Small dilatation pseudo-Anosov mapping classes
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1 Minimum dilatation problem

Let $\phi : S \to S$ be a pseudo-Anosov mapping class on an oriented surface $S = S_{g,n}$ of genus g and n punctures. The *dilatation* $\lambda(\phi)$ is the expansion factor of ϕ along the stable transverse measured singular foliation associated to ϕ, and is a Perron algebraic unit greater than one. The set of dilatations for a fixed S is discrete [17].

Let $\mathcal{P}(S)$ be the set of all pseudo-Anosov mapping classes on S. Let $\delta(S)$ be the minimum dilatation for $\phi \in \mathcal{P}(S)$. Let $P_{g,n}$ be the set of pseudo-Anosov mapping classes on $S_{g,n}$ with dilatation equal to $\delta(S_{g,n})$.

The minimum dilatation problem (cf. [15, 14, 3]) can be stated as follows.

Problem 1 (Minimum Dilatation Problem I) *What is the behavior of $\delta(S_{g,n})$ as a function of g and n?*

The exact value of $\delta(S_{g,n})$ is not known except for very small cases (for example, for closed surfaces, the answer is only known for $g = 2$ [6]). However, more is known about the asymptotic behavior of $\delta(S_{g,n})$ as a function of g and n, and the topological Euler characteristic $\chi(S_{g,n})$.

Let $\mathcal{P} = \bigcup_S \mathcal{P}(S)$. The *normalized dilatation* is defined by

$$\begin{array}{rcl} L : \mathcal{P} & \to & \mathbb{R}^+ \\ (S, \phi) & \mapsto & \lambda(\phi)^{|\chi(S)|}. \end{array}$$

For $\ell > 1$, we say ϕ is *ℓ-small* if $L(\phi) \leq \ell$. Let $\mathcal{P}(\ell)$ be the set of ℓ-small pseudo-Anosov maps.

The current smallest known accumulation point of the image of L is

$$\ell_0 = \left(\frac{3 + \sqrt{5}}{2}\right)^2. \tag{1}$$

(See [8, 1, 12].)

Problem 2 (Assymptotic Minimum Dilatation Problem) *Is there an accumulation point for the image of L that is smaller than ℓ_0?*

One can also formulate the minimum dilatation problem from a geometric rather than numerical standpoint.

Problem 3 (Minimum Dilatation Problem II) *What do small dilatation mapping classes look like?*

In the remainder of this note, we will describe three constructions of mapping classes with small dilatation. These constructions all define mapping classes that can be thought of as *nearly* periodic. We begin in Section 2 by making precise a notion of deformations of mapping classes on arbitrary surfaces (cf. [16]), and show that to solve Problems 2 and 3 it suffices to investigate the deformation theory of mapping classes (cf, [4]). Two nearly periodic constructions are described in Section 3. These are obtained by combining a periodic mapping class, or a periodic mapping class relative to boundary, with a mapping class that is the identity outside a subsurface of bounded Euler characteristic. We give a third construction in Section 4 using generalized Coxeter graphs to construct periodic mapping classes that form the building block for nearly periodic examples. In Section 5 we discuss further questions concerning the singularities of a mapping class, and their orbits.

2 Three-manifolds, fibered faces and small dilatation mapping classes.

Given a hyperbolic 3-manifold M (possibly with cusps), let $\Psi(M)$ be the set (possibly empty) of fibrations of M (with connected fibers) over the circle S^1. Let $\Phi(M)$ be the set of monodromies of elements of $\Psi(M)$. By allowing M to vary, we obtain a new partition of the set of pseudo-Anosov mapping classes

$$\mathcal{P} = \bigcup_M \Phi(M).$$

For fixed M, the set $\Phi(M)$ partitions further. Let $|| \ ||$ be the Thurston norm on $H^1(M;\mathbb{R})$ defined in [16]. This norm has the property that if $\psi \in H^1(M;\mathbb{Z})$ is induced by a fibration of M over S^1, i.e., it is a *fibered element*, then the the topological Euler characteristic of the fiber surface $\chi(S)$ satisfies

$$||\psi|| = |\chi(S)|.$$

The unit norm ball for $|| \ ||$ is a convex polyhedron with vertices defined over the integers. For any open top dimensional face F, the primitive integral elements in the cone over F in $H^1(M;\mathbb{R})$ are either all fibered, or are all non-fibered. In the former case, F is called a *fibered face.* The primitive elements in the cone over F are in 1-1 correspondence with rational points on F.

For a fibered face F and subset $K \subset F$, let $\Phi(M,K)$ be the set of monodromies (S,ϕ) of the fibrations corresponding to rational points on K. Then the $\Phi(M,F)$, where F ranges over fibered faces of M, partition the set $\mathcal{P}$ of all pseudo-Anosov mapping classes on punctured oriented surfaces of finite type. Furthermore, by work of Fried [5] and McMullen [14] the normalized dilatation function L extends to a convex function on F going to infinity toward the boundary of F and has a unique minimum in the interior of F. It follows that if $K \subset F$ is a compact subset of F, then L is bounded on $\Phi(M,K)$, and hence $\Phi(M,K)$ defines a family of small dilatation pseudo-Anosov mapping classes.

A theorem of Farb-Leininger-Margalit [4] shows, essentially, that all small dilatation mapping classes are contained in $\Phi(M,K)$, for a finite set of pairs (M,K), as we will now explain. Consider the subcollection $\mathcal{P}^0 \subset \mathcal{P}$ consisting of elements (S,ϕ) whose stable and unstable foliations have no interior singularities. Given $(S,\phi) \in \mathcal{P}$, let S^0 be the complement of the interior singularities in S, and let ϕ^0 be the restriction of ϕ to S^0. Then we have the following.

Lemma 4 *The dilatations of (S,ϕ) and (S^0,ϕ^0) satisfy*

$$\lambda(\phi^0) = \lambda(\phi).$$

It follows that there is a surjective map

$$\mathcal{P} \to \mathcal{P}^0$$

that preserves dilatation and increases normalized dilatation. Let $\mathcal{P}^0(\ell)$ be the set of pseudo-Anosov mapping classes with normalized dilatation less than or equal to ℓ.

Theorem 5 (Farb-Leininger-Margalit [4]) *Given $\ell > 1$, there is a finite set of 3-manifolds $M_1, \dots, M_r$ so that*

$$\mathcal{P}^0(\ell) \subset \bigcup_{i=1}^{r} \Phi(M_i).$$

Remark 6 It follows from Theorem 5 that to understand the shape of all ℓ-small dilatation mapping it suffices to understand how mapping classes vary in $\Phi(M,K)$ for fixed M and K.

We also mention the following Corollary to Theorem 5.

Corollary 7 *If $P \subset \mathcal{P}^0(\ell)$ is any subset, then there is a 3-manifold M so that*

$$P \cap \Phi(M)$$

is infinite.

There has been extensive study, for example, of the so-called magic manifold as a potential manifold associated to small dilatation pseudo-Anoosv maps [**?**, 12, **?**].

Penner showed [15](cf. [14]) that there exists an $\ell > 1$ so that the elements of $P_{g,0}$ are ℓ-small for large enough g, Let $P^0_{g,n}$ be the elements of $P_{g,n}$ after removing singularities. By the Farb-Leininger-Margalit theorem, we have the following.

Corollary 8 *There is a finite set of M_i such that*

$$\bigcup_g P^0_{g,0} \ \subset\ \bigcup_{i=1}^r \Phi(M_i),$$

and there exists a 3-manifold M so that

$$\bigcup_g P^0_{g,0} \ \cap\ \Phi(M)$$

is an infinite set.

Tsai showed in [18] that for fixed $g \geq 2$, the set $\bigcup_n P_{g,n}$ is not ℓ-small for any ℓ. It is plausible, however, that Farb-Leininger-Margalit's finiteness theorem extends to families such as $\bigcup_n P_{g,n}$.

Question 9 *For which $g \geq 2$ does there exist a finite set of M_i so that*

$$\bigcup_{g,n} P^0_{g,n} \ \subset\ \bigcup_{i=1}^k \Phi(M_i) \quad ?$$

Non-hyperbolic Dehn fillings. Let $g \geq 2$ and consider $(S_{g,n}, \phi_{g,n}) \in P_{g,n}$. Let M be the mapping torus. Then either $\overline{\phi}$ is not pseudo-Anosov, and hence the corresponding Dehn filling of M is not hyperbolic, or $\overline{\phi}$ is pseudo-Anosov and we have

$$\lambda(\phi) \geq \lambda(\overline{\phi}) \geq \lambda(\phi_{g,0}) > 1.$$

The latter can only happen for a finite number of n, since for fixed g,

$$\lim_{n\to\infty} \lambda(\phi_{g,n}) = 1$$

(see [18]).

It follows that aside from a finite number of n, the Dehn filling $\overline{M}(\phi_{g,n})$ is non-hyperbolic. Thus, an affirmative answer to Question 9 implies that for each g there is a 3-manifold M such that

$$\Phi(M) \cap \left(\bigcup_n P_{g,n} \right)$$

is infinite (accumulating toward the boundaries of fibered faces of M) and this M admits an infinite number of non-hyperbolic Dehn fillings corresponding to minimum dilatation mapping classes.

Question 10 *Let S be a fixed surface with boundary, and let $\phi \in \mathcal{P}(S)$ be an element of minimum dilatation. Is the Dehn filling of the mapping torus of (S, ϕ) corresponding to ϕ always non-hyperbolic?*

If the answer to Question 10 is negative, it implies that for some $g \geq 2$, the sequence $\delta_{g,n}$ is not strictly monotone decreasing as a function of n (cf. [3]).

3 Two constructions of nearly periodic mapping classes with small dilatation.

It is reasonable to guess that small dilatation mapping classes should be "nearly" periodic. We give two descriptions of sequences of mapping classes that are of this form.

Penner-type sequences. Let $\phi \in \mathcal{P}(S)$ be a mapping class with the following properties:

(i) S admits a periodic map $R_k : S \to S$ of order k with fundamental domain a subsurface Σ with boundary,

(ii) there are two disjoint unions of arcs B^+ and B^- on the boundary of Σ so that

$$R_k(B^-) = B^+ = \Sigma \cap R_k(\Sigma),$$

(iii) $\eta : S \to S$ is the identity map outside of Σ,

(iv) γ is a simple-closed curve on $\Sigma \cup R_k\Sigma \cup \cdots R_k^s\Sigma$ with $s < k$, and

(v) $R^i\gamma$ is disjoint from γ for all $i \leq s$.

A sequences of mapping classes (S_k, ϕ_k) is of *Penner-type* if for some $R_k, \gamma, \eta, \Sigma, B^{\pm}$ as above,

$$\phi_k = R_k \circ \partial_\gamma \circ \eta,$$

where ∂_γ is the (right or left) Dehn twist centered at γ. Let $C = |\chi(\Sigma \cup \gamma)|$. We say that the Penner sequence has *support* bounded by C. Given a sequence of Penner-type, let $\overline{\Sigma} = S_k/R_k$, and let $\overline{\phi}$ be the composition of $\partial_{\overline{\gamma}} \circ \eta$, where $\overline{\gamma}$ is the image of γ in the quotient space $\overline{S}_k$.

Theorem 11 ([10]) *Let (S_k, ϕ_k) be a Penner-type sequence. Then (S_k, ϕ_k) is pseudo-Anosov for large k if and only if $(\overline{\Sigma}, \overline{\phi})$ is pseudo-Anosov. In this case, the normalized dilatations $L(S_k, \phi_k)$ converges to $L(\overline{\Sigma}, \overline{\phi})$ and hence is bounded.*

Question 12 (Farb-Leininger-Margalit) *Can any small dilatation mapping class be constructed as a composition of a periodic mapping class and a mapping class that is the identity outside a locus with bounded Euler characteristic?*

Twisted mapping classes. Let P_m be a closed $2m$-gon with alternate sides removed. Let (S_1, ϕ_1) and (S_2, ϕ_2) be two mapping classes with proper embeddings $P_m \subset S_i$, for $i = 1, 2$. Then the *Murasugi sum* of (S_1, ϕ_1) and (S_2, ϕ_2) equals (S, ϕ), where S is the result of gluing S_1 and S_2 along the corresponding mages of P_m and ϕ is the composition of the extensions of ϕ_1 and ϕ_2 by the identity on S.

In [9], we show the following.

Lemma 13 *For each m, there is a family of mapping classes (Σ_k, σ_k) so that*

(i) σ_k^{mk} is a composition of Dehn twists centered at boundary components of Σ_k,

(ii) there exist mk disjoint embedded copies of P_m in Σ_k, and

(iii) the mapping tori of (Σ_k, σ_k) are independent of k.

The surfaces Σ_k constructed in [9] come with a distinguished proper embedding of P_m. Let (S_0, ϕ_0) be any mapping class with a proper embedding of P_m in S_0. Let (S_k, ϕ_k) be the mapping classes obtained by Murasugi sum of $(S_0.\phi_0)$ with (Σ_k, σ_k) along P_m.

Lemma 14 ([7]) *The mapping tori for (S_k, ϕ_k) have homeomorphism type that is independent of k.*

Theorem 15 ([7]) *For any choice of (S_0, ϕ_0), the mapping classes (S_k, ϕ_k) correpsond to a convergent sequence on a fibered face (possibly converging to the boundary).*

Theorem 16 ([9]) *There exists (S_0, ϕ_0) so that (S_k, ϕ_k) converge to a point in the interior of a fibered face, and*

$$\log(\lambda(\phi_k)) \asymp \frac{1}{k}.$$

In particular, there is an (S_0, ϕ_0) so that by closing over the boundary of S_k, we obtain orientable mapping classes $(\overline{S}_k, \overline{\phi}_k)$ such that

$$\lim_{k\to\infty} \lambda(\overline{\phi}_k)^{g_k} = \frac{3+\sqrt{5}}{2},$$

where g_k is the genus of $\overline{S}_k$.

4 Small dilatation orientable pseudo-Anosov mapping classes from mixed-sign Coxeter graphs.

In this section, we construct small dilatation quasi-periodic mapping classes using generalized Coxeter graphs.

Let Γ be a simply-laced Coxeter graph with vertices $\mathcal{V}$ and a *sign-labeling* $\mathfrak{s} : \mathcal{V} \to \{\pm 1\}$. A *geometric realization* of Γ is a pair $(S, \mathcal{G})$, where S is a compact oriented surface, and $\mathcal{G}$ is a set of simple-closed curves on S in general position with a bijection $f : \mathcal{V} \to \mathcal{G}$ so that the geometric intersection matrix for $\{f(v) \mid v \in \mathcal{V}\}$ in S equals the incidence matrix for $\mathcal{V}$ on Γ. The geometric realization $(S, \mathcal{G})$ determines a map from the Artin group of Γ to the mapping class group of S given by sending generators of the Artin group to Dehn twists centered at the curves in $\mathcal{G}$. Let $\phi : S \to S$ be the composition of Dehn twists centered at the curves of $\mathcal{G}$ with respect to some ordering. The graph Γ determines $(S, \mathcal{G})$ once we add the following requirements:

(a) the realization $(S, \mathcal{G})$ respects a given fat graph structure on Γ;

(b) S has a deformation retract to the union of curves in $\mathcal{G}$;

(c) for a given ordering on the vertices $\{v_1, \dots, v_k\}$ of Γ, if $i < j$, then the algebraic intersection of the curves γ_i and γ_j is non-positive; and

(d) the ordering of $\mathcal{G}$ used to define ϕ is compatible with the ordering in (c).

Given a surface S with boundary, let $\overline{S}$ be the closed surface obtained by filling in the boundary components of S with disks. If ϕ is a mapping class on S, then let $\overline{\phi}$ be the isotopy class of the canonical extension of ϕ over $\overline{S}$. We call $(\overline{S}, \overline{\phi})$ the closure of the mapping class (S, ϕ).

Question 17 *For which g can the minimum dilatation orientable mapping classes on a closed surfaces of genus g be realized as the closure of a mixed-sign Coxeter mapping class?*

In [9], we show using results of [13] that minimum dilatation orientable mapping classes for genus $g = 2, 3, 4$ and 5 can be realized as the closures of mixed-sign Coxeter mapping classes.

The structure of the mixed-sign Coxeter mapping classes is strongly associated to properties of an associated reflection system, which we call the mixed-sign Coxeter system. These are defined in [9]. The key property is that the Coxeter element (a product of reflections) of the mixed-sign Coxeter system has spectral radius equal to the spectral radius of the homological action of the corresponding mapping class (a corresponding product of parabolic elements). One expects small dilatation mapping classes to come from Coxeter graphs that are the join of a small Coxeter graph with a Coxeter element of spectral radius 1.

Consider the graph in Figure 1. The positively signed (or *classical*) Coxteter system associated to this graph is of higher rank type [2], and in particular none of its Coxeter elements have finite order. When the vertices of this graph are all given negative signs, however, and the vertices are ordered from top to bottom, the Coxeter element has finite order, but the Coxeter group can have infinite order, as is true for the Coxeter graph in Figure 1. One can see this by noticing that the graph contains bipartite Coxeter subgraphs that are non-spherical or affine.

The example in Figure 1 can be generalized to graphs with $m \times m$ vertices for $m \geq 2$ (see [9]). Broadly speaking, mixed-sign Coxeter graphs provide a larger set of examples of periodic mapping classes than in the classical case. These mapping classes may in turn be used to construct further examples of small dilatation mapping classes.

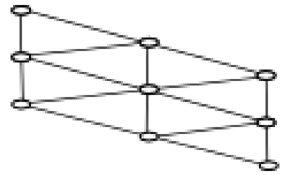

Figure 1: A negatively signed graph with finite order Coxeter element.

Problem 18 *Classify mixed-sign Coxeter graphs. In particular, which mixed-sign Coxeter graphs have a Coxeter element of finite order?*

5 Singularities of mapping classes

We conclude this note with some further questions concerning the shape of small dilatation mapping classes (S, ϕ). These concern the associated local Euclidean structure on S so that ϕ stretches in one direction by $\lambda > 1$ and in the other by $\frac{1}{\lambda}$.

Let $P_{*,0} = \bigcup_g P_{g,0}$. In [8], we find a sequence $(S_g, \phi_g) \in \mathcal{P}$, where S_g is a closed surface of $g \geq 2$, and $L(S_g, \phi_g)$ converges to ℓ_0. For these examples, (S_g, ϕ_g) has either 2 or 4 singularities.

Question 19 *Is there a bound on the number of singularities of elements of $P_{*,0}$?*

By Theorem 5, we know, for example, that there is a finite collection of hyperbolic 3-manifolds M_i such that the elements of $P_{*,0}$ are, after removing singularities, contained in $\Phi(M_i)$ for some i. Since the number of orbits of the singularities an element of $\Phi(M_i)$ equals the number of cusps of M_i, this means that the number of orbits must be bounded.

Question 20 *What is the maximum number of orbits of singularities for $(S, \phi) \in P_{*,0}$?*

Now consider $P_{g,*} = \bigcup_n P_{g,n}$. If Question 9 has an affirmative answer, then again, we see that the number of orbits of the singularities of $P_{g,n}$ must be bounded. On the other hand, by a theorem of Thurston, a hyperbolic 3-manifold with a single cusp has at most a finite number of non-hyperbolic Dehn fillings. Thus, an affirmative answer to Question 9 would imply that for fixed g there are an infinite number of elements of $P_{g,*}$ with punctures lying in more than one orbit. For $g = 0$, the smallest known examples have one orbit (see [11]).

The following questions are analogs of Question 19 and Question 20 for the punctured case.

Question 21 *For each fixed g, is there a bound on the number of interior singularities of elements $(S, \phi) \in \mathcal{P}_{g,n}$?*

Question 22 *For each fixed g is there a bound on the number of orbits of punctures for $(S, \phi) \in P^0_{g,n}$?*

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