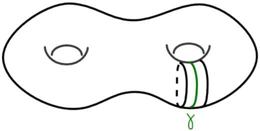
(Nielsen, Thurston) $\phi \in \mathrm{Mod}(S)$ is either

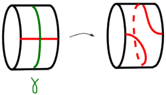
- *periodic* $(\exists k, \phi^k = \mathrm{id})$

  Example: a rotation

- *reducible* $(\exists k, \exists \gamma \subset S$ ess. s.c.c., such so that $\phi^k(\gamma) = \gamma.)$

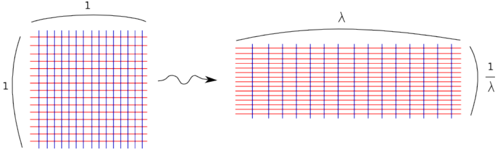
  Example: Dehn twist (Remark: These generate $\mathrm{Mod}(S)$)

- *pseudo-Anosov* ...

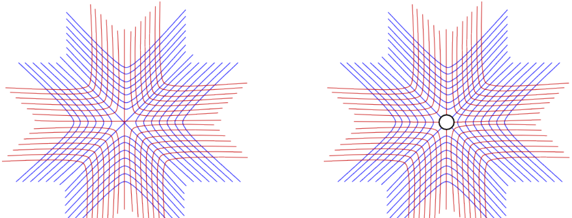
pseudo-Anosov mapping classes

(S, ϕ) is *pseudo-Anosov* if there is a local flat structure outside a finite set of *singularities* preserved by ϕ, defined by a transverse measured singular foliatations $(\mathcal{F}^{\pm}, \nu^{\pm})$, such that ϕ permutes the singularities, and $\phi_*(\mathcal{F}^{\pm}, \nu^{\pm}) = (\mathcal{F}^{\pm}, \lambda^{\pm 1}\nu^{\pm})$.

Near smooth points:



Near singularities and boundary components:



Properties of pseudo-Anosov maps

- Homological versus geometric dilatation
- Horizontal and vertical theories of mapping classes

Homological versus geometric dilatation

(S, ϕ) is pseudo-Anosov iff $\forall$ ess. s.c.c. C on S, and $\forall$ Riemannian metric on S,

$$|\phi^n(C)|^{\frac{1}{n}} \to \lambda(= \lambda_{\text{geo}}) = \text{spectral radius}(T) > 1,$$

where T is a Perron-Frobenius matrix.

Compare: the *homological dilatation* of a mapping class (S, ϕ)

$$\lambda_{\text{hom}} := \text{spectral radius}(\phi_* : H_1(S; \mathbb{R}) \to H_1(S; \mathbb{R})).$$

These can be computed using the Alexander polynomial.

Properties:

- $\lambda_{\text{hom}}(\phi) \leq \lambda_{\text{geo}}(\phi)$
- λ_{geo} is a Perron unit.
- The house of any algebraic integer can be realized as $\lambda_{\text{hom}}(\phi)$ for some (S, ϕ) (Kanenobu)

Horizontal Theory: Teichmüller space, Moduli space

Fix S:

$$\mathcal{M}(S) = \mathcal{T}(S)/\mathrm{Mod}(S).$$

There is a one-to-one correspondence between pseudo-Anosov elements $\mathcal{P}(S)$ and closed Teichmüller geodesics in $\mathcal{M}(S)$, and

$$\{\lambda(\phi) : \phi \in \mathcal{P}(S)\} \xrightarrow{\log} \{\text{Teich. lengths of closed geodesics in } \mathcal{M}(S)\}$$

Problem: How to translate between properties of elements of $\mathcal{P}$ and Teichmüller geodesics.

Vertical Theory: Thurston's theory of fibered faces

Let $\mathcal{M}\text{od} = \bigcup_S \text{Mod}(S)$, $\mathcal{P} = \bigcup_S \mathcal{P}(S)$.

Given a fibered 3-manifold M, let

$$\Phi(M) = \{(S, \phi) \mid M = M_\phi := \text{Mapping Torus}(S, \phi)\}.$$

Theorem: M is hyperbolic $\Leftrightarrow \Phi(M) \subset \mathcal{P} \Leftrightarrow \Phi(M) \cap \mathcal{P} \neq \emptyset$.

(Assume hyperbolic from now on.)

There are natural inclusions:

$$\begin{aligned} \Phi(M) &\hookrightarrow H^1(M;\mathbb{Z}) = \text{Hom}(H_1(M;\mathbb{Z}),\mathbb{Z}) \\ &\hookrightarrow H^1(M;\mathbb{R}) = \mathbb{R}^{b_1(M)} \end{aligned}$$

i.e., $\Phi(M)$ can be identified with a subset of the integer lattice points in $\mathbb{R}^{b_1}$.

Thurston norm ball

Thurston norm: $\exists$ a norm $|| \, ||$ on $H^1(M;\mathbb{R})$ with the properties:

- If $(S,\phi) \in \Phi(M)$, then $||(S,\phi)|| = |\chi(S)|$.
- Thurston norm ball $\{\alpha \in H^1(M;\mathbb{R}) \ : \ ||\alpha|| \leq 1\}$ is a convex polyhedron with integer vertices.
- For each top-dimensional face F of $H^1(M;\mathbb{R})$, let

$$\Phi(M,F) = (F \cdot \mathbb{R}^+) \cap \Phi(M).$$

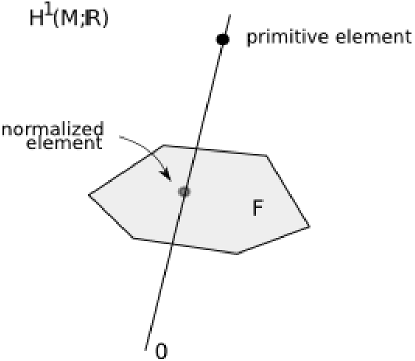
Then $\Phi(M,F)$ is either empty or

$$\Phi(M,F) = (F \cdot \mathbb{R}^+) \cap H^1(M;\mathbb{Z})^{\text{prim}}.$$

In the latter case, F is called a *fibered face*.

Fibered faces and dilatations

The monodromies $\Phi(M, F)$ of a hyperbolic 3-manifold M associated to a fibered face F define a subpartition of $\mathcal{P}$ with topological structure.



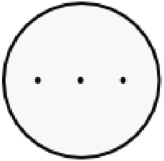
- $\Phi(M, F)$
  $\longleftrightarrow$ rational points on F.
- The fibered faces F are polyhedra of dimension $d = \dim H^1(M; \mathbb{R})$ -1.

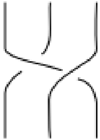
(Fried, McMullen) $L(S, \phi) = \lambda(\phi)^{|\chi(S)|}$ extends to a continuous, convex function on F, going to infinity toward the boundary of F.

Example: Simplest pseudo-Anosov braid monodromy

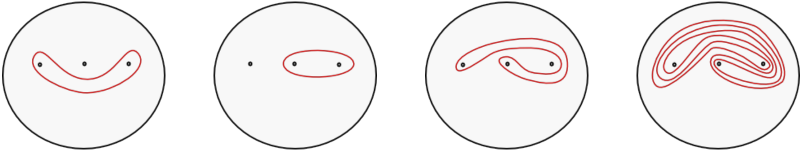
Identify $S = S_{0,4}$ with a disk with three interior boundary components.

Let (S, ϕ_0) be the braid monodromy corresponding to the braid $\sigma_1 \sigma_2^{-1}$ in terms of the standard braid generators.





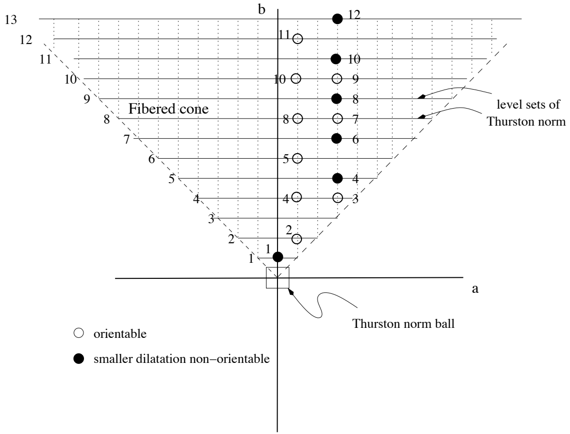
Action of 3 iterations of ϕ on an essential simple closed curve.



Example: Fibered face for simplest hyperbolic braid

Mapping torus:
$M = S \times [0,1]/(x,1) \sim (\phi_0(x), 0)$

$H_1(M;\mathbb{R})$



II. Minimum dilatation problem

Minimum dilatation problem.

Question: (Penner, Farb) What is the minimum dilatation

$$\delta(S) = \min\{\lambda(\phi) \mid \phi \in \mathcal{P}(S)\} \qquad ?$$

(Ko-Los-Song: $S_{0,5}$, Ham-Song: $S_{0,6}$, Cho-Ham: $S_{2,0}$, Lanneau-Thiffeault: orientable mapping classes)

Question: (Penner, McMullen, Farb) What is the behavior of the normalized dilatation

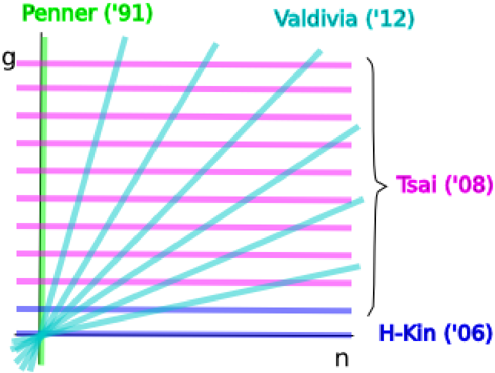
$$\delta(S_{g,0})^g$$

or more generally

$$\delta(S_{g,n})^{\chi(S_{g,n})} \qquad ?$$

Problem: Describe the "shape" of "small" pseudo-Anosov mapping classes.
(dynamics, number theoretic properties of dilatations)

Asymptotic behavior



The normalized minimum dilatations $\delta(S_{g,n})^{|\chi(S_{g,n})|}$ are bounded along blue/green rays, and unbounded along red rays.

Small dilatation maps

We will say (S, ϕ) has *P-small dilatation*, if $\lambda(\phi)^{|\chi(S)|} \leq P$.

(Farb-Leininger-Margalit '08) To understand all P-small dilatation mapping classes, it suffices to study the monodromy of a finite number of hyperbolic 3-manifolds.

Smallest known accumulation point for $\delta(S_{g,n})^{\chi(S_{g,n})}$ is

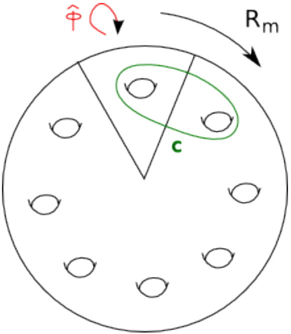
$$\ell_0 = \left(\frac{3+\sqrt{5}}{2}\right)^2 = (1 + \text{golden mean})^2$$

achieved by simplest pseudo-Anosov braid. (H, A-D, K-T '09-'10)

Question: Is ℓ_0 the smallest accumulation point?

III. Two models for small dilatation mapping classes

Type I: Quasi-periodic (wedge-type) mapping classes (Penner)

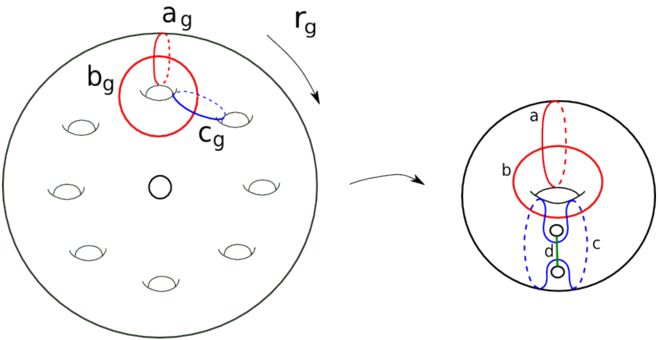


Theorem (Penner, Valdivia, H)

Let (S_m, ϕ_m) be the mapping class with $\phi_m = R_m \circ \delta_C \circ \widehat{\phi}$. Then if for some m (S_m, ϕ_m) is pseudo-Anosov, then they all are, and the normalized dilatation $\lambda(\phi_m)^m$ is bounded.

Example: Penner's sequence (sketch of proof)

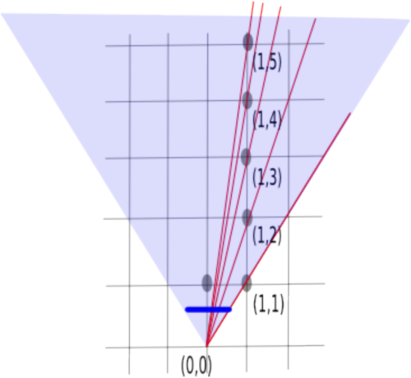
The mapping classes (S_m, ϕ_m) form a convergent sequence on a fibered face.



Let $\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g}$, $\phi = \delta_c \delta_b^{-1} \delta_a$.

$$(S_{g,2}, \phi_g) \longrightarrow (S_{1,2}, \phi)$$

Associated sequence on fibered face



Polynomial invariants:

Alexander polynomial $\Delta = \Delta_M$

$$\lambda_{\text{hom}}(\phi_\alpha) = |\Delta^\alpha|$$

Teichmüller polynomial

$\Theta = \Theta_{M,F}$ (McMullen)

$$\lambda_{\text{geo}}(\phi_\alpha) = |\Theta^\alpha|.$$

For Penner's sequence:

$$\lambda(\phi_g) = |x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1|$$

$$\lambda(\phi_g)^g \to \frac{7+3\sqrt{5}}{2}.$$

(Fried, McMullen)

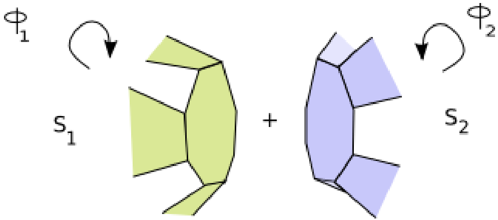
Type II: Murasugi sum and twisted mapping classes

For $i = 1, 2$, let (S_i, ϕ_i) be two mapping classes.

Let P_{2k} be a $2k$-gon.

Let $\alpha_i : P_{2k} \hookrightarrow S_i$ embeddings so that the jth edge of P_{2k} maps into the boundary of S_i whenever $i = j$ (mod 2).

The Murasugi sum (S, ϕ) is given as follows:



$$S = S_1 \bigcup_{\alpha_1(x)=\alpha_2(x)} S_2,$$

$\phi = \tilde{\phi}_2 \circ \tilde{\phi}_1$, where $\tilde{\phi}_i$ are the extensions to S by the identity on $S \setminus S_i$.

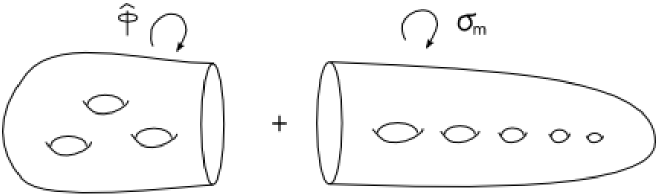
Twisted mapping classes

Let (Σ_m, σ_m) be a sequence of mapping classes, such that

$$(\sigma_m)^m = \delta_{b_1} \cdots \delta_{b_s}$$

where $b_1, \ldots, b_s$ are boundary components of Σ_m.

Let (S_m, ϕ_m) be the mapping classes obtained by Murasugi sum of (Σ_m, σ_m) and some fixed $(S, \widehat{\phi})$. We call (S_m, ϕ_m) a *twisted sequence* of mapping classes.



Theorem (H)

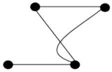
There exists $(S, \widehat{\phi})$ *and* (Σ_m, σ_m) *so that the twisted sequence* (S_m, ϕ_m) *satisfies*

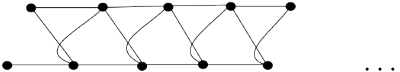
- $(S_m, \phi_m) \in \Phi(M, F)$, *for a single 3-manifold* M *and fibered face* F*;*
- *for* $m > 1$, $(\overline{S}_m, \overline{\phi}_m)$ *defines (orientable) pseudo-Anosov maps on closed genus* $g = 6m - 1$ *surfaces;*
- *The normalized dilatations converge:*

$$\lim_{m \to \infty} \lambda(\phi_m)^{\frac{|\chi(S_m)|}{2}} = \frac{(3 + \sqrt{5})}{2}.$$

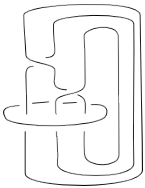
Alternate manifestations of this sequence of examples

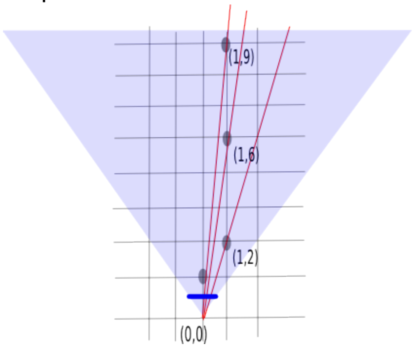
1. Coxeter mapping classes:





2. A sequence on a fibered face of the complement of the 6_{22} link converging to the simplest pseudo-Anosov braid.





Quasi-periodicity conjecture

Quasi-periodicity conjecture/question (Farb-Leininger-Margalit, McMullen) Is it possible to decompose all P-small elements of $\mathcal{P}$ as a composition of a periodic map and a map that is identity outside a subsurface Σ with $|\chi(\Sigma)| < K_P$?

Question: Can any P-small mapping class be built from a combination of the constructions of Type I and II, where $(S, \widehat{\phi})$ has bounded Euler characteristic depending on P?

Thank you!