Lehmer's Number and the Golden Mean

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Part I: Mandelbrot set

For c any complex constant, consider the polynomial function

$$f_c : z \mapsto z^2 + c$$

acting on the complex plane.
We can identify the the set of complex numbers $\mathbb{C}$ with the Euclidean plane $\mathbb{R}^2$:

$$\begin{aligned} z &= (x, y) \\ c &= (c_1, c_2) \\ f_c(x, y) &= (x^2 - y^2 + c_1, 2xy + c_2) \end{aligned}$$

c_1, c_2, x, y are real numbers

The dynamical behavior of the maps f_c fall into one of three classes:

1. there are (non-empty) open regions that are attracted to infinity and open regions that approach a periodic cycle (these regions are separated by what is called the *Julia set*);
2. there is no finite attractive periodic cycle, but there are points that stay bounded under iterations of the map (the points that stay bounded form the Julia set); and
3. all points are attracted to infinity (e.g., if $|c|$ is large enough).

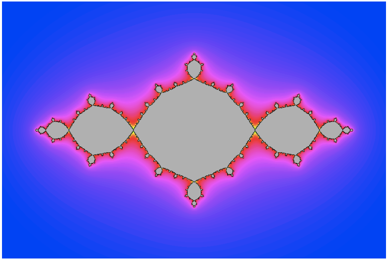

Figure: Julia set for $z \mapsto z^2 - 1$

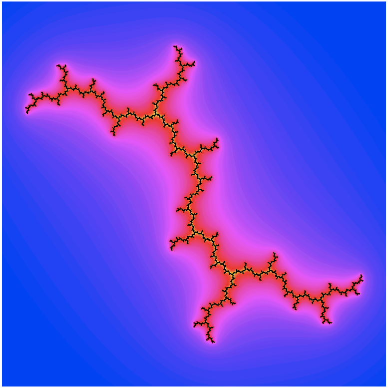

Figure: Julia set for $z \mapsto z^2 + i$

In the 1970's and 1980's, B. Mandelbrot working at IBM Watson Research Center asked, what happens when you plot the values of c for which the dynamics have Type 1?

(R. Brooks, J. Hubbard, A. Douady, D. Sullivan, others...)

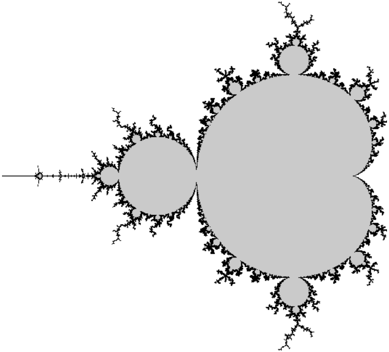

Figure: Mandelbrot Set (taken from Curt McMullen's web gallery)

Zooming in on the Mandelbrot Set. VIMEO

The video illustrates some themes that occur throughout mathematics.

- Building on simple rules, one can create and explore a world, in which even the most naive questions are difficult to solve.

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  - continuous parameter spaces for geometric structures
  - deformations of structures and corresponding deformations of invariants
- Deep investigation leads to complicated patterns from which certain essential recurring objects appear.

Part II: Lehmer's problem

Algebraic integers

Monic integer polynomial:

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0, \qquad a_i \in \mathbb{Z}.$$

Algebraic integer $\alpha \in \mathbb{C}$:

Solution to $p(\alpha) = 0$, for some monic integer polynomial

Some properties

The set of algebraic integers...

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- is dense in $\mathbb{C}$.

Example

Golden Mean: $\phi = 1 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cdots}}} \approx 1.618034$

ϕ is a root of $x^2 - x - 1$.

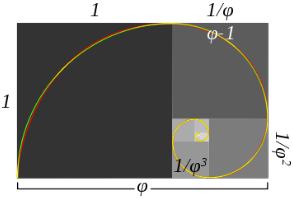

Figure: Acropolis in Athens, Greece, and spiraling squares

Location of zeros: roots of unity

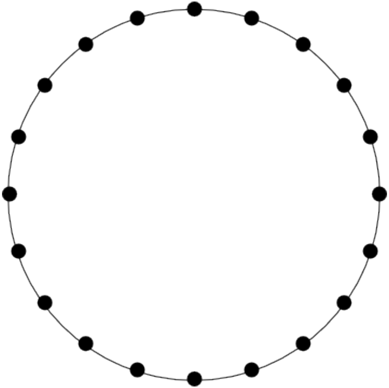

Figure: Roots of $p(x) = x^{20} - 1$.

Location of zeros: golden mean and its conjugate

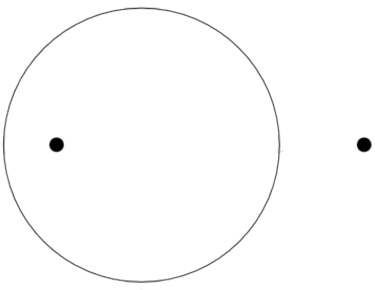

Figure: Roots of $p(x) = x^2 - x - 1$ and unit circle.

Location of zeros: a quintic polynomial

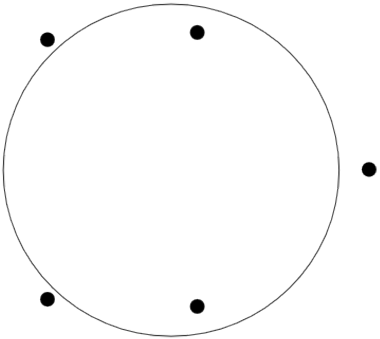

Figure: Roots of $p(x) = x^5 - x^2 - 1$ and unit circle.

Invariants of monic integer polynomials

Mahler measure: $\mathrm{Mah}(p(x)) = \Pi_{p(\mu)=0} \max\{|\mu|, 1\}$.

House: $|p(x)| = \max\{|\mu| \ : p(\mu) = 0\}$.

$N(p(x)) = \#\{\mu \ p(\mu) = 0, |\mu| > 1\}$.

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Question (Lehmer's problem, 1933)

Given $\delta > 0$, is there a monic integer polynomial $p(x)$ so that

$$1 < Mah(p(x)) < 1 + \delta?$$

Palindromic polynomials

palindromic polynomials:

$$p(x) = x^{2n} + a_{n-1}x^{2n-1} + \cdots + a_1x^{n+1} + a_1x^n + \cdots + a_{n-1}x + 1.$$

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Property: if μ is a root of $p(x)$, then so is $\frac{1}{\mu}$. We also say $p(x)$ is *reciprocal*.

Some subclasses of algebraic integers

Perron polynomials: $p(x)$ has a root $\alpha > 0$ such that $\alpha = |p(x)|$ and for all other roots μ of $p(x)$,

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Perron, Salem and **Pisot** numbers are the house of Perron, Salem and Pisot polynomials

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- (Smyth) The smallest Mahler measure for non-reciprocal polynomials is given by

$$\mu_P = \mathrm{Mah}(x^3 - x - 1) \approx 1.32472,$$

$x^3 - x - 1$ is a Pisot Polynomial.

Smallest Pisot number

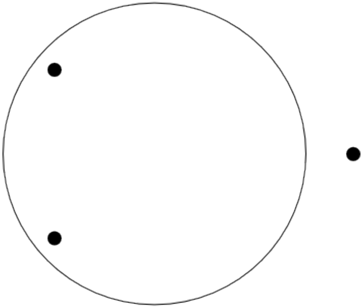

Figure: Roots of smallest Pisot polynomial $p_L(x) = x^3 - x - 1$.

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- Lehmer's number is smaller than the smallest non-reciprocal Mahler measure

$$\lambda_L = \mathrm{Mah}(p_L(x)) \approx 1.17628 < \mu_P,$$

  $p_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$ is a Salem polynomial.

Lehmer's number

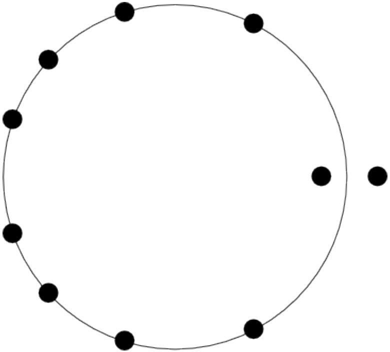

Figure: Roots of Lehmer's polynomial
$p_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1.$

Properties:

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- Open problem: Is there a polynomial $p(x)$ with

$$1 < \text{Mah}(p(x)) < \lambda_L?$$

Assume $p(x)$ is not a product of cyclotomic polynomials.

Still open questions:

- Is the minimum $\mathrm{Mah}(p(x))$ greater than one attained for $p(x)$ non-cyclotomic?
- Is the minimum attained by a Salem number?
- Is there a universal lower bound greater than 1 for

$$L(p(x)) = |p(x)|^{\deg(p(x))}?$$

Mapping classes on Surfaces

Let S be a compact surface of genus g with n boundary components.

Let $f : S \to S$ be a *mapping class*, that is, a self-homeomorphism taken up to isotopy relative to the boundary.

There are three types...

Nielsen-Thurston classification of mapping classes

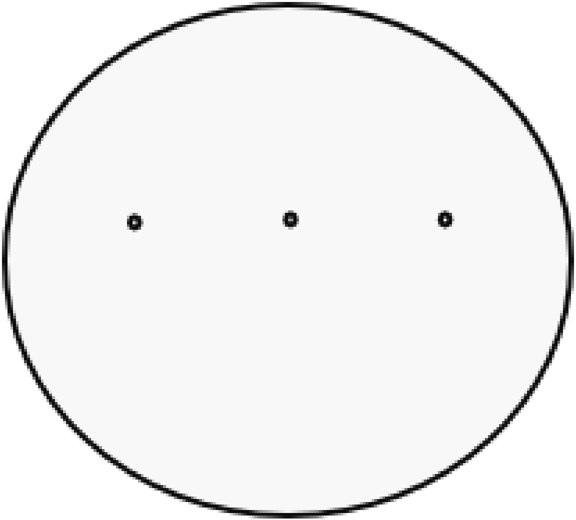
f is either

- *periodic* $f^n = \mathrm{id}$,
- *reducible* $f(\gamma) = \gamma$ for some essential simple closed curve $\gamma \subset S$, or
- *pseudo-Anosov* if $\ell_\omega(f^n(\gamma))$ has growth rate $\lambda > 1$, where λ does not depend on choice of Riemannian metric ω or γ. (can think of as a well-mixing property)

Action on essential simple closed curves

Example: a pseudo-Anosov map on the $S_{0,4}$, the sphere with 4 boundary components.

simplest hyperbolic braid:



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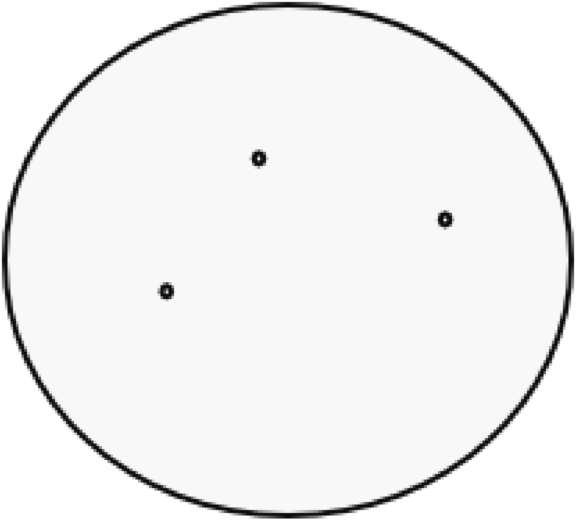

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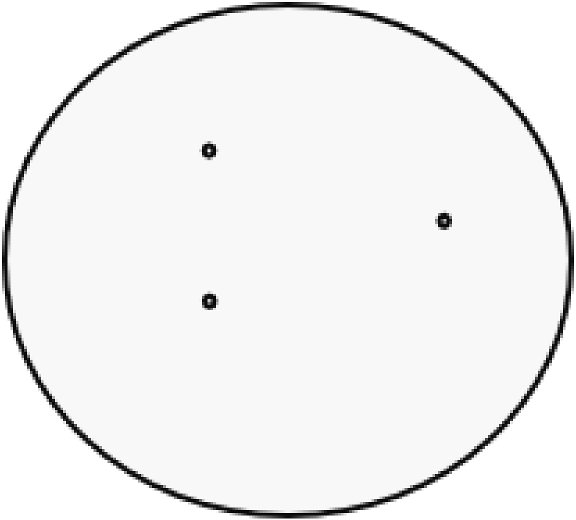

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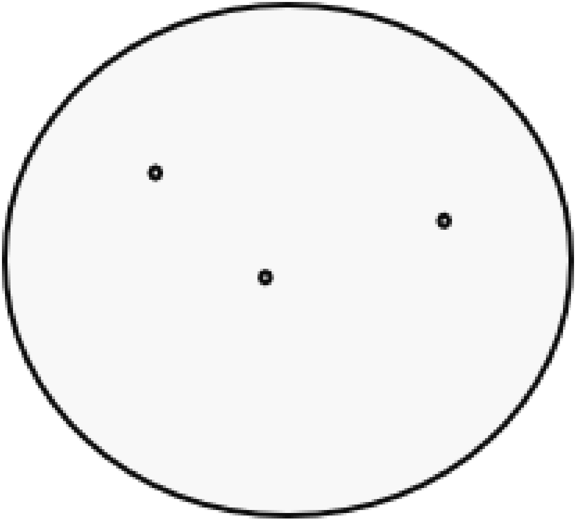

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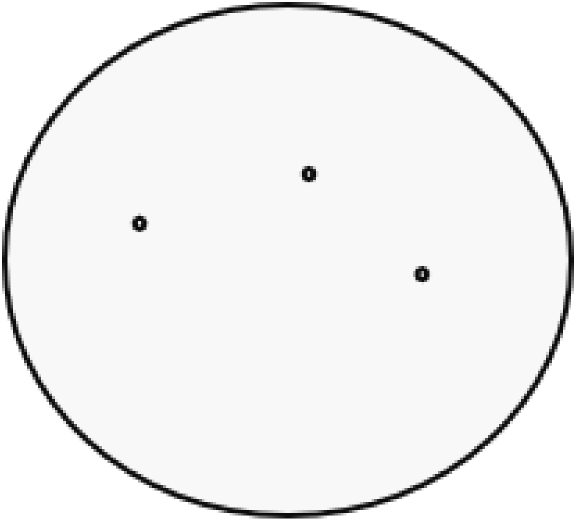

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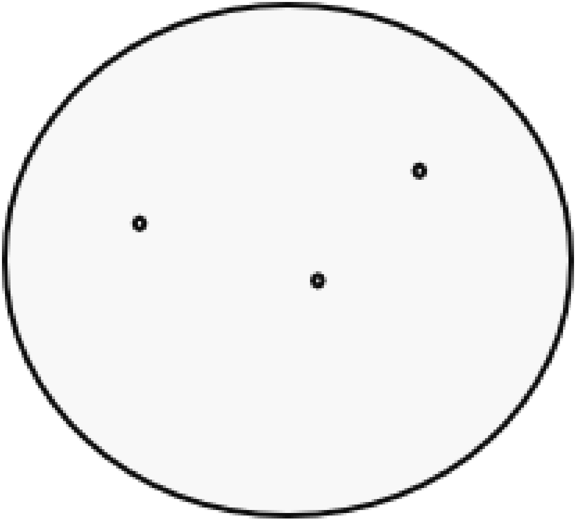

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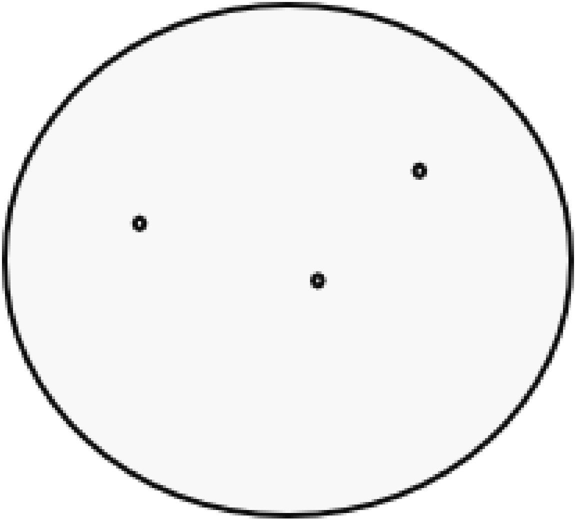

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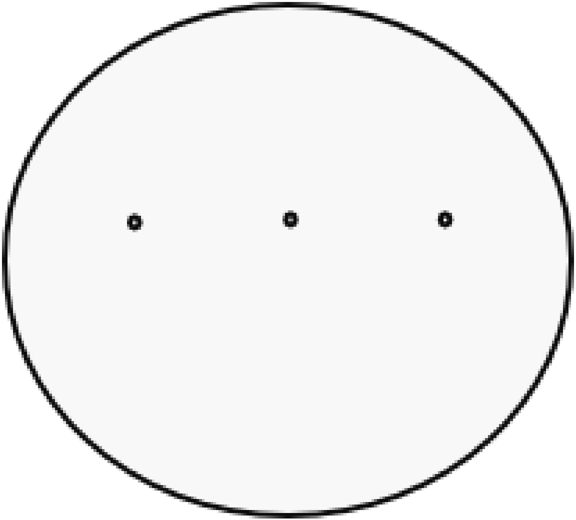
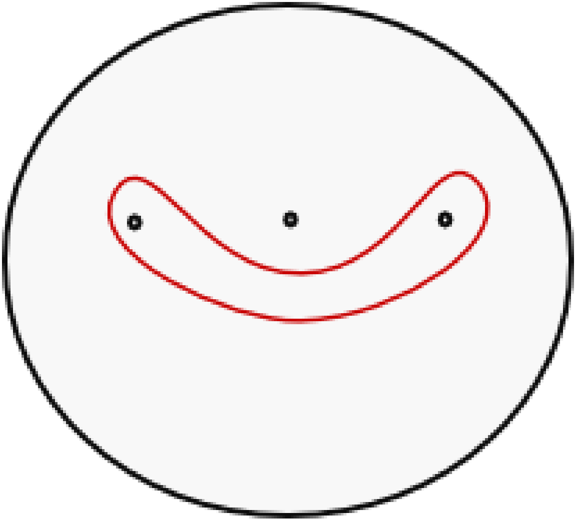


Action of the mapping class

Action of the mapping class on a simple closed curve.

Action on a simple closed curve:



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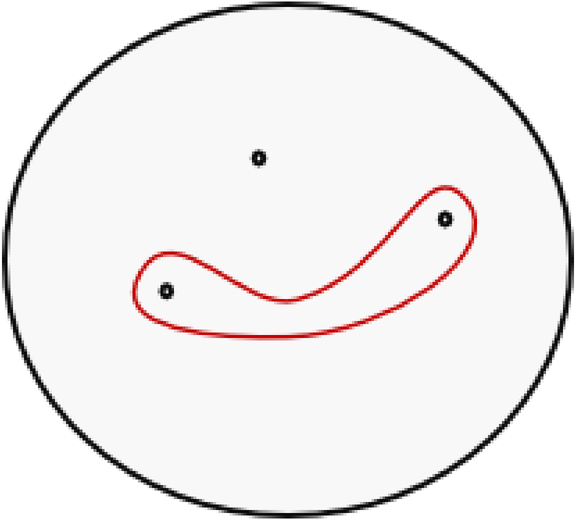

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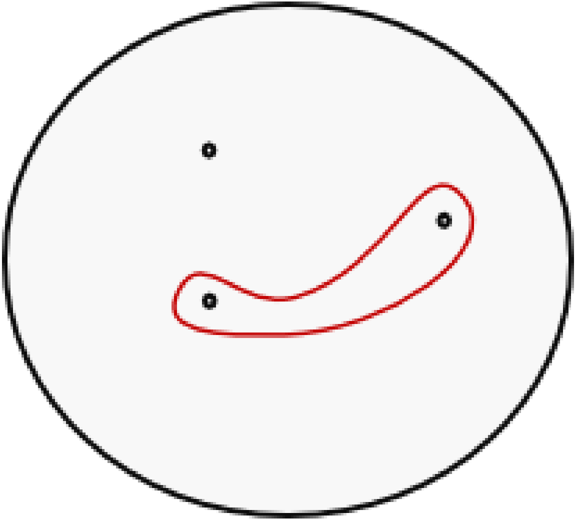

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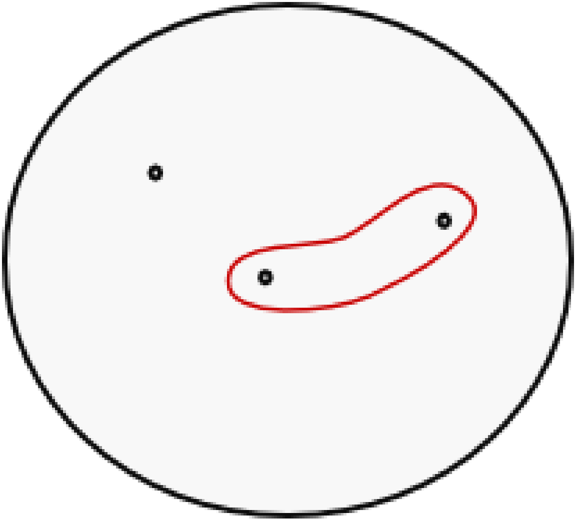

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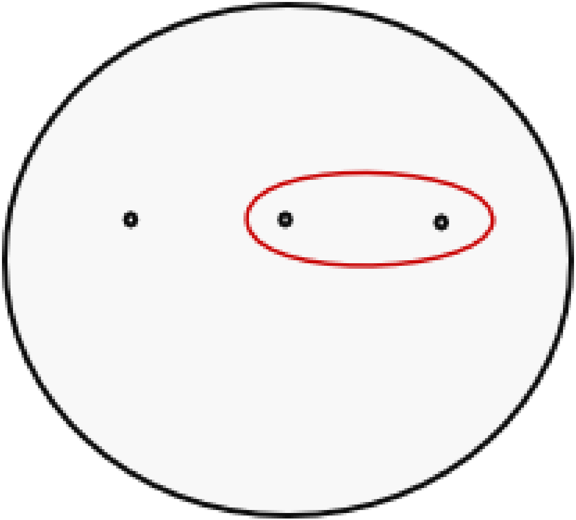

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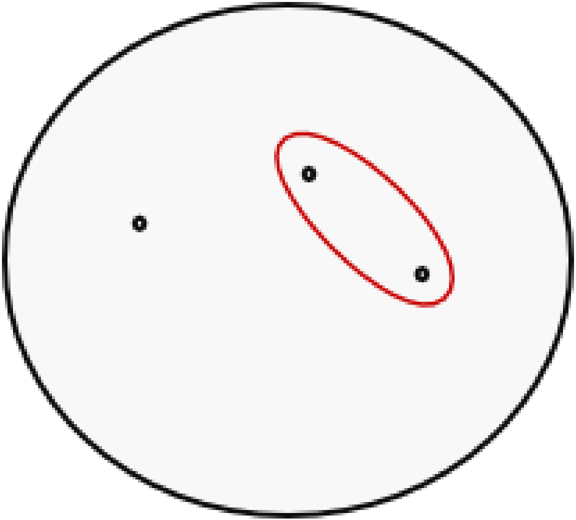

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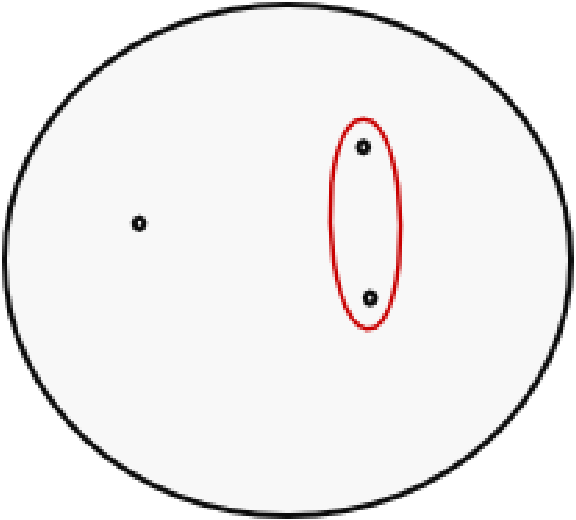

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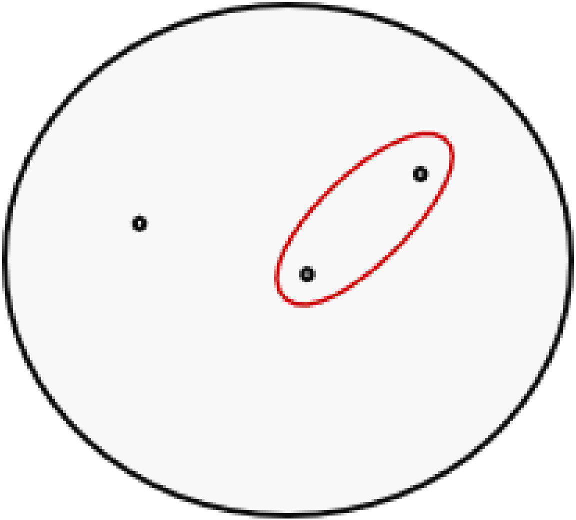
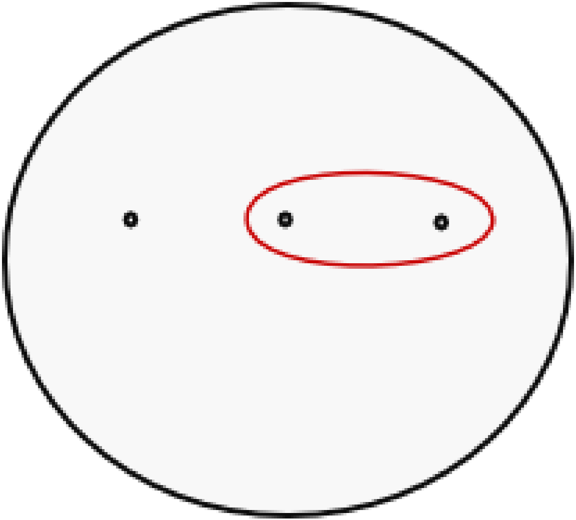


Action on a simple closed curve (one application of map):



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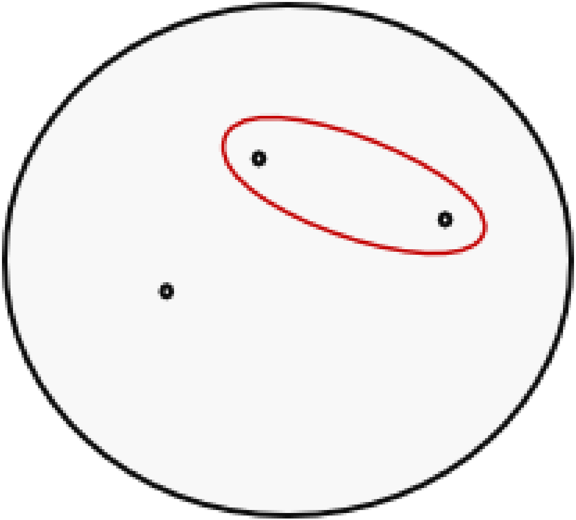

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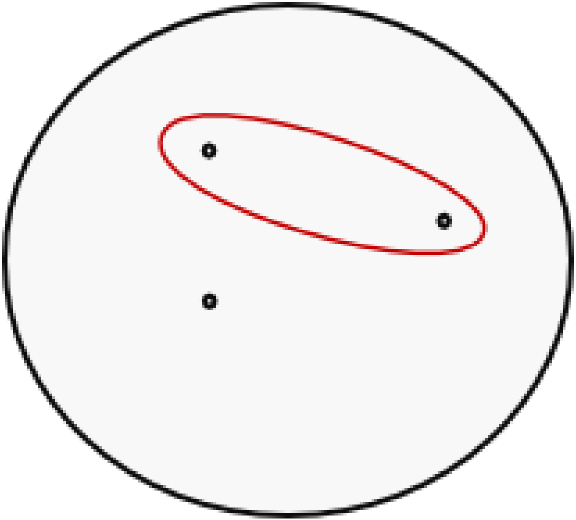

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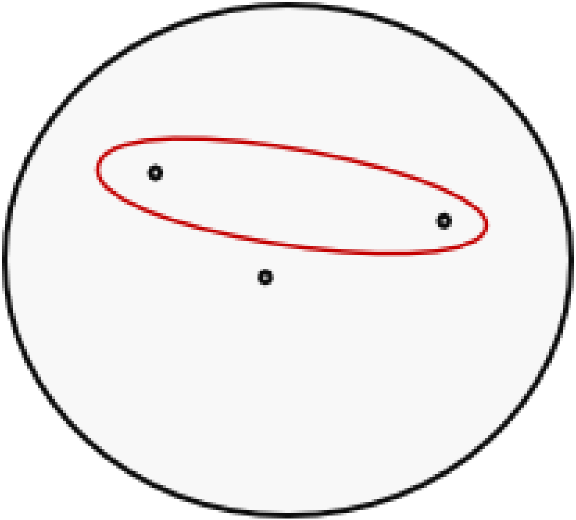

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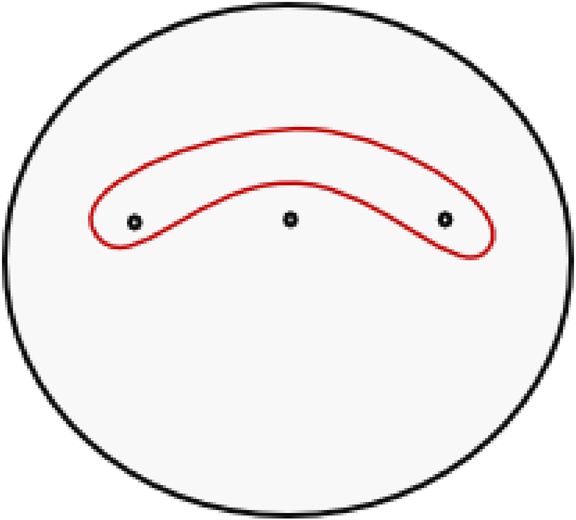

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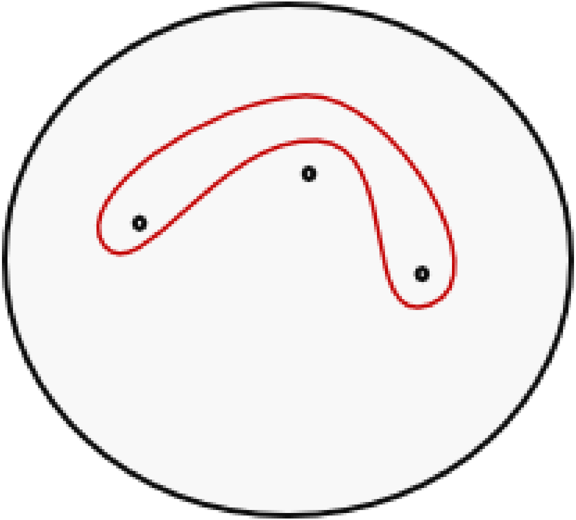

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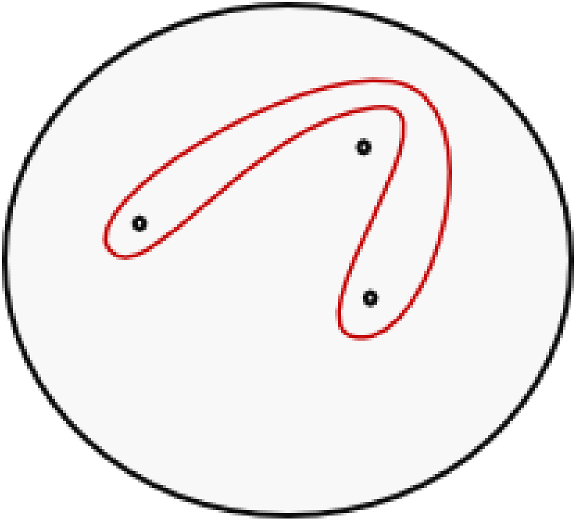

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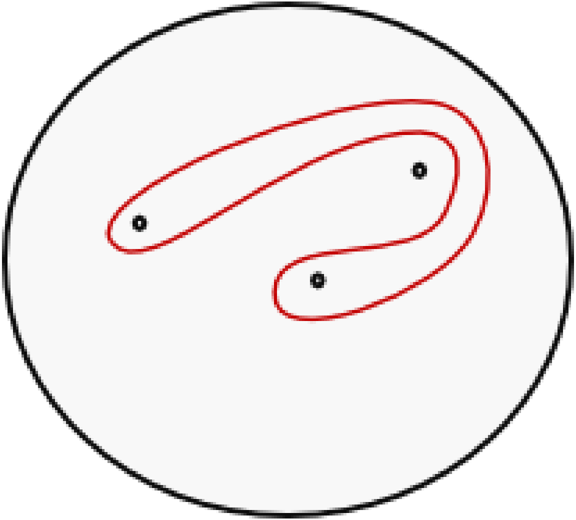
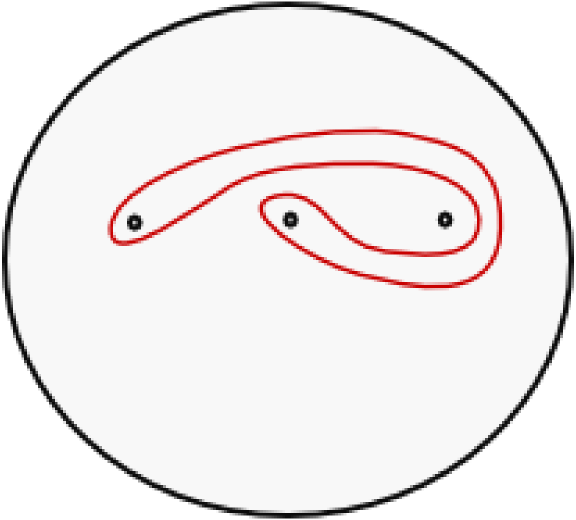


Action on a simple closed curve (2 applications of map):



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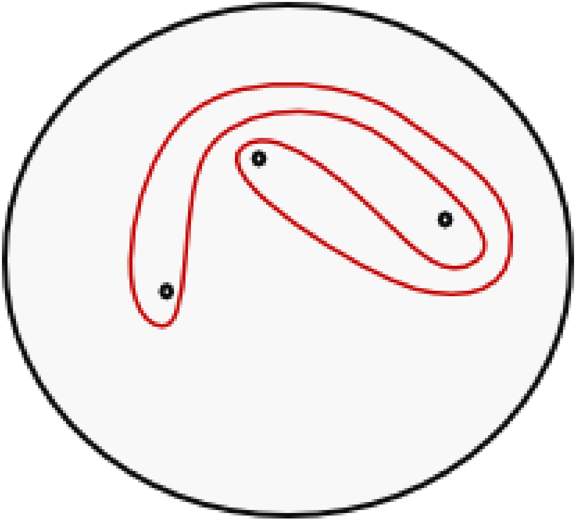

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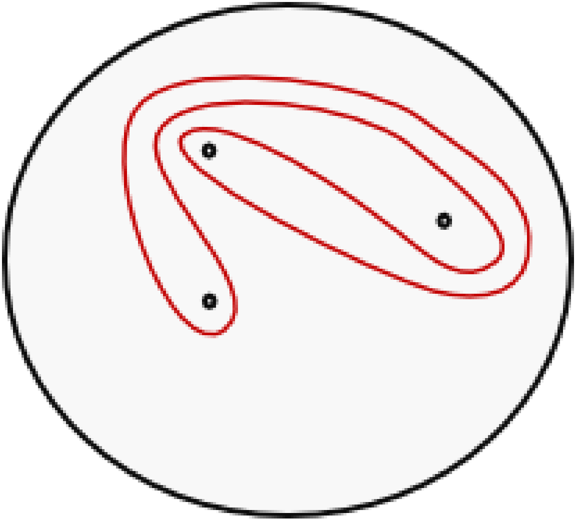

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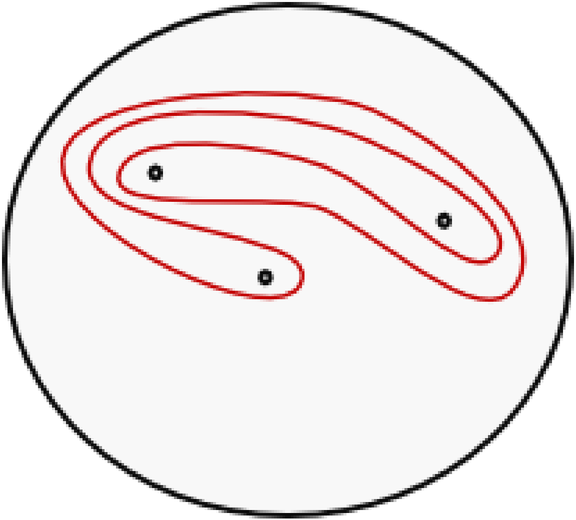

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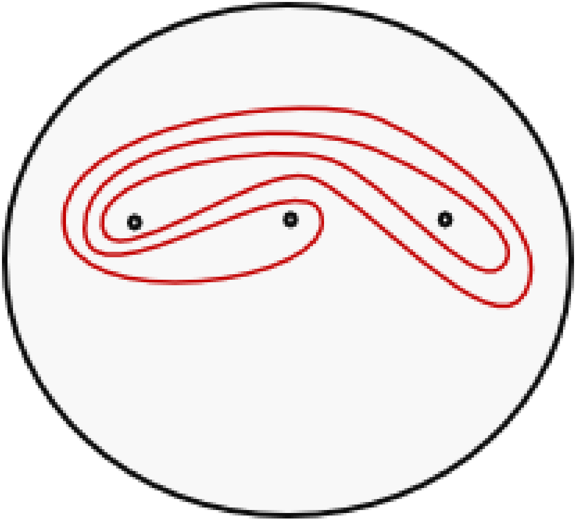

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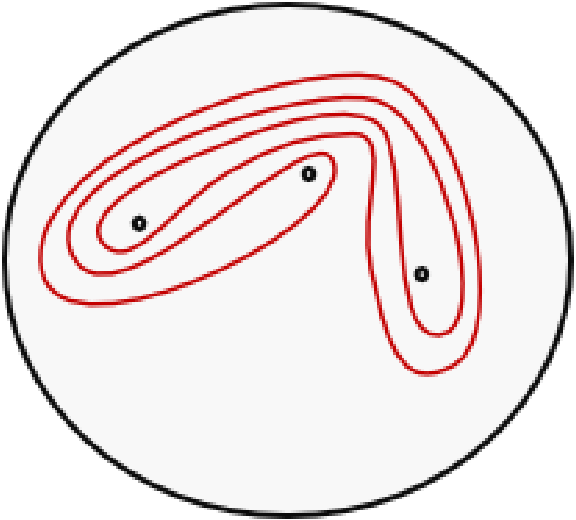

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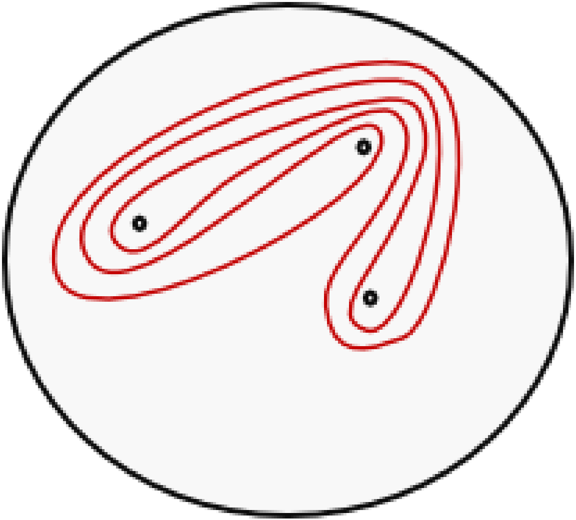

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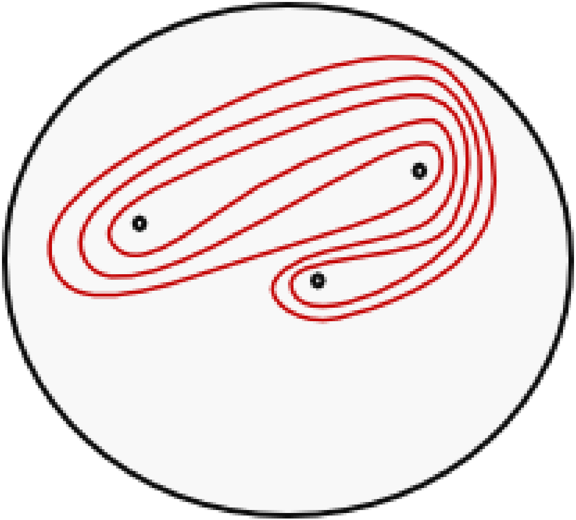

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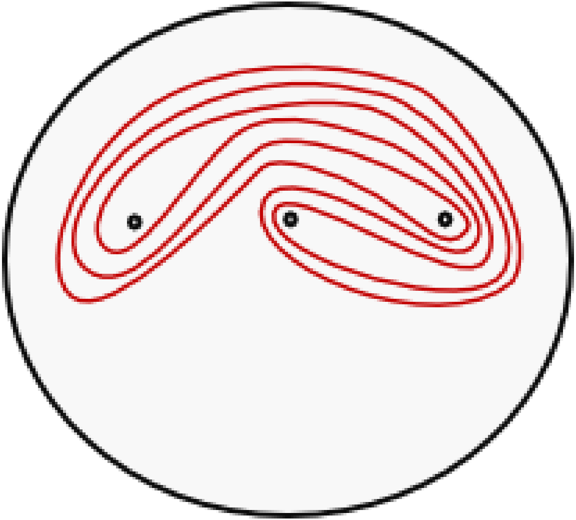
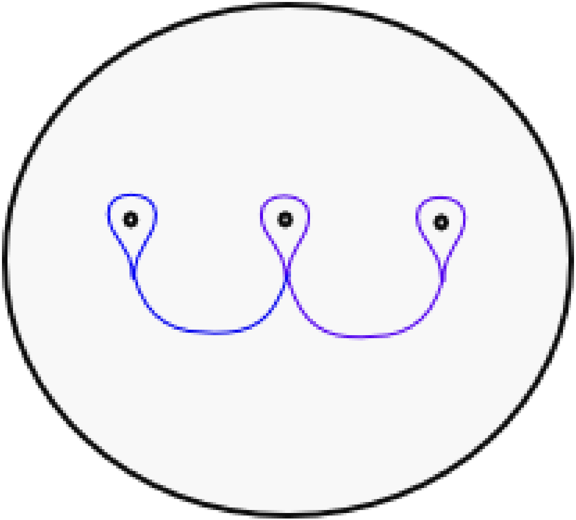
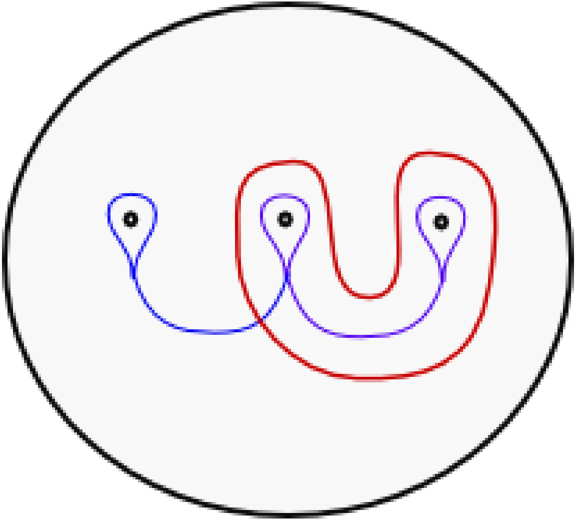


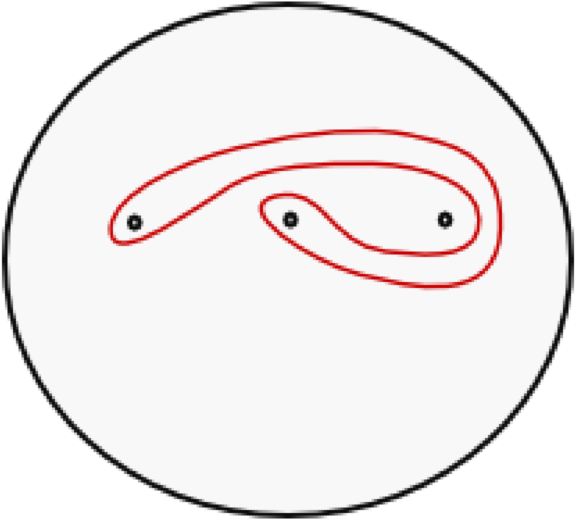
Train track compatible with simplest pseudo-Anosov braid



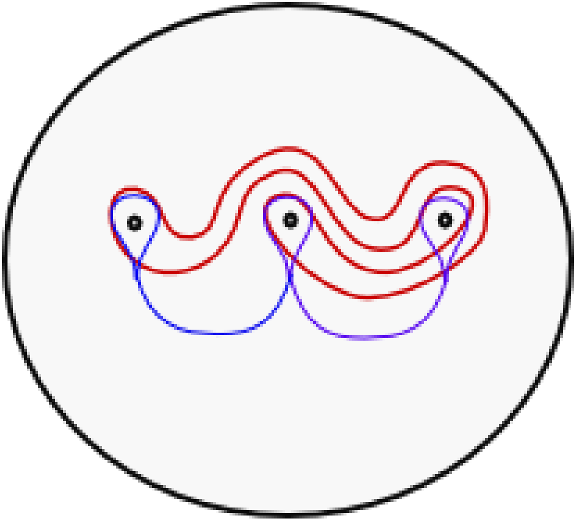
After 1st application of map (with train track):



Curve γ after 2nd application of map:



After 2nd application of map (with train track):



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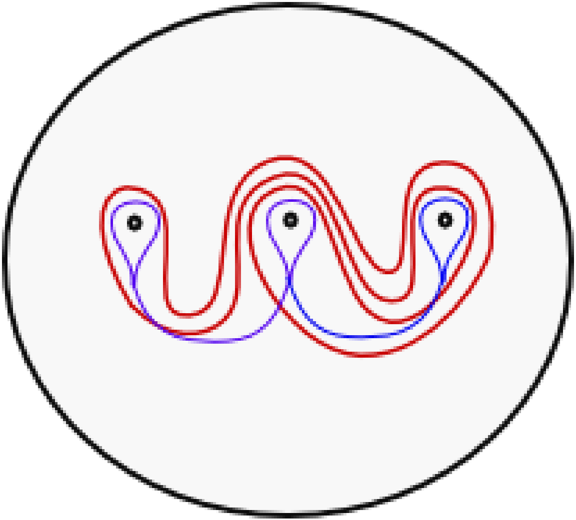
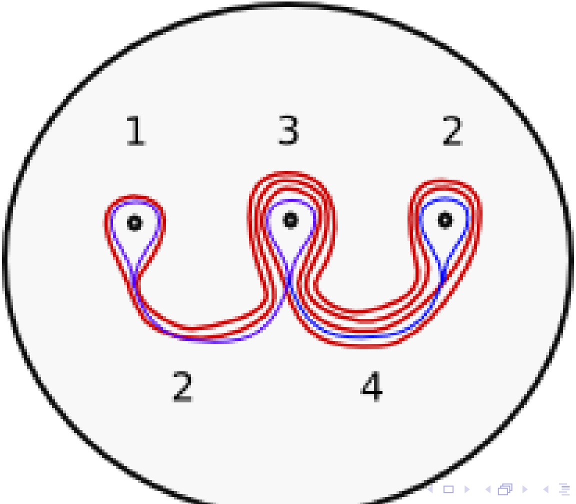
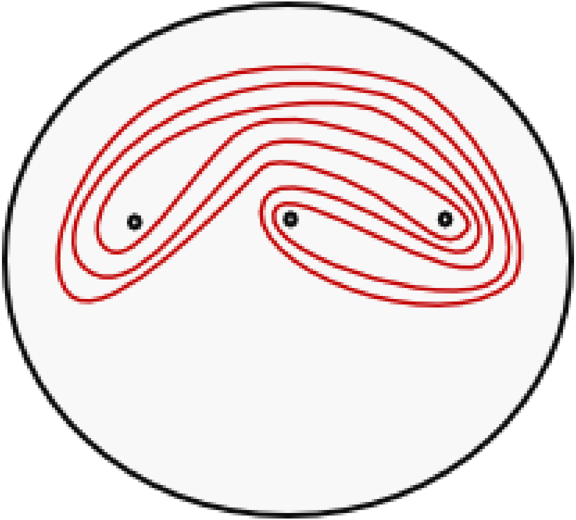


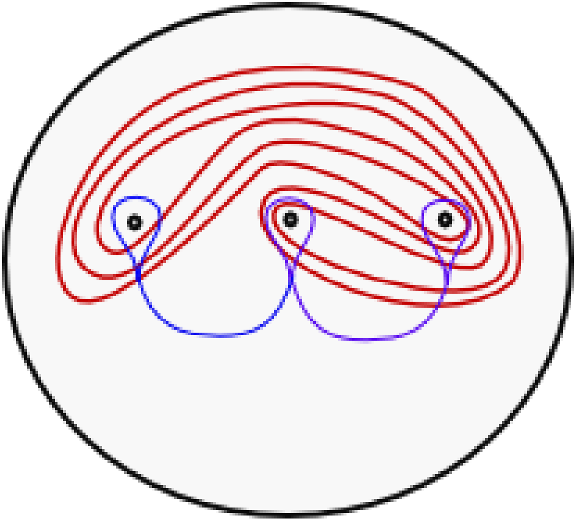
Train track with edge weights (after 2nd application of map):



Curve γ after 3rd application of map:



After 3rd application of map (with train track):



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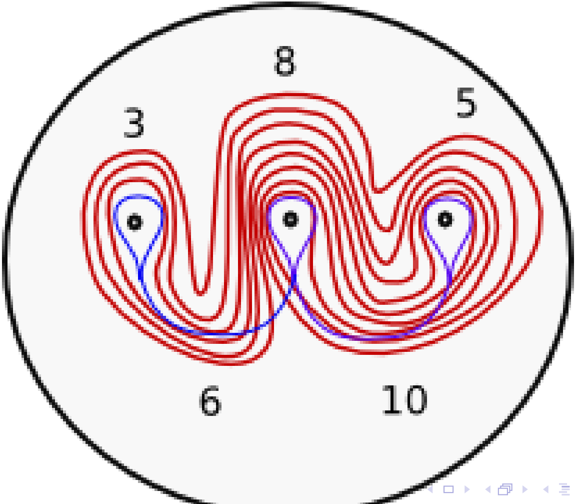
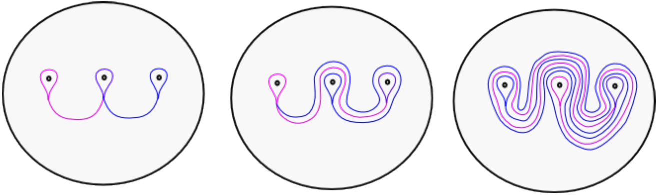
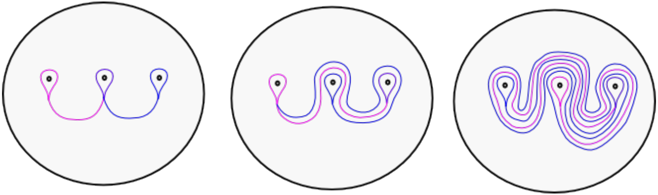


(Thurston) Pseudo-Anosov mapping classes have the property that for some train track, every essential simple closed curve is eventually *carried* on the train track. Thus the action of the mapping class on the train track determines the dilatation λ.

Transition matrix



Transition matrix



$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

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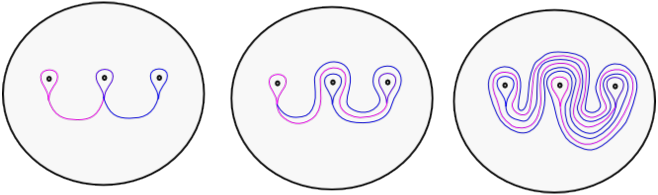

$$T = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}.$$

In our example,

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 4 \end{bmatrix} \mapsto \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

Consequence:

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Open question (W. Thurston): Are all Perron units realizable as the dilatation of a pseudo-Anosov mapping class?

Simplest hyperbolic braid and the golden mean

Define the *normalized dilatation* of a mapping class $f : S \to S$ by

$$L(S, f) = \lambda(f)^{|\chi(S)|}.$$

The smallest known accumulation point of $L(S, f)$ is

$$\left(\frac{3+\sqrt{5}}{2}\right)^2 = |x^2 - 3x + 1|^2 = (\text{golden mean} + 1)^2 = (\text{golden mean})^4,$$

realized by the simplest hyperbolic braid.

Lehmer's number and the golden mean

- The simplest hyperbolic braid generates an infinite family of mapping classes on different surfaces.

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- The simplest hyperbolic braid generates an infinite family of mapping classes on different surfaces.
- The function L behaves continuously under deformations in this family. (W. Thurston, D. Fried, C. McMullen)
- In this family, there is a mapping class with dilatation equal to Lehmer's number.

Lehmer's number and golden mean

An analysis of the family yields the (Teichmüller polynomial)

$$p(u,t) = u^2 - u(1 + t + t^{-1}) + 1.$$

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All the dilatations in the family can be computed using this polynomial. In particular,

$$\text{golden mean} + 1 = |p(x,1)| = |x^2 - 3x + 1|$$

and

$$\text{Lehmer's number} = |p(x^6, x)| = |x^{12} - x^7 - x^6 - x^5 + 1|.$$

Summary

In the context of normalized dilatations of pseudo-Anosov mapping classes, the 4th power of the golden mean is the conjectural minimum, and it is realized by the simplest pseudo-Anosov braid.

Lehmer's number appears in the family of dilatations naturally associated to the simplest pseudo-Anosov braid by "going up one dimension" and using the geometry of 3 dimensional manifolds.

Analogous to the Mandelbrot set, the golden mean and Lehmer's number occurs and recurs as one delves into questions about the dynamical complexity of algebraic integers and mapping classes of surfaces.

Thank you!

VIMEO