Alexander Stratifications of Character Varieties

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Abstract

Equations defining the jumping loci for the first cohomology group of one-dimensional representations of a finitely presented group Γ can be effectively computed using Fox calculus. In this paper, we give an exposition of Fox calculus in the language of group cohomology and in the language of finite abelian coverings of CW complexes. Work of Arapura and Simpson imply that if Γ is the fundamental group of a compact Kähler manifold, then the strata are finite unions of translated affine subtori. It follows that for Kähler groups the jumping loci must be defined by binomial ideals. As we will show, this is not the case for general finitely presented groups. Thus, the "binomial condition" can be used as a criterion for proving certain finitely presented groups are not Kähler.

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1 Introduction

Let X be homotopy equivalent to a finite CW complex and let Γ be the fundamental group of X. One would like to derive geometric properties of X from a finite presentation

$$\langle\, x_1, \ldots, x_r \;:\; R_1, \ldots, R_s \,\rangle$$

of Γ. Although the isomorphism problem is unsolvable for finite presentations, Fox calculus can be used to effectively compute invariants of Γ, up to second commutator, from the presentation. In this paper, we study a natural stratification of the character variety $\widehat{\Gamma}$ of Γ, associated to Alexander invariants, which we will call the *Alexander stratification.* We relate properties of the stratification to properties of unbranched coverings of X and to the existence of irrational pencils on X when X is a compact Kähler manifold. Furthermore, we obtain obstructions for a group Γ to be the fundamental group of a compact Kähler manifold.

This paper is organized as follows. In section 2, we give properties of the Alexander stratification as an invariant of arbitrary finitely presented groups. We begin with some notation and basic definitions of Fox calculus in section 2.1. In section 2.2, we relate the Alexander stratification to jumping loci for group cohomology and in section 2.3 we translate the definitions to the language of coherent sheaves. This allows one to look at Fox calculus as a natural way to get from a presentation of a group to a presentation of a canonically associated coherent sheaf, as we show in section 2.4. Another way to view the Fox calculus is geometrically, by looking at the CW complex associated to a finitely generated group. We show how the first Betti number of finite abelian coverings can be computed in terms of the Alexander strata in section 2.5.

In section 3, we relate group theoretic properties to properties of the Alexander stratification.

Of special interest to us in this paper are torsion translates of connected algebraic subgroups of $\widehat{\Gamma}$, we will call them *rational planes*, which sit inside the Alexander strata. In section 4, we show how these rational planes relate to geometric properties of X.

For example, in 4.1 we show that the first Betti number of finite abelian coverings of X depends only on a finite number of rational planes in the Alexander strata. This follows from a theorem of Laurent on the location of torsion points on an algebraic subset of an affine torus. When X is a compact Kähler manifold, we relate the rational planes to the existence of irrational pencils on X or on a finite unbranched covering of X. This gives a much weaker, but simpler version of a result proved by Beauville [Be] and Arapura [Ar1] which asserts that when X is a compact Kähler manifold the first Alexander stratum is a finite union of rational planes associated to the irrational pencils of X and of its finite coverings (see 4.2).

Simpson in [Sim] shows that if X is a compact Kähler manifold, then the Alexander strata for $\pi_1(X)$ are all finite unions of rational planes. Since the ideals defining the Alexander strata of a finitely presented group are computable and rational planes are zero sets of binomial ideals, one can test whether a group could not be the fundamental group of Kähler manifold in a practical way: by computing ideals defining the Alexander strata and showing that their radicals are not binomial ideals. In section 4.3 we use the above line of reasoning to obtain an obstruction for a finitely presented group of a certain form to be Kähler.

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2 Fox Calculus and Alexander Invariants

2.1 Notation.

For any group Γ, we denote by $\mathrm{ab}(\Gamma)$ the abelianization of Γ and

$$\mathrm{ab} : \Gamma \to \mathrm{ab}(\Gamma)$$

the abelianization map. By F_r, we mean the free group on r generators $x_1, \ldots, x_r$. For any ring $\mathcal{A}$, we let $\Lambda_r(\mathcal{A})$ be the ring of Laurent polynomials $\mathcal{A}[t_1^{\pm 1}, \ldots, t_s^{\pm 1}]$. When the ring $\mathcal{A}$ is understood, we will write Λ_r for $\Lambda_r(\mathcal{A})$.

Note that $\Lambda_r(\mathcal{A})$ is canonically isomorphic to the group ring $\mathcal{A}[\mathrm{ab}(\mathrm{F_r})]$ by the map $t_i \mapsto \mathrm{ab}(\mathrm{x_i})$. Let ab also denote the map

$$\mathrm{ab} : \mathrm{F_r} \to \Lambda_\mathrm{r}(\mathcal{A})$$

given by composing the abelianization map with the injection

$$\mathrm{ab}(\mathrm{F_r}) \to \mathcal{A}[\mathrm{ab}(\mathrm{F_r})] \cong \Lambda_\mathrm{r}(\mathcal{A}).$$

A finite presentation of a group Γ can be written in two ways. One is by

$$\langle\ F_r\ :\ \mathcal{R}\ \rangle,$$

where $\mathcal{R} \subset F_r$ is a finite subset. Then Γ is isomorphic to the quotient group

$$\Gamma = F_r/N(\mathcal{R}),$$

where $N(\mathcal{R})$ is the normal subgroup of F_r generated by $\mathcal{R}$. The other is by a sequence of homomorphisms

$$F_s \xrightarrow{\psi} F_r \xrightarrow{q} \Gamma,$$

where q is onto and the normalization of the image of ψ is the kernel of q.

Let $\widehat{\Gamma}$ be the group of characters of Γ. Then $\widehat{\Gamma}$ has the structure of an algebraic group with coordinate ring $\mathbb{C}[\mathrm{ab}(\Gamma)]$. (One can verify this by noting that that the closed points in $\mathrm{Spec}(\mathbb{C}[\mathrm{ab}(\Gamma)])$ correspond to homomorphisms from $\mathrm{ab}(\Gamma)$ to $\mathbb{C}^*$.) A presentation $\langle F_r\ :\ \mathcal{R}\rangle$ of Γ gives an embedding of $\widehat{\Gamma}$ in $\widehat{F_r}$. The latter can be canonically identified with the affine torus $(\mathbb{C}^*)^r$ as follows. To a character $\rho \in \widehat{F_r}$ we identify the point $(\rho(x_1), \ldots, \rho(x_r)) \in (\mathbb{C}^*)^r$. The image of $\widehat{\Gamma}$ in $(\mathbb{C}^*)^r$ is the zero set of the subset of $\Lambda_r(\mathbb{C})$ defined by

$$\{\ \mathrm{ab}(\mathrm{R}) - 1\ :\ \mathrm{R} \in \mathcal{R}\ \} \subset \mathbb{C}[\mathrm{ab}(\mathrm{F_r})] \cong \Lambda_\mathrm{r}(\mathbb{C}).$$

Given any homomorphism, $\alpha : \Gamma' \to \Gamma$ between two finitely presented groups, let $\widehat{\alpha} : \widehat{\Gamma} \to \widehat{\Gamma'}$ be the map given by composition. Let $\alpha_{\mathrm{ab}} : \mathrm{ab}(\Gamma) \to \mathrm{ab}(\Gamma')$ be the map canonically induced by α and let $\widehat{\alpha}^* : \mathbb{C}[\mathrm{ab}(\Gamma')] \to \mathbb{C}[\mathrm{ab}(\Gamma)]$ be the linear extension of α_{ab}. Then it is easy to verify that $\widehat{\alpha}$ is an algebraic morphism and $\widehat{\alpha}^*$ is the corresponding map on coordinate rings: $\widehat{\alpha}^*(f)(\rho) = f(\widehat{\alpha}(\rho))$, for $\rho \in \Gamma$ and $f \in \mathbb{C}[\mathrm{ab}(\Gamma')]$.

In [Fox], Fox develops a calculus to compute invariants, originally discovered by Alexander, of finitely presented groups. The calculus can be defined as follows: fix r and, for $i = 1, \ldots, r$, let

$$D_i : F_r \to \Lambda_r(\mathbb{Z})$$

be the map given by

$$\begin{array}{rcl} D_i(x_j) & = & \delta_{i,j}, \text{ and} \\ D_i(fg) & = & D_i(f) + \mathrm{ab(f)D_i(g)}. \end{array}$$

The map

$$D = (D_1, \ldots, D_r) : F_r \to \Lambda_r(\mathbb{Z})^r$$

is called the *Fox derivative* and the D_i are called the ith partials. Now let Γ be a group with finite presentation

$$\langle F_r \ : \ \mathcal{R} \rangle$$

and let $q : F_r \to \Gamma$ be the quotient map. The *Alexander matrix* of Γ is the $r \times s$ matrix of partials

$$M(F_r, \mathcal{R}) = [\ (\widehat{q})^* D_i(R_j)\ ].$$

For any $\rho \in \widehat{\Gamma}$, let $M(F_r, \mathcal{R})(\rho)$ be the $r \times s$ complex matrix given by evaluation on ρ and define

$$V_i(\Gamma) = \{\ \rho \in \widehat{\Gamma} \mid \mathrm{rank\ M(F_r, \mathcal{R})(\rho) < r - i}\ \}.$$

These are subvarieties of $\widehat{\Gamma}$ defined by the ideals of $(r-i) \times (r-i)$ minors of $M(F_r, \mathcal{R})$. We will call the nested sequence of algebraic subsets

$$\widehat{\Gamma} \supset V_1(\Gamma) \supset \ldots \supset V_r(\Gamma)$$

the *Alexander stratification* of Γ.

One can check that the Tietze transformations on group presentations give different Alexander matrices, but don't effect the $V_i(\Gamma)$. Hence the Alexander stratification is independent of the presentation. Later in section 2.4 (Corollary 2.4.3) we will prove the independence by other methods.

2.2 Jumping loci for group cohomology.

For any group Γ, let $C^1(\Gamma, \rho)$ be the set of *crossed homomorphisms* $f : \Gamma \to \mathbb{C}$ satisfying

$$f(g_1 g_2) = f(g_1) + \rho(g_1) f(g_2).$$

Then $C^1(\Gamma, \rho)$ is a vector space over $\mathbb{C}$. Note that for any $f \in C^1(\Gamma, \rho)$, $f(1) = 0$.

Here are two elementary lemmas, which will be useful throughout the paper.

Lemma 2.2.1 *Let $\alpha : \Gamma' \to \Gamma$ be a homomorphism of groups and let $\rho \in \widehat{\Gamma}$. Then right composition by α defines a vector space homomorphism*

$$T_\alpha : C^1(\Gamma, \rho) \to C^1(\Gamma', \widehat{\alpha}(\rho)).$$

Proof. Take any $f \in C^1(\Gamma, \rho)$. Then, for $g_1, g_2 \in \widehat{\Gamma}'$,

$$\begin{aligned} T_\alpha(f)(g_1 g_2) &= f(\alpha(g_1 g_2)) \\ &= f(\alpha(g_1)\alpha(g_2)) \\ &= f(\alpha(g_1)) + \rho(\alpha(g_1)) f(\alpha(g_2)) \\ &= T_\alpha(f)(g_1) + \widehat{\alpha}(\rho)(g_1)(T_\alpha(f))(g_2). \end{aligned}$$

Thus, $T_\alpha(f)$ is in $C^1(\Gamma', \widehat{\alpha}(\rho))$.

■

Lemma 2.2.2 *Let $g, x \in \Gamma$ and let $f \in C^1(\Gamma, \rho)$, for any $\rho \in \widehat{\Gamma}$. Then*

$$f(gxg^{-1}) = f(g)(1 - \rho(x)) + \rho(g) f(x).$$

Proof. This statement is easy to check by expanding the left hand side and noting that

$$f(g^{-1}) = -\rho(g)^{-1} f(g),$$

for any $g \in \Gamma$.

■

Let

$$U_i(\Gamma) = \{ \ \rho \in \widehat{\Gamma} \mid \dim C^1(\Gamma, \rho) > i \ \}.$$

This defines a nested sequence

$$\widehat{\Gamma} \supset U_0(\Gamma) \supset U_1(\Gamma) \supset \ldots.$$

In section 2.4 (Corollary 2.4.3), we will show that $U_i(\Gamma) = V_i(\Gamma)$, for all $i \in \mathbb{N}$.

Define, for $\rho \in \widehat{\Gamma}$,

$$B_1(\Gamma, \rho) = \{ \ f : \Gamma \to \mathbb{C} \mid f(g) = (\rho(g) - 1)c \text{ for some constant } c \in \mathbb{C} \ \}.$$

Then $B_1(\Gamma, \rho)$ is a subspace of $C^1(\Gamma, \rho)$. Define

$$\mathrm{H}^1(\Gamma, \rho) = \mathrm{C}^1(\Gamma, \rho)/\mathrm{B}^1(\Gamma, \rho).$$

This is the *first cohomology group of* Γ *with respect to the representation* ρ. Let

$$W_i(\Gamma) = \{ \ \rho \in \widehat{\Gamma} \mid \dim \mathrm{H}^1(\Gamma, \rho) \geq \mathrm{i} \ \},$$

for $i \in \mathbb{Z}_+$. We will call the $W_i(\Gamma)$ the *jumping loci* for the first cohomology of Γ. This defines a nested sequence

$$\widehat{\Gamma} = W_0(\Gamma) \supset W_1(\Gamma) \supset \ldots.$$

If $\rho = \hat{1}$ is the identity character in $\widehat{\Gamma}$, then $\rho(g) = 1$, for all $g \in \Gamma$. Thus, $B^1(\Gamma, \rho) = \{0\}$. Also, $C^1(\Gamma, \hat{1})$ is the set of all homomorphisms from Γ to $\mathbb{C}$ and is isomorphic to the abelianization of Γ tensored with $\mathbb{C}$. Thus,

$$\dim \mathrm{H}^1(\Gamma, \hat{1}) = \dim \mathrm{C}^1(\Gamma, \hat{1}) = \mathrm{d},$$

where d is the rank of the abelianization of Γ. If $\rho \neq \hat{1}$, then $B^1(\Gamma, \rho)$ is isomorphic to the field of constants $\mathbb{C}$, so

$$\dim C^1(\Gamma, \rho) = \dim \mathrm{H}^1(\Gamma, \rho) + 1.$$

We have thus shown the following.

Lemma 2.2.3 *The jumping loci $W_i(\Gamma)$ and the nested sequence $U_i(\Gamma)$ are related as follows:*

$$\begin{aligned} W_i(\Gamma) &= U_i(\Gamma) && \textit{for } i \neq d \\ W_i(\Gamma) &= U_i(\Gamma) \cup \{\hat{1}\} && \textit{for } i = d. \end{aligned}$$

Remark. The jumping loci could also have been defined using the cohomology of local systems. Let X be a topological space homotopy equivalent to a finite CW complex with $\pi_1(X) = \Gamma$. Let $\widetilde{X} \to X$ be the universal cover of X. Then for each $\rho \in \widehat{\Gamma}$, each $g \in \Gamma$ acts on $\widetilde{X} \times \mathbb{C}$ by its action as covering automorphism on $\widetilde{X}$ and by multiplication by $\rho(g)$ on $\mathbb{C}$. This defines a local system $\mathbb{C}_\rho \to X$ over X. Then $W_i(\Gamma)$ is the jumping loci for the rank of the cohomology group $\mathrm{H}^1(\mathrm{X}, \mathbb{C}_\rho)$ with coefficients in the local system $\mathbb{C}_\rho$.

2.3 Coherent sheaves over the character variety.

Let Γ be a finitely presented group and let $C^1(\Gamma, \rho)^\vee$ be the dual space of $C^1(\Gamma, \rho)$. We will construct sheaves $\mathcal{C}^1(\Gamma)$ and $\mathcal{C}^1(\Gamma)^\vee$ over $\widehat{\Gamma}$ whose stalks are $C^1(\Gamma, \rho)$ and $C^1(\Gamma, \rho)^\vee$, respectively. Then, the jumping loci $U_i(\Gamma)$ defined in the previous section, are just the jumping loci for the dimensions of stalks of $\mathcal{C}^1(\Gamma)$ and $\mathcal{C}^1(\Gamma)^\vee$. This just gives a translation of the previous section into the language of sheaves, but using this language we will show that a presentation for Γ induces a presentation of $\mathcal{C}^1(\Gamma)^\vee$ as a coherent sheaf such that the presentation map on sheaves is essentially the Alexander matrix.

We start by constructing $\mathcal{C}^1(F_r)$ for free groups.

Lemma 2.3.1 *For any r and $\rho \in \widehat{F_r}$, $C^1(F_r, \rho)$ is isomorphic to $\mathbb{C}^r$, and has a basis given by $\langle x_i \rangle_\rho$, where*

$$\langle x_i \rangle_\rho (x_j) = \delta_{i,j}.$$

Proof. By the product rule, elements of $C^1(F_r, \rho)$ only depend on what happens to the generators of F_r. Since there are no relations on F_r, any choice of values on the basis elements determines an element of $C^1(F_r, \rho)$.

∎

Let

$$E_r = \bigcup_{\rho \in \widehat{F_r}} C^1(F_r, \rho)$$

be the trivial $\mathbb{C}^r$-vector bundle over $\widehat{F_r}$ whose fiber over $\rho \in \widehat{F_r}$ is $C^1(F_r, \rho)$. For each generator x_i of F_r, define

$$\langle x_i \rangle : \widehat{F_r} \to E_r,$$

by $\langle x_i \rangle(\rho) = \langle x_i \rangle_\rho$. The maps $\langle x_1 \rangle, \dots, \langle x_r \rangle$ are global sections of E_r over $\widehat{F_r}$. Let $\mathcal{C}^1(F_r)$ be the corresponding sheaf of sections of the bundle $E_r \to \widehat{F_r}$.

The module M_r of global sections of $\mathcal{C}^1(F_r)$ is a free Λ_r-module of rank r, generated by $\langle x_1 \rangle, \dots, \langle x_r \rangle$, and $\mathcal{C}^1(F_r)$ is the sheaf associated to M_r (in the sense of [Ha], p.110).

Fix a presentation

$$F_s \xrightarrow{\psi} F_r \xrightarrow{q} \Gamma,$$

of Γ. This induces maps on character varieties

$$\begin{array}{ccccc} \Gamma & \xrightarrow{\widehat{q}} & \widehat{F_r} & \xrightarrow{\widehat{\psi}} & \widehat{F_s} \\ & & \| & & \| \\ & & (\mathbb{C}^*)^r & & (\mathbb{C}^*)^s. \end{array}$$

Let $\mathcal{C}^1(F_r)_\Gamma$ and $\mathcal{C}^1(F_s)_\Gamma$ be the pullbacks of $\mathcal{C}^1(F_r)$ and $\mathcal{C}^1(F_s)$ over $\widehat{\Gamma}$. These are the sheafs associated to the modules:

$$M_r(\Gamma) = M_r \otimes_{\mathbb{C}[\mathrm{ab}(\mathrm{F_r})]} \mathbb{C}[\mathrm{ab}(\Gamma)] \cong \mathbb{C}[\mathrm{ab}(\Gamma)]^{\mathrm{r}}$$

and

$$M_s(\Gamma) = M_s \otimes_{\mathbb{C}[\mathrm{ab}(\mathrm{F_s})]} \mathbb{C}[\mathrm{ab}(\Gamma)] \cong \mathbb{C}[\mathrm{ab}(\Gamma)]^{\mathrm{s}},$$

respectively.

Let

$$\mathcal{T}_\psi : \mathcal{C}^1(F_r)_\Gamma \to \mathcal{C}^1(F_s)_\Gamma$$

be the homomorphism of sheaves defined by composing sections by ψ. For any $\rho \in \widehat{\Gamma}$, the stalk of $\mathcal{C}^1(F_r)_\Gamma$ over ρ is given by $C^1(F_r, \widehat{q}(\rho))$. Since $q \circ \psi$ is the trivial map, the stalk of $\mathcal{C}^1(F_s)_\Gamma$ over ρ is given by $C^1(F_s, \widehat{1})$. For any $\rho \in \widehat{\Gamma}$, the map on stalks determined by $\mathcal{T}_\psi$ is the map

$$(\mathcal{T}_\psi)_\rho : C^1(F_r, \widehat{q}(\rho)) \to C^1(F_s, \widehat{1})$$

defined by $(\mathcal{T}_\psi)_\rho(f) = f \circ \psi$.

Let $M_\Gamma(F_r, \mathcal{R})$ be the sub $\mathbb{C}[\mathrm{ab}(\Gamma)]$-module of $M_r(\Gamma)$ given by the kernel of the map

$$\begin{array}{rcl} M_r(\Gamma) & \to & M_s(\Gamma) \\ f \otimes g & \mapsto & (f \circ \psi) \otimes g \end{array}$$

Let $\mathcal{C}^1(\Gamma)$ be the kernel of $\mathcal{T}_\psi$. That is, $\mathcal{C}^1(\Gamma)$ is the sheaf associated to $M_\Gamma(F_r, \mathcal{R})$.

Lemma 2.3.2 *The stalk of $\mathcal{C}^1(\Gamma)$ over $\rho \in \widehat{\Gamma}$ is isomorphic to $C^1(\Gamma,\rho)$.*

Proof. We need to show that the kernel of $(\mathcal{T}_\psi)_\rho$ is isomorphic to $C^1(\Gamma,\rho)$. Let

$$(T_q)_\rho : C^1(\Gamma,\rho) \to C^1(F_r, \widehat{q}(\rho))$$

be the homomorphism given by composing with q as in Lemma 2.2.1. Since q is surjective, it follows that $(T_q)_\rho$ is injective. The composition $T_\psi \circ (T_q)_\rho$ is right composition by $\psi \circ q$, which is trivial, so the image of $(T_q)_\rho$ lies in the kernel of Ψ. Now suppose, $f \in C^1(F_r, \widehat{q}(\rho))$ is in the kernel of Ψ. Then f is trivial on $\psi(F_s)$. Since $\widehat{q}(\rho)$ is trivial on $\psi(F_s)$, Lemma 2.2.2 implies that f is trivial on the normalization of $\psi(F_s)$ in F_r. Thus, f induces a map from Γ to $\mathbb{C}$ which is twisted by ρ.

■

Lemma 2.3.3 *Let $\alpha : \Gamma' \to \Gamma$ be a homomorphism of groups and let $\widehat{\alpha} : \widehat{\Gamma} \to \widehat{\Gamma}'$ be the corresponding morphism on character varieties. Let $\mathcal{C}(\Gamma)$ and $\mathcal{C}(\Gamma')$ be the sheaves associated to Γ and Γ' and let $\mathcal{C}(\Gamma')_\Gamma$ be the pullback of $\mathcal{C}(\Gamma')$ over $\widehat{\Gamma}$. Then the map $\mathcal{T}_\alpha : \mathcal{C}(\Gamma) \to \mathcal{C}(\Gamma')_\Gamma$ defined by composing sections by α is a homomorphism of sheaves.*

Proof. The statement follows from Lemma 2.2.1.

■

Corollary 2.3.4 *There are exact sequences of sheaves*

$$0 \to \mathcal{C}^1(\Gamma) \longrightarrow \mathcal{C}^1(F_r)_\Gamma \xrightarrow{\mathcal{T}_\psi} \mathcal{C}^1(F_s)_\Gamma$$

and

$$\mathcal{C}^1(F_s)_\Gamma^\vee \xrightarrow{\mathcal{T}_\psi^{\vee}} \mathcal{C}^1(F_r)_\Gamma^\vee \to \mathcal{C}^1(\Gamma)^\vee \to 0.$$

We have seen that the modules of holomorphic sections of $\mathcal{C}^1(F_r)$ and $\mathcal{C}^1(F_s)$ are freely generated over $\mathcal{C}[\mathrm{ab}(\Gamma)]$ of ranks r and s, respectively. Similarly, the dual sheaves $\mathcal{C}^1(F_r)^\vee$ and $\mathcal{C}^1(F_s)^\vee$ are freely generated. This gives $\mathcal{C}^1(\Gamma)^\vee$ the structure of a coherent sheaf. In section 2.4 we will show that the Alexander Matrix gives a presentation for global sections of $\mathcal{C}^1(\Gamma)^\vee$.

2.4 Jumping loci and the Alexander stratification.

In this section, we show that for a given group Γ, the jumping loci $U_i(\Gamma)$ defined in 2.2 is the same as the Alexander stratification $V_i(\Gamma)$.

For any group Γ, there is an exact bilinear pairing

$$(\mathbb{C}\Gamma)_\rho \times C^1(\Gamma,\rho) \to \mathbb{C}$$

where

$$(\mathbb{C}\Gamma)_\rho = \mathbb{C}\Gamma / \{g_1 g_2 - g_1 - \rho(g_1) g_2 | g_1, g_2 \in \Gamma\},$$

and the pairing is given by

$$[g, f] = f(g).$$

The pairing determines a $\mathbb{C}$-linear map

$$\Phi[\Gamma]_\rho : (\mathbb{C}\Gamma)_\rho \to C^1(\Gamma, \rho)^\vee,$$

where, for $g \in (\mathbb{C}\Gamma)_\rho$ and $f \in C^1(\Gamma, \rho)$, $\Phi[\Gamma]_\rho(f)(g) = [g, f] = f(g)$.

Lemma 2.4.1 *Let $\alpha : \Gamma' \to \Gamma$ be a group homomorphism. For each $\rho \in \widehat{\Gamma}$, we have a commutative diagram*

$$\begin{array}{ccc} (\mathbb{C}\Gamma')_{\widehat{\alpha}(\rho)} & \xrightarrow{\Phi[\Gamma']_{\widehat{\alpha}(\rho)}} & C^1(\Gamma', \widehat{\alpha}(\rho))^\vee \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle T_\alpha^\vee} \\ (\mathbb{C}\Gamma)_\rho & \xrightarrow{\Phi[\Gamma]_\rho} & C^1(\Gamma, \rho)^\vee \end{array}$$

where $T_\alpha^\vee$ is the dual map to $T_\alpha : C^1(\Gamma, \rho) \to C^1(\Gamma', \widehat{\alpha}(\rho))$.

Proof. For $g \in (\mathbb{C}\Gamma')_{\widehat{\alpha}}(\rho)$ and $f \in C^1(\Gamma, \rho)$, the pairing $[,]$ gives

$$[g, T_\alpha(f)] = T_\alpha(f)(g) = f(\alpha(g)) = [\alpha(g), f].$$

■

Let ${M_r}^\vee$ be the global holomorphic sections of $\mathcal{C}^1(F_r)^\vee$. Define

$$\Phi : \mathbb{C}F_r \to {M_r}^\vee$$

by

$$\begin{aligned} \Phi(x_i) &= \langle x_i \rangle^\vee \\ \Phi(g_1 g_2) &= \Phi(g_1) + \mathrm{ab(g_1)\Phi(g_2) \qquad for\ g_1, g_2 \in F_r}, \end{aligned}$$

where

$$\langle x_i \rangle_\rho^\vee : C^1(F_r, \rho) \to \mathbb{C}$$

is given by

$$\langle x_i \rangle_\rho^\vee (\langle x_j \rangle_\rho) = \delta_{i,j}.$$

Define, for any $\rho \in \widehat{F_r}$ and $g \in \mathbb{C}F_r$, with image g_ρ in $(\mathbb{C}F_r)_\rho$, $\Phi_\rho(g_\rho) = \Phi_\rho(g)(\rho) \in C^1(F_r, \rho)^\vee$, where

$$\Phi_\rho(g)(\rho)(f) = f(g)$$

for all $f \in C^1(F_r, \rho)$. Then $\Psi_\rho = \Psi[F_r]_\rho$.

Since ${M_r}^\vee$ is generated freely by the global sections

$$\langle x_1 \rangle^\vee, \ldots, \langle x_r \rangle^\vee$$

as a $\Lambda_r(\mathbb{C})$-module, we can identify ${M_r}^\vee$ with $\Lambda_r(\mathbb{C})^r$. Thus, the map Φ is the extension of the Fox derivative

$$D : F_r \to \Lambda_r(\mathbb{Z})^r$$

in the obvious way to $\mathbb{C}[F_r] \to \Lambda_r(\mathbb{C})^r$.

Let $\mathcal{D}_r(\mathcal{R})$ be the sub Λ_r-module of $\Lambda_r(\mathbb{C})^r$ spanned by $\Phi(\mathcal{R})$. For $\rho \in \widehat{\Gamma}$, let $\mathcal{D}_r(\mathcal{R})(\rho)$ be the subspace of $\mathbb{C}^r$ spanned by the vectors obtained by evaluating the r-tuples of functions in $\Phi(\mathcal{R})$ at ρ.

Lemma 2.4.2 *Let $\langle F_r : \mathcal{R} \rangle$ be a presentation for Γ. For each $\rho \in \widehat{\Gamma}$, the dimension of $C^1(\Gamma, \rho)$ is given by*

$$r - \dim(\mathcal{D}_r(\mathcal{R})(\rho)).$$

Proof. Let

$$F_s \xrightarrow{\psi} F_r \xrightarrow{q} \Gamma$$

be the sequence of maps determined by the presentation. Then, for each $\rho \in \widehat{\Gamma}$, by Corollary 2.3.4, there is an exact sequence

$$C^1(F_s, \widehat{1})^\vee \xrightarrow{{\mathcal{T}_\psi}^\vee} C^1(F_r, \widehat{q}(\rho))^\vee \xrightarrow{{\mathcal{T}_q}^\vee} C^1(\Gamma, \rho)^\vee \longrightarrow 0.$$

By Lemma 2.4.1, the following diagram commutes:

$$\begin{array}{ccc}
(\mathbb{C}F_s)_{\widehat{1}} & \xrightarrow{\Phi[F_s]_{\widehat{1}}} & C^1(F_s, \widehat{1})^\vee \\
\downarrow \psi & & \downarrow {\mathcal{T}_\psi}^\vee \\
(\mathbb{C}F_r)_{\widehat{q}(\rho)} & \xrightarrow{\Phi[F_r]_{\widehat{q}(\rho)}} & C^1(F_r, \widehat{q}(\rho))^\vee \\
\downarrow q & & \downarrow {\mathcal{T}_q}^\vee \\
(\mathbb{C}\Gamma) & \xrightarrow{\Phi[\Gamma]_\rho} & C^1(\Gamma, \rho)^\vee
\end{array}$$

Thus,

$$\dim C^1(\Gamma, \rho) \quad = \quad \dim C^1(F_r, \widehat{q}(\rho)) - \dim(\mathrm{image}({\mathcal{T}_\psi}^\vee)).$$

Since $\Phi[F_s]_{\widehat{1}}$ is onto

$$\begin{aligned}
\mathrm{image}({\mathcal{T}_\psi}^\vee) &= \mathrm{image}(\Phi[F_s]_{\widehat{1}} \circ {\mathcal{T}_\psi}^\vee) \\
&= \mathrm{image}(\Phi[F_r]_{\widehat{q}(\rho)} \circ \psi)
\end{aligned}$$

For any ρ, $C^1(F_r, \widehat{q}(\rho))$ is isomorphic to $\mathbb{C}^r$. Putting this together, we have

$$\begin{aligned}
\dim C^1(\Gamma, \rho) &= r - \dim \Phi[F_r]_{\widehat{q}(\rho)}(\mathcal{R}) \\
&= r - \dim \mathcal{D}_r(\mathcal{R})(\rho).
\end{aligned}$$

■

Corollary 2.4.3 *For any finitely presented group Γ, the jumping loci $U_i(\Gamma)$ for the cohomology of Γ is the same as the Alexander stratification $V_i(\Gamma)$.*

2.5 Abelian coverings of finite CW complexes.

In this section we explain the Fox calculus and Alexander stratification in terms of finite abelian coverings of a finite CW complex. The relations between homology of coverings of a $K(\Gamma,1)$ and the group cohomology of Γ are well known (see, for example, [Br]). The results of this section come from looking at Fox calculus from this point of view.

Let X be a finite CW complex and let $\Gamma=\pi_1(X)$. Suppose Γ has presentation given by $\langle x_1,\dots,x_r : R_1,\dots,R_s\rangle$. Then X is homotopy equivalent to a CW complex with cell decomposition whose tail end is given by

$$\dots\supset\Sigma_2\supset\Sigma_1\supset\Sigma_0,$$

where Σ_0 consists of a point P, Σ_1 is a bouquet of r oriented circles S^1 joined at P. Identify F with $\pi_1(\Sigma_1)$ so that each x_i is the positively oriented loop around the i-th circle. Each R_i defines a homotopy class of map from S^1 to Σ_1. The 2-skeleton Σ_2 is the union of s disks attached along their boundaries to Σ_1 by maps in the homotopy class defined by $R_1,\dots,R_s$.

Let $\alpha:\Gamma\to G$ be any epimorphism of Γ to a finite abelian group G. Let $\tau_\alpha : X_\alpha\to X$ be the regular unbranched covering determined by α with G acting as group of covering automorphisms. Our aim is to show how Fox calculus can be used to compute the first Betti number of X_α. Choose a basepoint $1P\in\tau_\alpha^{-1}(P)$. For each i-chain $\sigma\in\Sigma_i$ and $g\in G$, let $g\sigma$ be the the component of its preimage which passes through gP. For each generating i-cell in Σ_i, there are exactly G copies of isomorphic cells in its preimage. Thus X_α has a cell decomposition

$$\dots\supset\Sigma_{2,\alpha}\supset\Sigma_{1,\alpha}\supset\Sigma_{0,\alpha},$$

where the i-cells in $\Sigma_{i,\alpha}$ are given by the set $\{g\sigma\ :\ g\in G,\sigma \text{ an } i\text{-cell in } \Sigma_i\}$. With this notation if σ attaches to $\Sigma_{i-1,\alpha}$ according to the homotopy class of mapping $f:\partial\sigma\to\Sigma_{i-1}$, where $\partial\sigma$ is the boundary of σ, then $g\sigma$ attaches to $\Sigma_{i-1,\alpha}$ by the map $f':\partial g\sigma\to\Sigma_{i-1,\alpha}$ lifting f at the basepoint gP.

Let C_i be the i-chains on X and let $C_{i,\alpha}$ be the i-chains on X_α. Then there is a commutative diagram for the chain complexes for X and X_α:

$$\begin{array}{ccccccccc}
\dots & \longrightarrow & C_{2,\alpha} & \xrightarrow{\delta_{2,\alpha}} & C_{1,\alpha} & \xrightarrow{\delta_{1,\alpha}} & C_{0,\alpha} & \xrightarrow{\epsilon} & \mathbb{Z}\\
 & & \downarrow{\scriptstyle\tau_\alpha} & & \downarrow{\scriptstyle\tau_\alpha} & & \downarrow{\scriptstyle\tau_\alpha} & & \\
\dots & \longrightarrow & C_2 & \xrightarrow{\delta_2} & C_1 & \xrightarrow{\delta_1} & C_0, & &
\end{array}$$

where the map ϵ is the augmentation map

$$\epsilon(\sum_{g\in G}(a_g g))=\sum_{g\in G}a_g.$$

Let $\langle x_1\rangle_\alpha,\dots,\langle x_r\rangle_\alpha$ be the elements of $C_{1,\alpha}$ given by lifting $x_1,\dots,x_r$, considered as loops on Σ_1, to 1-chains on $\Sigma_{1,\alpha}$ with basepoint $1P$. Then $C_{1,\alpha}$ can be identified with $\mathbb{C}[G]^r$, with basis $\langle x_1\rangle,\dots,\langle x_r\rangle$ and $C_{0,\alpha}$ can be identified with $\mathbb{C}[G]$, where each $g\in G$ corresponds to gP.

The above commutative diagram can be rewritten as

$$\begin{array}{ccccccccc} \dots & \longrightarrow & \mathbb{Z}[G]^s & \stackrel{\delta_{2,\alpha}}{\longrightarrow} & \mathbb{Z}[G]^r & \stackrel{\delta_{1,\alpha}}{\longrightarrow} & \mathbb{Z}[G] & \stackrel{\epsilon}{\longrightarrow} & \mathbb{Z} \\ & & \big\downarrow \tau_\alpha & & \big\downarrow \tau_\alpha & & \big\downarrow \tau_\alpha & & \\ \dots & \longrightarrow & \mathbb{Z}^s & \stackrel{\delta_2}{\longrightarrow} & \mathbb{Z}^r & \stackrel{\delta_1}{\longrightarrow} & \mathbb{Z}. & & \end{array} \tag{1}$$

For any finite set S, let $|S|$ denote its order. The map ϵ is surjective, so we have the formula

$$\begin{aligned} b_1(X_\alpha) &= \text{nullity}(\delta_{1,\alpha}) - \text{rank}(\delta_{2,\alpha}) \\ &= (r-1)|G| + 1 - \text{rank}(\delta_{2,\alpha}), \end{aligned} \tag{2}$$

where $b_1(X_\alpha)$ is the rank of $\ker \delta_{1,\alpha}/\text{image}(\delta_{2,\alpha})$ and is the rank of $H_1(X_\alpha;\mathbb{Z})$. We will rewrite this formula in terms of the Alexander stratification.

Lemma 2.5.1 *The map $\delta_{1,\alpha}$ is given by*

$$\delta_{1,\alpha}(\sum_{i=1}^r f_i\langle x_i\rangle_\alpha) = \sum_{i=1}^r f_i\widehat{q_\alpha}^*(t_i - 1).$$

Proof. It's enough to notice that the lift of x_i to $C_{1,\alpha}$ at the basepoint $1P$ has end point $\widehat{q_\alpha}^*(t_i)P$.

∎

We will now relate the map $\delta_{2,\alpha}$ with the Fox derivative.

Recall that Σ_1 equals a bouquet of r circles $\wedge_r S^1$. Let $\tau : \mathcal{L}_r \to \wedge_r S^1$ be the universal abelian covering. Then $\mathcal{L}_r$ is a lattice on r generators with $\text{ab}(\text{F}_\text{r})$ acting as covering automorphisms. The vertices of the lattice can be identified with $\text{ab}(\text{F}_\text{r})$. Let $K_\alpha = \ker(\alpha \circ q) \subset F_r$ and let $\widetilde{K_\alpha}$ be its image in $\text{ab}(\text{F}_\text{r})$. Then $\Sigma_{1,\alpha} = \mathcal{L}_r/\widetilde{K_\alpha}$ and we have a commutative diagram

$$\begin{array}{ccc} \mathcal{L}_r & \stackrel{\eta_\alpha}{\longrightarrow} & \Sigma_{1,\alpha} \\ \big\downarrow \tau & & \big\downarrow \tau_\alpha \\ \wedge_r S^1 & = & \Sigma_1 \end{array}$$

where $\eta_\alpha : \mathcal{L}_r \to \Sigma_{1,\alpha}$ is the quotient map. Let $(\eta_\alpha)_* : C_1(\mathcal{L}_r) \to C_1(\Sigma_{1,\alpha})$ be the induced map on one chains. Then identifying $C_1(\mathcal{L}_r)$ with $\mathbb{Z}[\text{ab}(\text{F}_\text{r})]^\text{r}$ and $C_1(\Sigma_{1,\alpha})$ with $\mathbb{Z}[G]^r$, we have $(\eta_\alpha)_* = (\widehat{q_\alpha}^*)^r$.

Choose $1\tilde{P} \in \tau^{-1}(P)$. Let $C_1(\mathcal{L}_r)$ be the 1-chains on $\mathcal{L}_r$. Let $\langle x_1\rangle, \dots, \langle x_r\rangle$ be the lifts of $x_1, \dots, x_r$ to $C_1(\mathcal{L}_r)$ at the base point $1\tilde{P}$. This determines an identification of $C_1(\mathcal{L}_r)$ with $\Lambda(\mathbb{Z})^r$ and determines a choice of homotopy lifting map $\ell : \pi_1(\Sigma_1) \to C_1(\mathcal{L}_r)$.

Lemma 2.5.2 *The identifications $F_r = \pi_1(\Sigma_1)$ and $\Lambda_r(\mathbb{Z}) = C_1(\mathcal{L}_r)$, make the following diagram commute*

$$\begin{array}{ccc} \pi_1(\Sigma_1) & \stackrel{\ell}{\longrightarrow} & C_1(\mathcal{L}_r) \\ \| & & \| \\ F_r & \stackrel{D}{\longrightarrow} & \Lambda_r(\mathbb{Z}). \end{array}$$

Proof. By definition, both maps ℓ and D send x_i to $\langle x_i \rangle$, for $i = 1, \ldots, r$. We have left to check products. Let $f, g \in F_r$, be thought of as loops on $\wedge_r S^1$. Then the lift of f has endpoint ab(f). Therefore, $\ell(fg) = \ell(f) + \mathrm{ab(f)}\ell(\mathrm{g})$. Since these rules are the same as those for the Fox derivative map, the maps must be the same.

■

Corollary 2.5.3 *Let Γ be a finitely presented group with presentation $\langle F_r : \mathcal{R} \rangle$. Let $\alpha : \Gamma \to G$ be an epimorphism to a finite abelian group G. Let $M(F_r, \mathcal{R})_\alpha$ be the matrix $M(F_r, \mathcal{R})$ with $\widehat{q}_\alpha^*$ applied to all the entries. Then*

$$\begin{array}{ccc} C_{2,\alpha} & \stackrel{\delta_{2,\alpha}}{\longrightarrow} & C_{1,\alpha} \\ \| & & \| \\ \mathbb{Z}[G]^s & \stackrel{M(F_r,\mathcal{R})_\alpha}{\longrightarrow} & \mathbb{Z}[G]^r. \end{array}$$

Proof. Let $\sigma_1, \ldots, \sigma_s$ be the s disks generating the 2-cells C_2. For each $i = 1, \ldots, s$ and $g \in G$, let $g\sigma_i$ denote the lift of σ_i at gP. Let $R_1, \ldots, R_s$ be the elements of $\mathcal{R}$. By Lemma 2.5.2, the boundary $\partial\sigma_i$ maps to $D(R_i)$ in $C_1(\mathcal{L}_r)$. Thus, the boundary of $g\sigma_i$ equals $gD(R_i)$, and for $g_1, \ldots, g_s \in \mathbb{Z}[G]$,

$$\delta_{\alpha,2}(\sum_{i=1}^{s} g_i\sigma_i) = \sum_{i=1}^{s} g_i D(R_i).$$

This is the same as the application of $M(F_r, \mathcal{R})_\alpha$ on the s-tuple $(g_1, \ldots, g_s)$.

■

We now give a formula for the first Betti number $b_1(X_\alpha)$ in terms of the Alexander stratification in the case where G is finite.

Tensor the top row in diagram (1) by $\mathbb{C}$. Then the action of G on $\mathbb{C}[G]$ diagonalizes to get

$$\mathbb{C}[G] \cong \bigoplus_{\rho \in \widehat{G}} \mathbb{C}[G]_\rho,$$

where $\mathbb{C}[G]_\rho$ is a one-dimensional subspace of $\mathbb{C}[G]$ and $g \in G$ acts on $\mathbb{C}[G]_\rho$ by multiplication by $\rho(g)$.

The top row of diagram (1) becomes

$$\bigoplus_{\rho \in \widehat{G}} \mathbb{C}[G]_\rho^s \stackrel{\delta_{\alpha,2}}{\longrightarrow} \bigoplus_{\rho \in \widehat{G}} \mathbb{C}[G]_\rho^r \stackrel{\delta_{\alpha,1}}{\longrightarrow} \bigoplus_{\rho \in \widehat{G}} \mathbb{C}[G]_\rho \stackrel{\epsilon}{\longrightarrow} \mathbb{C}.$$

The map $\delta_{\alpha,2}$ considered as a matrix $M(F_r, \mathcal{R})_\alpha$, as in Lemma 2.5.3, decomposes into blocks

$$M(F_r, \mathcal{R})_\alpha = \bigoplus_{\rho \in \widehat{G}} M(F_r, \mathcal{R})_\alpha(\rho),$$

where, if $M(F_r,\mathcal{R})_\alpha = [f_{i,j}]$, then $M(F_r,\mathcal{R})_\alpha(\rho) = [f_{i,j}(\rho)]$. We thus have the following formula for the rank of $M(F_r,\mathcal{R})_\alpha$:

$$\mathrm{rank}(\mathrm{M}(\mathrm{F_r},\mathcal{R})_\alpha) \quad = \quad \sum_{\rho\in\widehat{G}} \mathrm{rank}(\mathrm{M}(\mathrm{F_r},\mathcal{R})_\alpha(\rho)). \tag{3}$$

Recall that the Alexander stratification $V_i(\Gamma)$ was defined to be the zero set in $\widehat{\Gamma}$ of the $(r-i)\times(r-i)$ ideals of $M(F_r,\mathcal{R})$. For any $\rho\in\widehat{G}$, $M(F_r,\mathcal{R})_\alpha(\rho) = M(F_r,\mathcal{R})(\widehat{\alpha}(\rho)) = M(F_r,\mathcal{R})(\widehat{q_\alpha}(\rho))$, since $\widetilde{\alpha}(f)(\rho) = f(\widehat{\alpha}(\rho))$ and $\widetilde{q_\alpha}(f)(\rho) = f(\widehat{q_\alpha}(\rho))$.

We thus have the following Lemma.

Lemma 2.5.4 *For $\rho\in\widehat{G}$, $\widehat{\alpha}(\rho)\in V_i(\Gamma)$ if and only if* $\mathrm{rank}(\mathrm{M}(\mathrm{F_r},\mathcal{R})_\alpha(\rho)) < \mathrm{r}-\mathrm{i}$.

For each $i=0,\ldots,r-1$, let $\chi_{V_i(\Gamma)}$ be the indicator function for $V_i(\Gamma)$. Then, for $\rho\in\widehat{G}$, we have

$$\mathrm{rank}(\mathrm{M}(\mathrm{F_r},\mathcal{R})_\alpha(\rho)) = \mathrm{r} - \sum_{\mathrm{i}=0}^{\mathrm{r}-1} \chi_{\mathrm{V_i}(\widehat{\Gamma})}(\widehat{\alpha}(\rho)). \tag{4}$$

Lemma 2.5.5 *For the special character $\widehat{1}$,*

$$\mathrm{rank}(\mathrm{M}(\mathrm{F_r},\mathcal{R})_\alpha(\widehat{1})) = \mathrm{r} - \mathrm{b_1}(\mathrm{X})$$

and $\mathrm{rank}(\mathrm{M}(\mathrm{F_r},\mathcal{R})_\alpha(\widehat{1})) = \mathrm{r}$ *if and only if $\widehat{\Gamma} = \{\widehat{1}\}$ and Γ has no nontrivial abelian quotients.*

Proof. The group G acts trivially on $\Lambda_{\alpha,\widehat{1}}$. Thus, in the commutative diagram

$$\begin{array}{ccccc}
\Lambda^s_{\alpha,\widehat{1}} & \xrightarrow{M(F_r,\mathcal{R})_\alpha(\widehat{1})} & \Lambda^r_{\alpha,\widehat{1}} & \xrightarrow{\delta_\alpha(\widehat{1})} & \Lambda_{\alpha,\widehat{1}} \\
\downarrow & & \downarrow & & \downarrow \\
(\mathbb{C})^s & \xrightarrow{\delta_2} & (\mathbb{C})^r & \xrightarrow{\delta_1} & \mathbb{C}
\end{array}$$

the vertical arrows are isomorphisms.

We thus have

$$\begin{aligned}
\mathrm{rank}(\mathrm{M}(\mathrm{F_r},\mathcal{R})_\alpha(\widehat{1})) &= \mathrm{rank}(\delta_2) \\
&= r - b_1(X).
\end{aligned}$$

■

Proposition 2.5.6 *Let Γ be a finitely presented group and let $\alpha : \Gamma \to G$ be an epimorphism where G is a finite abelian group. Let $\widehat{\alpha} : \widehat{G} \hookrightarrow \widehat{\Gamma}$ be the inclusion map induced by α. Then*

$$b_1(X_\alpha) = b_1(X) + \sum_{i=1}^{r-1} |V_i(\Gamma) \cap \widehat{\alpha}(\widehat{G} \setminus \widehat{1})|.$$

Proof. Starting with formula (2) and Corollary 2.5.3, we have

$$\begin{aligned} b_1(X_\alpha) &= (r-1)|G| + 1 - \mathrm{rank}(\mathrm{M}(\mathrm{F_r}, \mathcal{R})_\alpha) \\ &= r - \mathrm{rank}(\mathrm{M}(\mathrm{F_r}, \mathcal{R})_\alpha(\widehat{1})) + \sum_{\rho \in \widehat{\mathrm{G}} \setminus \widehat{1}} (\mathrm{r}-1) - \mathrm{rank}(\mathrm{M}(\mathrm{F_r}, \mathcal{R})_\alpha(\rho)). \end{aligned}$$

By Lemma 2.5.5, the left hand summand equals $b_1(X)$ and by (4) the right hand side can be written in terms of the indicator functions:

$$b_1(X_\alpha) = b_1(X) + \sum_{\rho \in \widehat{G} \setminus \widehat{1}} \sum_{i=1}^{r-1} \chi_{V_i(\widehat{\Gamma})}(\widehat{\alpha}(\rho))$$

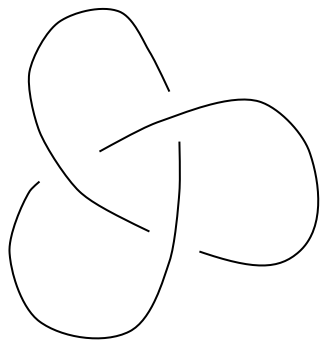
and the claim follows.

■

Corollary 2.5.7 *Let $\Gamma = \pi_1(X)$ be a finitely presented group and $\alpha : \Gamma \to G$ an epimorphism to a finite abelian group G, as above. Then*

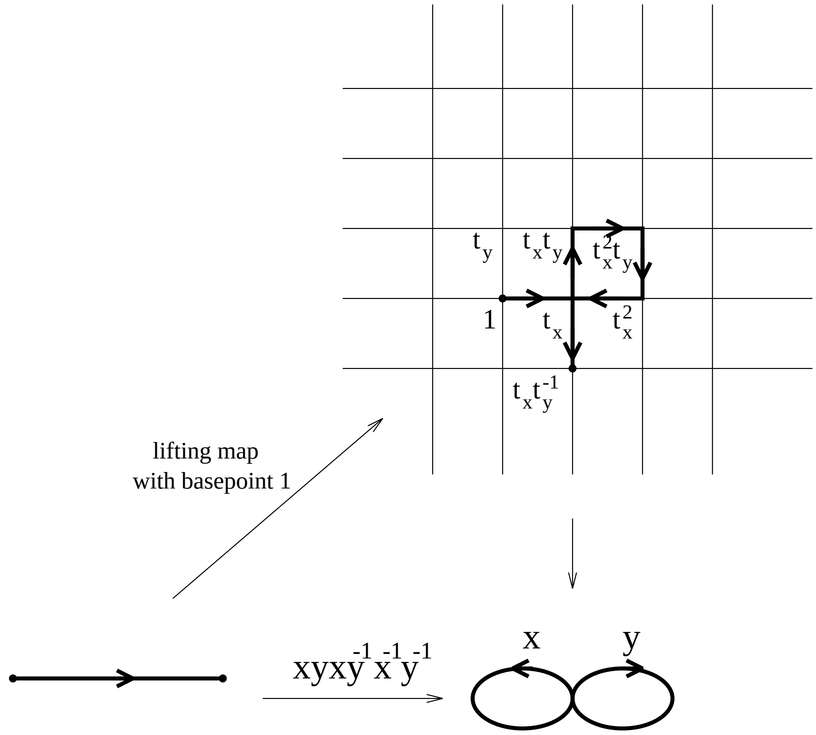
$$b_1(X_\alpha) = \sum_{i=1}^{r} |W_i(\Gamma) \cap \widehat{\alpha}(\widehat{G})|.$$

Example. We illustrate the above exposition using the well known case of the trefoil knot in the three sphere S^3:



One presentation of the fundamental group of the complement is $\Gamma = \langle x, y : xyxy^{-1}x^{-1}y^{-1} \rangle$. Then Σ_1 is a bouquet of two circles and $F = \pi_1(\Sigma_1)$ has two generators x, y one for each positive loop around the circles. The maximal abelian covering of Σ_1 is the lattice $\mathcal{L}_2$. Now take the relation $R =$

$xyxy^{-1}x^{-1}y^{-1} \in F$. The lift of R at the origin of the lattice is drawn in the figure below.



Note that the order in which the path segments are taken does not matter in computing the 1-chain. One can verify that $D(R)$ is the 1-chain defined by

$$(1 - t_x + t_x t_y)\langle x\rangle + (-t_x t_y^{-1} + t_x - t_x^2)\langle y\rangle.$$

Thus, the Alexander matrix for the relation R is

$$M(F_r, \mathcal{R}) = \begin{bmatrix} 1 - t + t^2 \\ -1 + t - t^2 \end{bmatrix}.$$

Here t_x and t_y both map to the generator t of $\mathbb{Z}$ under the abelianization of Γ.

The Alexander stratification of Γ is thus given by

$$\begin{array}{rcl} V_0(\Gamma) & = & \widehat{\Gamma} = \mathbb{C}^*; \\ V_1(\Gamma) & = & V(1 - t + t^2); \\ V_i(\Gamma) & = & \emptyset \qquad \text{for } i \geq 2. \end{array}$$

Note that the torsion points on $V_1(\Gamma)$ are the two primitive 6th roots of unity $\exp(\pm 2\pi/6)$.

Now let $\alpha : \Gamma \to G$ be any epimorphism onto an abelian group. Then since $\mathrm{ab}(\Gamma) \cong \mathbb{Z}$, G must be a cyclic group of order n for some n. This means the image of $\widehat{\alpha} : \widehat{G} \to \mathbb{C}^*$ is the set of n-th roots of unity in $\mathbb{C}^*$. Let X_n be the

n-cyclic unbranched covering of the complement of the trefoil corresponding to the map $\alpha = \alpha_n$. By Proposition 2.5.6,

$$b_1(X_n) = \begin{cases} 3 & \text{if } 6|n \\ 1 & \text{otherwise.} \end{cases}$$

3 Group theoretic constructions and Alexander invariants.

3.1 Group homomorphisms.

Let Γ and Γ' be finitely presented groups and let $\alpha : \Gamma' \to \Gamma$ be a group homomorphism. In this section, we look at what can be said about the Alexander strata of the groups Γ and Γ' in terms of α.

Lemma 3.1.1 *The homomorphism*

$$T_\alpha : C^1(\Gamma, \rho) \to C^1(\Gamma', \widehat{\alpha}(\rho))$$

given by composition with α induces a homomorphism

$$\widetilde{T_\alpha} : H^1(\Gamma, \rho) \to H^1(\Gamma', \widehat{\alpha}(\rho)).$$

Proof. It suffices to show that if f is an element of $B^1(\Gamma, \rho)$, then $T_\alpha(f)$ is an element of $B^1(\Gamma', \widehat{\alpha}(\rho))$.

For any $f \in B^1(\Gamma, \rho)$, there is a constant $c \in \mathbb{C}$ such that for all $g \in \Gamma$,

$$f(g) = (1 - \rho(g))c.$$

Then, for any $g' \in \Gamma'$,

$$\begin{aligned} T_\alpha(f)(g') &= (1 - \rho(\alpha(g'))c \\ &= (1 - \widehat{\alpha}(\rho)(g'))c. \end{aligned}$$

Thus, $T_\alpha(f)$ is in $B^1(\Gamma', \widehat{\alpha}(\rho))$.

■

The following lemma follows easily from the definitions.

Lemma 3.1.2 *If $\alpha : \Gamma' \to \Gamma$ is a group homomorphism, then (1) implies (2) and (2) implies (3), where (1), (2), and (3) are the following statements.*

(1) $\widetilde{T_\alpha} : H^1(\Gamma, \rho) \to H^1(\Gamma', \widehat{\alpha}(\rho))$ *is injective;*

(2) $\dim H^1(\Gamma, \rho) \leq \dim H^1(\Gamma', \widehat{\alpha}(\rho))$, *for all* $\rho \in \widehat{\Gamma}$*; and*

(3) $\widehat{\alpha}(W_i(\Gamma)) \subset W_i(\Gamma')$.

Proposition 3.1.3 *If $\alpha : \Gamma' \to \Gamma$ is an epmiorphism, then*

$$\widetilde{T_\alpha} : H^1(\Gamma, \rho) \to H^1(\Gamma', \rho)$$

is injective. Furthermore,

$$\widehat{\alpha}(V_i(\Gamma)) \subset V_i(\Gamma').$$

Proof. To show the first statement we need to show that if $T_\alpha(f) \in B^1(\Gamma', \widehat{\alpha}(\rho))$ for some $\rho \in \widehat{\Gamma}$, then $f \in B^1(\Gamma, \rho)$.

If $f \in C^1(\Gamma, \rho)$ and $T_\alpha(f) \in B^1(\Gamma', \widehat{\alpha}(\rho))$, then for some $c \in \mathbb{C}$ and all $g' \in \Gamma'$ we have

$$T_\alpha(f) = (1 - \widehat{\alpha}(\rho)(g'))c.$$

Take $g \in \Gamma$. Since α is surjective, there is a $g' \in \Gamma'$ so that $\alpha(g') = g$. Thus,

$$\begin{aligned} f(g) &= f(\alpha(g')) \\ &= T_\alpha(f)(g') \\ &= (1 - \widehat{\alpha}(\rho)(g'))c \\ &= (1 - \rho(\alpha(g'))c \\ &= (1 - \rho(g))c. \end{aligned}$$

Since this holds for all $g \in \Gamma$, f is in $B^1(\Gamma, \rho)$.

The second statement follows from Lemma 3.1.2, Lemma 2.2.3 and Corollary 2.4.3, since $\widehat{\alpha}$ is injective and sends the trivial character to the trivial character.

■

Proposition 3.1.4 *If $\alpha : \Gamma' \to \Gamma$ is a monomorphism whose image has finite index in Γ, then, for any $\rho \in \widehat{\Gamma}$,*

$$\widetilde{T_\alpha} : H^1(\Gamma, \rho) \to H^1(\Gamma', \widehat{\alpha}(\rho))$$

is injective.

Proof. We can assume that Γ' is a subgroup of Γ. Take any $\rho \in \widehat{\Gamma}$. We can think of $\widehat{\alpha}(\rho)$ as the restriction of the representation ρ on Γ to the subgroup Γ'.

The map $\widetilde{T_\alpha}$ is then the restriction map

$$\mathrm{res}^{\Gamma}_{\Gamma'} : \mathrm{H}^1(\Gamma, \rho) \to \mathrm{H}^1(\Gamma', \widehat{\alpha}(\rho))$$

in the notation of Brown ([Br], III.9). Furthermore, one can define a *transfer map*

$$\mathrm{cor}^{\Gamma}_{\Gamma'} : \mathrm{H}^1(\Gamma', \widehat{\alpha}(\rho)) \to \mathrm{H}^1(\Gamma, \rho)$$

with the property that

$$\mathrm{cor}^{\Gamma}_{\Gamma'} \circ \mathrm{res}^{\Gamma}_{\Gamma'} : \mathrm{H}^1(\Gamma, \rho) \to \mathrm{H}^1(\Gamma, \rho)$$

is multiplication by the index $[\Gamma : \Gamma']$ of Γ' in Γ (see [Br], Proposition 9.5).

This implies that $\mathrm{res}^{\Gamma}_{\Gamma'}$ is injective.

■

Note that Proposition 3.1.4 does not hold if $\alpha(\Gamma)$ does not have finite index. For example, let $\alpha : F_1(= \mathbb{Z}) \hookrightarrow F_2$ be the inclusion of the free group on one generator into that free group on two generators, sending the generator of F_1 to the first generator of F_2. Then for any $\rho \in \widehat{F_2}$,

$$\dim\ H^1(F_2, \rho) = 2 > 1 = \dim\ H^1(F_1, \widehat{\alpha}(\rho)).$$

3.2 Free products.

In this section, we treat free products of finitely presented groups. The easiest case is a free group. Since there are no relations, it is easy to see that

$$V_i(F_r) = \widehat{F_r} = (\mathbb{C}^*)^r$$

for $i = 1, \ldots, r-1$ and is empty for $i \geq r$. Thus,

$$W_i(F_r) = \begin{cases} (\mathbb{C}^*)^r & \text{if } i = 1, \ldots, r-1, \\ \{\widehat{1}\} & \text{if } i = r \end{cases}$$

and is empty for $i > r$.

Proposition 3.2.1 *If* $\Gamma = \Gamma_1 * \ldots * \Gamma_k$ *is a free product of* k *finitely presented groups, then*

$$V_i(\Gamma) = \sum_{i_1+\ldots+i_k} V_{i_1}(\Gamma_1) \oplus \ldots \oplus V_{i_k}(\Gamma_k).$$

Proof. We first do the case $k = 2$. Suppose Γ is isomorphic to the free product $\Gamma_1 * \Gamma_2$, where Γ_1 and Γ_2 are finitely presented groups with presentations $\langle F_{r_1}, \mathcal{R}_1\rangle$ and $\langle F_{r_2}, \mathcal{R}_2\rangle$, respectively. Suppose $\mathcal{R}_1 = \{R_1, \ldots, R_{s_1}\}$ and $\mathcal{R}_2 = \{S_1, \ldots, S_{s_2}\}$. Then, setting $r = r_1 + r_2$ and noting the isomorphism $F_r \cong F_{r_1} * F_{r_2}$, Γ has the finite presentation $\langle F_r, \mathcal{R}\rangle$ where $\mathcal{R} = \{R_1, \ldots, R_{s_1}, S_1, \ldots, S_{s_2}\}$.

The character group $\widehat{F_r}$ splits into the product $\widehat{F_r} = \widehat{F_{r_1}} \times \widehat{F_{r_2}}$. Thus, each $\rho \in \widehat{\Gamma}$ can be written as $\rho = (\rho_1, \rho_2)$, where $\rho_1 \in \widehat{F_{r_1}}$ and $\rho_2 \in \widehat{F_{r_2}}$. The vector space $\mathcal{D}_r(\mathcal{R})(\rho)$ splits into a direct sum $\mathcal{D}(\mathcal{R})(\rho) = \mathcal{D}(\mathcal{R}_1)(\rho_1) \oplus \mathcal{D}(\mathcal{R}_2)(\rho_2)$ so we have

$$\dim \mathcal{D}(\mathcal{R})(\rho) = \dim \mathcal{D}(\mathcal{R}_1)(\rho_1) + \dim \mathcal{D}(\mathcal{R}_2)(\rho_2).$$

The rest follows by induction.

■

3.3 Direct products.

In this section we deal with groups Γ which are finite products of finitely presented groups.

Lemma 3.3.1 *Let* Γ *be the direct product of free groups* $F_{r_1} \times \ldots \times F_{r_k}$. *Let* $q_i : \Gamma \to F_{r_i}$ *be the projections. Let* $r = r_1 + \ldots + r_k$ *and let* $m = \max\{r_1, \ldots, r_k\}$. *Then*

$$V_i(\Gamma) = \begin{cases} \bigcup_{i<r_j} \widehat{q}_j(\widehat{F_{r_j}}) & \text{if } 1 \leq i < m; \\ \{\widehat{1}\} & \text{if } m \leq i < r; \\ \emptyset & \text{if } i \geq r. \end{cases}$$

Proof. We know from section 3.2 that

$$V_i(F_{r_j}) = \begin{cases} \widehat{F_{r_j}} & \text{for } i < r_j; \\ \emptyset & \text{for } i \geq r_j. \end{cases}$$

By Proposition 3.1.3, the epimorphisms $q_j : \Gamma \to F_{r_j}$ give inclusions

$$\widehat{q}_j(\widehat{F_{r_j}}) \subset V_i(\Gamma)$$

for all j such that $i < r_j$. This gives the inclusion

$$\bigcup_{i<r_j} \widehat{q}_j(\widehat{F_{r_j}}) \subset V_i(\Gamma)$$

for all $i < m$.

Let $x_{i,1}, \ldots, x_{i,r_i}$ be the generators for F_{r_i}, for $i = 1, \ldots, k$. Let $F_r = F_{r_1} * \ldots * F_{r_k}$. For $i, j = 1, \ldots, k$, $i < j$, $\ell = 1, \ldots, r_i$ and $m = 1, \ldots, r_j$, let $R_{i,\ell,j,m} = [x_{i,\ell}, x_{j,m}]$. Let

$$\mathcal{R} = \{\ R_{i,\ell,j,m}\ :\ i \neq j\ \}.$$

Then $\langle F_r, \mathcal{R}\rangle$ is a presentation for Γ. Let Λ_r be the Laurent polynomials in the generators $t_{i,\ell}$, $i = 1, \ldots, k$, $\ell = 1, \ldots, r_i$ and associate this to the ring of functions on $\widehat{F_r} = \widehat{\Gamma}$ by sending $x_{i,\ell}$ to $t_{i,\ell}$.

We have

$$D(R_{i,\ell,j,m}) = (1 - t_{j,m})\langle x_{i,\ell}\rangle + (t_{i,\ell} - 1)\langle x_{j,m}\rangle.$$

It immediately follows that $M(F_r, \mathcal{R})(\widehat{1})$ is the zero matrix, so $\widehat{1} \in V_i(\Gamma)$ for $i < r$ and $\widehat{1} \notin V_i(\Gamma)$ for $i \geq r$.

Now consider $\rho \in \widehat{F_r} = \widehat{\Gamma}$ with $\rho \neq \widehat{1}$. We will show that if $\rho \in \widehat{q}_i(F_{r_i})$ then $\rho \in V_n(\Gamma)$ for $n < r_i$ and $\rho \notin V_n(\Gamma)$ for $n \geq r_i$. If $\rho \notin \widehat{q}_i(F_{r_i})$ for any i, then we will show that $\rho \notin V_1(\Gamma)$.

Let $\rho_{i,\ell}$, $i = 1, \ldots, k$ and $\ell = 1, \ldots, r_i$, be the component of ρ corresponding to the generator $t_{i,\ell}$ in Λ_r. For each $i = 1, \ldots, k$, let $s_i = r_1 + \ldots + \widehat{r_i} + \ldots + r_k$.

Take $\rho \in \widehat{q}_i(F_{r_i})$. We know from Proposition 3.1.3 that $\rho \in V_n(\Gamma)$ for $n < r_i$. Also, $\rho_{j,m} = 1$, for all $j = 1, \ldots, \hat{\imath}, \ldots, k$. Since $\rho \neq \widehat{1}$, $\rho_{i,\ell} \neq 1$ for some ℓ. Consider the $s_i \times s_i$ minor of $M(F_r, \mathcal{R})(\rho)$ with rows corresponding to the generators $\langle x_{j,m}\rangle$ and columns corresponding to generators $R_{i,\ell,j,m}$, where $j = 1, \ldots, \hat{\imath}, \ldots, k$ and $m = 1, \ldots, r_j$. This is the $s_i \times s_i$ matrix

$$(1 - \rho_{i,\ell})I_{s_i}$$

where I_{s_i} is the $s_i \times s_i$ identity matrix. Thus, rank $M(F_r, \mathcal{R})(\rho) \geq s_i$. This means that $\rho \notin V_n(\Gamma)$ for $n \geq (r - s_i) = r_i$.

Now take $\rho \notin \widehat{q}_i(F_{r_i})$ for any i. Then, for some i and j with $i \neq j$, and some ℓ and m, we have $\rho_{i,\ell} \neq 1$ and $\rho_{j,m} \neq 1$. Consider the minor of $M(F_r, \mathcal{R})(\rho)$ with columns corresponding to all generators except $x_{i,\ell}$, and rows corresponding to relations $R_{i,\ell,j',m'}$, where $j' = 1, \ldots, \hat{\imath}, \ldots, k$ and $m' = 1, \ldots, r_{j'}$, and $R_{i,\ell',j,m}$, where $\ell' = 1, \ldots, \hat{\ell}, \ldots, r_i$. This is the $r - 1 \times r - 1$ matrix

$$\begin{bmatrix} \pm(1 - \rho_{i,\ell})I_{s_i} & 0 \\ 0 & \pm(1 - \rho_{j,m})I_{r_i - 1} \end{bmatrix}$$

which has rank $r - 1$. Thus, ρ is not in $V_1(\Gamma)$.

■

Corollary 3.3.2 *Let Γ be the direct product of finitely presented groups*

$$\Gamma = \Gamma_1 \times \ldots \times \Gamma_k$$

with $r_1, \ldots, r_k$ generators, respectively. Let

$$P = F_{r_1} \times \ldots \times F_{r_k}.$$

Then

$$V_i(\Gamma) \subset V_i(P)$$

for each i and, in particular,

$$V_i(\Gamma) \subset \{\widehat{1}\}$$

if $\max\{r_1, \ldots, r_k\} \leq i$.

Proof. This follows from Lemma 3.1.2 and Proposition 3.1.3.

■

In particular, if Γ is abelian, we have the following result.

Corollary 3.3.3 *If Γ is an abelian group, then*

$$V_i(\Gamma) = \begin{cases} \{\widehat{1}\} & \text{if } 1 \leq \mathrm{i} < \mathrm{rank}(\Gamma) \\ \emptyset & \text{otherwise.} \end{cases}$$

Here $\mathrm{rank}(\Gamma)$ *means the rank of the abelianization of* Γ.

4 Applications.

Let X be any topological space homotopy equivalent to a finite CW complex with fundamental group Γ. In this section, we will study the role that rational planes in the Alexander strata $V_i(\Gamma)$ and the jumping loci $W_i(\Gamma)$ relate to the the geometry of X.

4.1 Betti numbers of abelian coverings.

Let X be homotopy equivalent to a finite CW complex. Let $\Gamma = \pi_1(X)$. We will relate the first Betti number of finite abelian coverings of X to rational planes in the jumping loci $W_i(\Gamma)$.

Let $\alpha : \Gamma \to G$ be an epimorphism onto a finite abelian group G. Assume that Γ is generated by r elements. Then by Corollary 2.5.7, we have

$$b_1(X_\alpha) = \sum_{i=1}^{r} |W_i(\Gamma) \cap \widehat{\alpha}(\widehat{G})|.$$

Since G is finite, all points in $\alpha(\widehat{G})$ have finite order. Thus, to compute $b_1(X_\alpha)$ for finite abelian coverings X_α, we need only know about the torsion points on $W_i(\Gamma)$.

The position of torsion points $\mathrm{Tor(V)}$ for any algebraic subset $V \subset (\mathbb{C}^*)^r$ is described by the following result due to Laurent [La].

Theorem 4.1.1 *(Laurent) If $V \subset (\mathbb{C}^*)^r$ is any algebraic subset, then there exist rational planes $P_1, \dots, P_k$ in $(\mathbb{C}^*)^r$ such that $P_i \subset V$ for each $i = 1, \dots, k$ and*

$$\mathrm{Tor(V)} = \bigcup_{\mathrm{i=1}}^{\mathrm{k}} \mathrm{Tor(P_i)}.$$

From this theorem it follows that, to any finitely presented group Γ, we can associate a collection of finite sets of rational planes $\mathcal{P}_i$, such that

$$\mathrm{Tor(V_i(\Gamma))} = \bigcup_{\mathrm{P} \in \mathcal{P}_\mathrm{i}} \mathrm{Tor(P)}.$$

We thus have the following.

Corollary 4.1.2 *The rank of co-abelian, finite index subgroups of a finitely presented group Γ depends only on the rational planes contained in the Alexander strata $V_i(\Gamma)$.*

4.2 Existence of irrational pencils.

Let X be a compact Kähler manifold. An *irrational pencil on X* is a surjective morphism

$$X \to C_g,$$

where C_g is a Riemann surface of genus $g \geq 2$. In this section, we will discuss the relation between properties of the Alexander stratification for $\Gamma = \pi_1(X)$ and the existence of irrational pencils on X.

Let Γ_g be the fundamental group of C_g. Then Γ_g has presentation $\langle F_{2g}, R_g \rangle$, where R_g is the single element

$$[x_1, x_{g+1}][x_2, x_{g+2}] \dots [x_g, x_{2g}].$$

The Fox derivative of R_g is given by

$$D(R_g) = \sum_{i=1}^{g} (t_i - 1)\langle x_i \rangle + \sum_{i=g+1}^{2g} (1 - t_i)\langle x_i \rangle.$$

Thus, we have

$$V_i(\Gamma_g) = \begin{cases} \widehat{\Gamma_g} \cong (\mathbb{C}^*)^{2g} & \text{if } 1 \leq i < 2g-1; \\ \{\hat{1}\} & \text{if } i = 2g-1; \\ \emptyset & \text{if } i > 2g-1. \end{cases}$$

and for the jumping loci

$$W_i(\Gamma_g) = \begin{cases} \widehat{\Gamma_g} \cong (\mathbb{C}^*)^{2g} & \text{if } 1 \leq i < 2g-1; \\ \{\hat{1}\} & \text{if } 2g-1 \leq i \leq 2g; \\ \emptyset & \text{if } i > 2g. \end{cases}$$

Given an irrational pencil $X \to C_g$, the Stein factorization gives a map

$$X \to C_h \to C_g,$$

where the map from C_h to C_g is a finite surjective morphism and X has connected fibers. Then $h \geq g$ and there is a surjective group homomorphism

$$\pi_1(X) \to \Gamma_h.$$

By Proposition 3.1.3, this implies that there is an inclusion

$$W_i(\Gamma_h) \to W_i(\pi_1(X)),$$

for all i.

We can thus conclude the following.

Proposition 4.2.1 *If X has an irrational pencil of genus g, then for some $h \geq g$, $W_i(\pi_1(X))$ contains an affine subtorus of dimension $2h$, for $i = 1, \ldots, 2h-2$.*

The question arises, do the maximal affine subtori in $W_i(\pi_1(X))$ all come from irrational pencils? This was answered in the affirmative by Beauville [Be] for $W_1(\pi_1(X))$ (see also [G-L],[Ar1] and [Cat].) This shows that the irrational pencils on X only depend on the topological type of X (see also [Siu]).

Now suppose $V \subset W_i(\Gamma)$ is a translate of an affine subtorus by a character $\rho \in \widehat{\Gamma}$ of finite order. Then, since Γ is finitely generated, the image of ρ is finite in $\mathbb{C}^*$. Let $\widetilde{X} \to X$ be the finite abelian unbranched covering associated to this map. Then the corresponding map on fundamental groups

$$\alpha : \pi_1(\widetilde{X}) \to \pi_1(X)$$

has image equal to the kernel of ρ. Thus, $\widehat{\alpha}(\rho)$ is the trivial character in $\widehat{\pi_1(\widetilde{X})}$ and $\widehat{\alpha}(V)$ is a connected subgroup, i.e., an affine subtorus of $\widehat{\pi_1(\widetilde{X})}$.

As we discuss in the next section, a theorem of Simpson shows that all the jumping loci $W_i(\pi_1(X))$ are finite unions of rational planes. This leads us to the following question:

Question. Can all the rational planes in the jumping loci $W_i(\pi_1(X))$ be explained by irrational pencils on X or on finite abelian coverings of X?

4.3 Binomial criterion for Kähler groups.

If Γ is a group such that there is an isomorphism $\Gamma \cong \pi_1(X)$ for some compact Kähler manifold X, we will say that Γ is *Kähler.*

A *binomial ideal* in $\Lambda_r(\mathbb{C})$ is an ideal generated by *binomial* elements of the form

$$t^\lambda - u$$

where $\lambda \in \mathbb{Z}^r$, $t^\lambda = t_1^{\lambda_1} \ldots t_r^{\lambda_r}$ and $u \in \mathbb{C}$ is a unit.

The following is straightforward.

Lemma 4.3.1 *If $V \subset (\mathbb{C}^*)^r$ is a rational plane then V is defined by a binomial ideal where the units u are roots of unity.*

In ([Ar1], Theorem 1), Arapura shows that $W_i(\Gamma)$ is a finite union of unitary translates of affine tori. Simpson ([Sim], Theorem 4.2) extends Arapura's result, showing that the $W_i(\Gamma)$ are actually translates of rational tori.

Theorem 4.3.2 *(Simpson) If Γ is Kähler, then $W_i(\Gamma)$ is a finite union of rational planes for all i.*

Corollary 4.3.3 *If Γ is Kähler, then any irreducible component of $V_i(\Gamma)$ is defined by a binomial ideal.*

Proof. By Lemma 2.2.3, $V_i(\Gamma)$ equals $W_i(\Gamma)$ except when i equals the rank of the abelianization of Γ. Suppose the latter holds. Then, again by Lemma 2.2.3, $V_i(\Gamma)$ is $W_i(\Gamma)$ minus the identity character $\widehat{1}$. But $V_i(\Gamma)$ is a closed algebraic set, so $\widehat{1}$ is an isolated component of $W_i(\Gamma)$. Thus, since $W_i(\Gamma)$ is a finite union of rational planes, so is $V_i(\Gamma)$. The rest follows from Lemma 4.3.1.

■

Remark. Stated in terms of the ideals of minors (also known as Alexander ideals or fitting ideals) of an Alexander matrix, Simpson's theorem implies a property of the radical of these ideals for Kähler groups. Subtler and interesting questions can be asked about the fitting ideals themselves. We leave this as a topic for further research.

Let R_g be the standard relation for $\pi_1(C_g)$, where C_g is a Riemann surface of genus g. It is possible from Corollary 4.3.3 to make many examples of nonKähler finitely presented groups. For example, we have the following Proposition (cf. [Ar2], [Gro], [Sim]).

Proposition 4.3.4 *Let $g \geq 2$ and let*

$$\Gamma = \langle x_1, \ldots, x_{2g} : S_1, \ldots, S_s \rangle,$$

where

$$S_i = u_{i,1} R_g u_{i,1}^{-1} \ldots u_{i,k_i} R_g u_{i,k_i}^{-1},$$

for $i = 1, \ldots, s$. Let

$$p_i = \mathrm{ab}(\mathrm{u}_{\mathrm{i},1}) + \ldots + \mathrm{ab}(\mathrm{u}_{\mathrm{i},\mathrm{k}_\mathrm{i}})$$

considered as a polynomial in Λ_r. Then, if Γ is Kähler, the set of common zeros $V(p_1, \ldots, p_s)$ must be defined by binomial ideals.

Proof. The Fox derivative $D : F_{2g} \to \mathbb{Z}[\mathrm{ab}(\mathrm{F}_{2\mathrm{g}})]^{2\mathrm{g}}$ takes each S_i to

$$D(S_i) = (\mathrm{ab}(\mathrm{u}_{\mathrm{i},1}) + \ldots + \mathrm{ab}(\mathrm{u}_{\mathrm{i},\mathrm{k}_\mathrm{i}}))\mathrm{D}(\mathrm{R}_\mathrm{g}).$$

Thus, the ith row of the Alexander matrix $M(F_{2g}, \mathcal{R})$ equals $M(F_{2g}, R_g)$, considered as row vector, multiplied by p_i. It follows that the rank of $M(F_{2g}, \mathcal{R})$ is at most 1 and equals 1 outside of the set of common zeros of $p_1, \ldots, p_s$ and the point $(1, \ldots, 1)$. The rest is a consequence of Corollary 4.3.3.

■

Example. Fix $g \geq 3$, and let Γ be given by

$$\Gamma = \langle x_1, \ldots, x_{2g} : S_1, S_2 \rangle,$$

where

$$S_1 = x_1 R_{2g} x_1^{-1} \ldots x_g R_{2g} x_g^{-1}$$

and

$$S_2 = x_{g+1} R_{2g} x_{g+1}^{-1} \ldots x_{2g} R_{2g} x_{2g}^{-1}.$$

Then

$$\begin{aligned} D(S_1) &= (t_1 + \ldots + t_g) D(R_g) \\ D(S_2) &= (t_{g+1} + \ldots + t_{2g}) D(R_g) \end{aligned}$$

which implies that $V_1(\Gamma)$ contains $\widehat{1}$ and the points in

$$V(t_1 + \ldots + t_g) \cap V(t_{g+1} + \ldots + t_{2g}).$$

This is isomorphic to the product of the hypersurface in $(\mathbb{C}^*)^g$ defined by $V = V(t_1 + \ldots + t_g)$ with itself. Since $g \geq 3$, this hypersurface is not defined by a binomial ideal. Thus, Γ is not Kähler.

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