

The augmented deformation space of rational maps

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ABSTRACT. It was recently shown that the Epstein deformation space of marked rational maps with prescribed combinatorial and dynamical structure can be disconnected. For example, the family of quadratic rational maps with a periodic critical cycle of order 4 and an extra critical value not lying in this cycle has a deformation space with infinitely many components. We study the structure of the augmented deformation space for this example, and show, in particular, that the closure of deformation space in augmented deformation space is also disconnected in this case.

In celebration of Lê Dũng Tráng's 70th birthday

1. Introduction

Let A and B be two finite subsets of the 2-dimensional sphere S^2 such that $|A|, |B| \geq 3$, and let $f, \iota : (S^2, A) \rightarrow (S^2, B)$ be two maps of pairs:

$$f : (S^2, A) \rightarrow (S^2, B)$$

a branched covering with branch values contained in B , and

$$\iota : (S^2, A) \hookrightarrow (S^2, B)$$

a homeomorphism identifying domain and range, and A with a subset of B .

The *Epstein deformation space* $\mathcal{D}_{f,\iota}$ is defined as the equalizer of the induced maps on Teichmüller spaces

$$f^*, \iota^* : \mathcal{T}_{(S^2, B)} \rightarrow \mathcal{T}_{(S^2, A)}.$$

In unpublished work from the 1990s, A. Epstein showed that if f is not Lattès^[1], then $\mathcal{D}_{f,\iota}$ is either empty or a complex $|B| - |A|$ dimensional submanifold of $\mathcal{T}_{(S^2, B)}$ [7]. This generalizes a seminal result of W. Thurston who showed that in the *postcritically finite* case, when $A = B$, either $\mathcal{D}_{f,\iota}$ is empty, f is a Lattès example, or $\mathcal{D}_{f,\iota}$ contains exactly one point [6]. In particular, $\mathcal{D}_{f,\iota}$ is always connected in the postcritically finite case. When f is not post-critically finite, then $\mathcal{D}_{f,\iota}$ need not be connected [11], but so far there is only one known class of examples.

PROBLEM 1.1. What are necessary and sufficient conditions for $\mathcal{D}_{f,\iota}$ to be connected?

This work was partially supported by a grant from the Simons Foundation #426722.

¹See [15] for definitions.

The counter-example to connectedness in [\[11\]](#) is part of a class of quadratic rational maps studied by Milnor in [\[14\]](#). These are denoted $\text{Per}_n(0)$ and have one periodic n -cycle containing a single critical point, and the behavior of the other critical value under iteration is unspecified. If in addition the extra critical value does not lie in the first critical n -cycle, the rational map is said to be in $\text{Per}_n(0)^*$. Given an element $F \in \text{Per}_n(0)^*$ let $f, \iota : (S^2, A) \rightarrow (S^2, B)$ be defined so that f is the topological covering underlying F , ι is a identification of domain and range of f so that $A \subset B$, A is the critical n -cycle, and B is the union of A and the extra critical point. For $n = 3$, $\mathcal{T}_{(S^2, A)}$ is a singleton set, and hence $\mathcal{D}_{f, \iota}$ is the entire space $\mathcal{T}_{(S^2, B)}$. For $n = 4$, the work in [\[11\]](#) shows that $\mathcal{D}_{f, \iota}$ is not connected, and in fact has infinitely many connected components. This leads to the following question.

QUESTION 1.2. Does $\mathcal{D}_{f, \iota}$ have a natural connected closure?

The points in $\mathcal{D}_{f, \iota}$ are sometimes called the *dynamical points* in the space of complex structures on $\mathcal{T}_{(S^2, B)}$. The *augmented deformation space* $\mathcal{AD}_{f, \iota}$, or *ideal dynamical points*, is the subset of the Bers augmented deformation space $\mathcal{AT}_{(S^2, B)}$ (see [\[2\]](#)) defined as the equalizer of extensions

$$\tilde{f}^*, \tilde{\iota}^* : \mathcal{AT}_{(S^2, B)} \rightarrow \mathcal{AT}_{(S^2, A)}$$

of f^* and ι^* .

At the time of this writing, it is not known whether $\mathcal{AD}_{f, \iota}$ is connected in the $\text{Per}_4(0)^*$ case. Our goal in this paper is to extend the techniques of [\[11\]](#), and apply them to give a partial description of the structure of $\mathcal{AD}_{f, \iota}$ and the closure of $\mathcal{D}_{f, \iota}$ within it. We prove the following theorem.

THEOREM 1.3. *For (f, ι) associated to an element of $\text{Per}_4(0)^*$, the closure of $\mathcal{D}_{f, \iota}$ in augmented deformation space $\mathcal{AD}_{f, \iota}$ is not connected.*

1.1. Some background and ideas behind the proofs. The question of whether and when $\mathcal{D}_{f, \iota}$ is connected has roots in work of Thurston from the 1980s, in which he showed that if

$$F : \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

is a non-Lattès rational map from the complex projective line to itself with a finite post-critical set

$$\bigcup_{n=1}^{\infty} F^{(n)}(\text{Crit}_F),$$

and $f : (S^2, P) \rightarrow (S^2, P)$ is the corresponding branched covering of pointed spheres with domain and range identified, and postcritical set P , then the lifting map on holomorphic markings defines a contracting map on Teichmüller space

$$f^* : \mathcal{T}_{(S^2, P)} \rightarrow \mathcal{T}_{(S^2, P)}.$$

Thus, f^* has a unique fixed point, and hence $\mathcal{D}_{f, \iota}$ is a singleton set.

This classical result suggests that there could be a contracting flow in the general case when the identification map $\iota : (S^2, A) \rightarrow (S^2, B)$ is a strict inclusion on A . If such a flow exists, there are two possibilities: one is that $\mathcal{D}_{f, \iota}$ is connected, and the other is that some points may be pushed out to the boundary of $\mathcal{T}_{(S^2, B)}$. This suggests looking at dynamical elements of the augmented Teichmüller space to find a natural connected completion of $\mathcal{D}_{f, \iota}$.

The proof in [11], that $\mathcal{D}_{f,\iota}$ can be disconnected translates the question about flows on Teichmüller spaces to one about the topology of algebraic varieties. A key ingredient is to define an intermediate covering \mathcal{M}_f of $\mathcal{T}_{(S^2,B)} \rightarrow \mathcal{M}_{(S^2,B)}$ that is a natural quotient of the space of marked rational maps topologically equal to f . The projection $\mathcal{M}_f \rightarrow \mathcal{M}_{(S^2,B)}$ is a finite covering, and hence has the structure of a quasi-projective variety.

In the case when $F \in \text{Per}_4(0)^*$, the space \mathcal{M}_f is isomorphic to a Zariski dense subset of $\mathbb{P}^1 \times \mathbb{P}^1$:

$$\mathcal{M}_f = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathcal{Z} \cup \mathcal{L}$$

where $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathcal{L} = \mathcal{M}_{(S^2,A)} \times \mathcal{M}_{(S^2,A)}$, and \mathcal{Z} indicates the locus where the both critical points of f lie in the same periodic cycle. With respect to this parameterization, the image of $\mathcal{D}_{f,\iota}$ in \mathcal{M}_f is the diagonal

$$\mathcal{V}_{f,\iota} = \{(x, x) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid (x, x) \notin \mathcal{Z} \cup \mathcal{L}\}.$$

Let L_f be the group of covering automorphisms of $\mathcal{T}_{(S^2,B)}$ over $\mathcal{M}_{f,\iota}$. It was shown in [11] that the projection of $\mathcal{D}_{f,\iota}$ on $\mathcal{V}_{f,\iota}$ is a regular covering with covering automorphism group a proper subgroup $S_{f,\iota} \subset L_f$. In particular, $S_{f,\iota}$ acts transitively on fibers of $\mathcal{D}_{f,\iota} \rightarrow \mathcal{V}_{f,\iota}$.

Choose a basepoint $d_0 \in \mathcal{D}_{f,\iota}$ and let v_0 be its image in $\mathcal{V}_{f,\iota}$. Then we can identify L_f with $\pi_1(\mathcal{M}_f, v_0)$ and $S_{f,\iota}$ with a subgroup of the fundamental group. Let $E_{f,\iota}$ be the image of $\pi_1(\mathcal{V}_{f,\iota}, v_0) \rightarrow \pi_1(\mathcal{M}_f, v_0)$ induced by the inclusion map. Then $E_{f,\iota}$ is the subgroup of $S_{f,\iota}$ that fixes the component of $\mathcal{D}_{f,\iota}$ containing d_0 ; $E_{f,\iota}$ has infinite index in $S_{f,\iota}$; and hence $\mathcal{D}_{f,\iota}$ has infinitely many components. In other words, the image of $\pi_1(\mathcal{V}_{f,\iota}, v_0)$ is not sufficiently large in $\pi_1(\mathcal{M}_f, v_0)$. One way to try to rectify this is to put both \mathcal{M}_f and $\mathcal{V}_{f,\iota}$ into a larger ambient space.

Let $\mathcal{AT}_{(S^2,B)}$ be the augmented Teichmüller space of (S^2, B) . Then L_f extends to an action on $\mathcal{AT}_{(S^2,B)}$ giving a quotient \mathcal{AM}_f and a commutative diagram

$$\begin{array}{ccc} \mathcal{T}_{(S^2,B)} & \hookrightarrow & \mathcal{AT}_{(S^2,B)} \\ \downarrow /L_f & & \downarrow /L_f \\ \mathcal{M}_f & \hookrightarrow & \mathcal{AM}_f. \end{array}$$

Similarly, $S_{f,\iota}$ acts on $\mathcal{AD}_{f,\iota}$ with quotient denoted $\mathcal{AV}_{f,\iota}$. The spaces $\mathcal{AV}_{f,\iota}$ and \mathcal{AM}_f have the advantage of being algebraic geometric sets, and can be studied via singularity theory.

We define a connected pure 1-dimensional algebraic subset X of a blowup $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ of $\mathbb{P}^1 \times \mathbb{P}^1$ and an embedding $X \rightarrow \mathcal{AV}_{f,\iota}$ that is surjective except possibly a finite set of points (Proposition 4.2). By studying properties of X and its embedding in $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ we prove Theorem 1.3

1.2. Organization. In Section 2.1, we give necessary definitions of Teichmüller and moduli spaces for rational maps, and their augmented versions. In Section 3, we prove some general properties of complements of plane algebraic curves and their blowups. In Section 4, we apply these ideas to the $\text{Per}_4(0)^*$ case, and prove Theorem 1.3

1.3. Further questions. There are still many open questions along the lines of this investigation. So far, the $\text{Per}_4(0)^*$ example is the only known case when $\mathcal{D}_{f,\iota}$ is disconnected. Are there others? Is $\mathcal{AD}_{f,\iota}$ connected in the $\text{Per}_4(0)^*$ case, and is $\mathcal{AD}_{f,\iota}$ connected in general? The analysis in this paper suggest general approaches to these questions, which we leave for further investigation.

1.4. Some comments on notation. This paper grew out of the ideas in [11], but some of the notation has changed. Most notably, we refer to the Teichmüller space \mathcal{T}_f of rational maps topologically equivalent to f , and its corresponding Moduli space \mathcal{M}_f . Thus, Epstein's deformation space $\mathcal{D}_{f,\iota}$ will be considered as a subspace of \mathcal{T}_f rather than of the isomorphic space $\mathcal{T}_{(S^2,B)}$ (also known as the *parameter space*). The space \mathcal{M}_f may be more familiarly known as a connected component of a Hurwitz space of rational maps [16] and was denoted by \mathcal{W}_f in [11]. A smaller change is the use of $\mathcal{D}_{f,\iota}$ for the deformation space \mathcal{D}_f to emphasize the dependence on both the topological branched covering of pairs f and the identification ι . Similarly, we changed the notation of S_f to $S_{f,\iota}$. These ease our transition from a discussion of deformation space to augmented deformation space.

1.5. Acknowledgments. I would like to thank X. Buff, S. Koch, C. McMullen, R. Ramadas, and L. D. Tráng for helpful references and discussions, and the anonymous referee for their careful reading and useful comments.

2. Background definitions

In this section, we recall definitions and properties of Teichmüller space, moduli space and deformation space for rational maps and their augmented versions. For more details about the general theory see, for example, [5], [2], [12], [16] and [17]. We also recall definitions of liftables L_f and special liftables $S_{f,\iota}$ from [11], and examine their extensions to the augmented spaces.

2.1. Teichmüller spaces for rational maps. Let A be a finite subset of 3 or more points of the topological 2-sphere S^2 . The *Teichmüller space* $\mathcal{T}_{(S^2,A)}$ of *holomorphic markings* on (S^2, A) is the collection of orientation preserving homeomorphisms

$$\phi : (S^2, A) \rightarrow (\mathbb{P}^1, \phi(A))$$

defined up to post-composition by automorphisms of \mathbb{P}^1 (i.e., Möbius transformations) and pre-composition by self-homeomorphisms of S^2 that are isotopic rel A to the identity.

Similarly, given a finite branched covering of pairs $f : (S^2, A) \rightarrow (S^2, B)$, where $\infty > |A|, |B| \geq 3$ and B contains the branch values (or critical values) of f , we define the *Teichmüller space* \mathcal{T}_f of *holomorphic markings* on (f, A, B) as the set of commutative diagrams

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ f \downarrow & & \downarrow F \\ (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$

also denoted by (ϕ, ψ, F) , where F is a rational map, ϕ and ψ are homeomorphisms. Two triples (ϕ, ψ, F) and (ϕ_1, ψ_1, F_1) in \mathcal{T}_f are *equivalent* if there are homeomorphisms $\alpha : (S^2, A) \rightarrow (S^2, A)$ isotopic to the identity map rel A and

$\beta : (S^2, B) \rightarrow (S^2, B)$ isotopic to the identity rel B , and biholomorphic maps $\mu, \nu : \mathbb{P}^1 \rightarrow \mathbb{P}^1$ so that the diagram

$$\begin{array}{ccccccc} (\mathbb{P}^1, \psi_1(A)) & \xleftarrow{\psi_1} & (S^2, A) & \xrightarrow{\alpha} & (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ F_1 \downarrow & & f \downarrow & & f \downarrow & & F \downarrow \\ (\mathbb{P}^1, \phi_1(B)) & \xleftarrow{\phi_1} & (S^2, B) & \xrightarrow{\beta} & (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$

commutes, and $\mu = \psi \circ \alpha \circ \psi_1^{-1}$ and $\nu = \phi \circ \beta \circ \phi_1^{-1}$.

By these definitions, \mathcal{T}_f comes with natural surjections

$$\begin{array}{ccc} & \mathcal{T}_f & \\ q \swarrow & & \searrow p_U \\ \mathcal{T}_{(S^2, B)} & & \mathcal{T}_{(S^2, A)} \end{array}$$

recording the holomorphic markings of the domain space (p_U) and the range space (q). Furthermore, \mathcal{T}_f can be thought of as the graph of a *lifting map*

$$f^* : \mathcal{T}_{(S^2, B)} \rightarrow \mathcal{T}_{(S^2, A)}$$

defined by f . Thus, q is an isomorphism of holomorphic spaces.

We create an iteration scheme from f by partially identifying the domain (S^2, A) and range (S^2, B) of f . That is, we fix a homeomorphism $\iota : S^2 \rightarrow S^2$ that restricts to an inclusion $\iota|_A : A \hookrightarrow B$. Then we have a map of pairs

$$\iota : (S^2, A) \rightarrow (S^2, B).$$

Let $p_L = \iota^* \circ q$, where $\iota^* : \mathcal{T}_{(S^2, B)} \rightarrow \mathcal{T}_{(S^2, A)}$ is the map that takes $\phi \in \mathcal{T}_{(S^2, B)}$ to the class in $\mathcal{T}_{(S^2, A)}$ defined by

$$\phi \circ \iota : (S^2, A) \rightarrow (\mathbb{P}^1, (\phi \circ \iota)(A)) = (\mathbb{P}^1, \phi(A)).$$

Then the elements of \mathcal{T}_f can be thought of as the holomorphic markings of the branched covering f , and p_U and p_L record the induced marked holomorphic structures (S^2, A) on the domain and range.

The Epstein's *deformation space* $\mathcal{D}_{f, \iota}$ is the subspace of \mathcal{T}_f consisting of holomorphic structures on the covering and base of f that are equivalent relative to A . That is, $\mathcal{D}_{f, \iota}$ consists of the triples (ϕ, ψ, F) so that $\phi^{-1} \circ \psi$ is isotopic to the identity relative to A , or equivalently

$$f^* \phi = \iota^* \psi.$$

Another way to say this is that $\mathcal{D}_{f, \iota}$ is the equalizer in \mathcal{T}_f of the maps p_U and p_L , that is

$$\mathcal{D}_{f, \iota} = \{(\phi, \psi, F) \in \mathcal{T}_f \mid p_U(\phi, \psi, F) = p_L(\phi, \psi, F)\}.$$

We think of $\mathcal{D}_{f, \iota}$ as the *dynamical Teichmüller space* for (f, ι) .

2.2. Moduli spaces. Let $\mathcal{M}_{(S^2, A)}$ be the space of embeddings

$$A \hookrightarrow \mathbb{P}^1$$

up to post-composition by a Möbius transformation. Then the restriction map $[\phi] \mapsto [\phi|_A]$ defines a regular covering map

$$\mathcal{T}_{(S^2, A)} \rightarrow \mathcal{M}_{(S^2, A)}$$

with covering automorphism group equal to the *mapping class group* $\text{Mod}(S^2, A)$ of orientation preserving homeomorphisms $h : S^2 \rightarrow S^2$ that fix the points of A up to isotopy rel A . (Unlike the case for Teichmüller spaces and moduli space of general surfaces, $\text{Mod}(S^2, A)$ on $\mathcal{T}_{(S^2, A)}$ acts without fixed points.)

Similarly let \mathcal{M}_f be the space of commutative diagrams

$$\begin{array}{ccc} A & \xrightarrow{j} & \mathbb{P}^1 \\ f|_A \downarrow & & \downarrow F \\ B & \xrightarrow{i} & \mathbb{P}^1. \end{array}$$

where (i, j, F) is defined up to modifications of the domain and range of F by Möbius transformations. Then the map $(\phi, \psi, F) \mapsto (\phi|_B, \psi|_A, F)$ defines a covering map

$$\mathcal{T}_f \rightarrow \mathcal{M}_f,$$

with covering automorphism group $L_f \subset \text{Mod}(S^2, B)$ called the subgroup of *liftables* consisting of elements $h \in \text{Mod}(S^2, B)$ such that for some $h' \in \text{Mod}(S^2, A)$ the diagram

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{h'} & (S^2, A) \\ f \downarrow & & \downarrow f \\ (S^2, B) & \xrightarrow{h} & (S^2, B) \end{array}$$

commutes.

REMARK 2.1. Since A and B are assumed to contain at least three points, $f : (S^2, A) \rightarrow (S^2, B)$ can have no non-trivial covering automorphisms. In the degree 2 case, this follows from the fact that f must have exactly two branch points and two branch values. Any other marked point would lie in the unbranched part of the covering. Even in the case when the degree of f is greater than 2, the fact that f may be realized as a rational map implies that all covering automorphisms must be conjugate to a Möbius transformation. If there are at least three points in A , then the Möbius transformation must be the identity.

By the definition of L_f and since f has no non-trivial covering automorphisms, f defines a unique lifting map

$$f^\# : L_f \rightarrow \text{Mod}(S^2, A)$$

where $f^\#h = h'$.

The following Proposition was shown using slightly different language in [13] (cf. [11] Proposition 2.2). For the convenience of the reader, we include a proof below.

PROPOSITION 2.2. *The map $\mathcal{T}_f \rightarrow \mathcal{M}_f$ is a regular covering with automorphism group L_f .*

PROOF. Let $h \in L_f$. Take any $(\phi, \psi, F) \in \mathcal{T}_f$. Then we have the commutative diagram

$$(2.1) \quad \begin{array}{ccccc} (S^2, A) & \xleftarrow{f^\#h} & (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ f \downarrow & & f \downarrow & & \downarrow F \\ (S^2, B) & \xleftarrow{h} & (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \mathcal{B}). \end{array}$$

Since h and $f^\#h$ fix B and A , respectively, (ϕ_B, ψ_A, F) and $(\phi \circ h, \psi \circ f^\#h, F)$ map to the same element in \mathcal{M}_f .

Conversely, suppose (ϕ, ψ, F) and (ϕ_1, ψ_1, F_1) both map to equivalent elements in \mathcal{M}_f . Then by definition, there are embeddings $i : B \rightarrow \mathbb{P}^1$ and $j : A \rightarrow \mathbb{P}^1$ so that $i = \phi|_B = \phi_1|_B$, $j = \psi|_A = \psi_1|_A$ and there are Möbius transformations μ and ν such that $F = \mu^{-1} \circ F_1 \circ \nu$. We can assume for simplicity that $F = F_1$ by replacing (ϕ_1, ψ_1, F_1) by the equivalent element $(\mu^{-1} \circ \phi_1, \nu^{-1} \circ \psi_1, F)$ in \mathcal{T}_f . Then the common image of (ϕ, ψ, F) and (ϕ_1, ψ_1, F) is some (i, j, F) in \mathcal{M}_f . Thus the images of ϕ and ϕ_1 are the same in $\mathcal{M}_{(S^2, B)}$, the images of ψ and ψ_1 are the same in $\mathcal{M}_{(S^2, A)}$, and hence there are mapping classes $h \in \text{Mod}(S^2, B)$ and $f^\#h \in \text{Mod}(S^2, A)$ such that $\phi = h \circ \phi_1$ and $\psi = f^\#h \circ \psi_1$. The maps h and $f^\#h$ complete a diagram of the form (2.1), and thus $h \in L_f$. \square

Let $\bar{q} : \mathcal{M}_f \rightarrow \mathcal{M}_{(S^2, B)}$ be the map sending (i, j, F) to the equivalence class containing i in $\mathcal{M}_{(S^2, B)}$; let $\bar{p}_U : \mathcal{M}_f \rightarrow \mathcal{M}_{(S^2, A)}$ be the map sending (i, j, F) to the equivalence class containing j in $\mathcal{M}_{(S^2, A)}$; let $\bar{\tau}^* : \mathcal{M}_{(S^2, B)} \rightarrow \mathcal{M}_{(S^2, A)}$ be the forgetful map sending an inclusion $i : B \rightarrow \mathbb{P}^1$ to $i \circ \iota|_A$; and let $\bar{p}_L = \bar{\tau}^* \circ \bar{q}$. Then we have the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{T}_f & & \\ & \swarrow \simeq & \downarrow \rho & \searrow p_U & \\ & \mathcal{T}_{(S^2, B)} & & & \mathcal{T}_{(S^2, A)} \\ & \downarrow / \text{Mod}(S^2, B) & \searrow / L_f & & \downarrow / \text{Mod}(S^2, A) \\ & \mathcal{M}_{(S^2, B)} & \mathcal{M}_f & & \mathcal{M}_{(S^2, A)} \\ & \swarrow \bar{q} & \downarrow \rho & \searrow \bar{p}_U & \\ & \mathcal{M}_{(S^2, B)} & & & \mathcal{M}_{(S^2, A)} \end{array}$$

All vertical arrows and the left three diagonal arrows are unbranched covering maps. The right diagonal arrows may not be surjective (see [3]). Let $\mathcal{V}_{f, \iota}$ be the image of the deformation space $\mathcal{D}_{f, \iota}$ in \mathcal{M}_f . Then we have

$$\mathcal{V}_{f, \iota} = \text{Eq}(\bar{p}_L, \bar{p}_U).$$

2.3. Stabilizer of deformation space. Fix a basepoint $d_0 \in \mathcal{D}_{f, \iota}$, and let $m_0 \in \mathcal{M}_{f, \iota}$ be the image of d_0 under the map ρ . Then we have an identification

$$\ell : \pi_1(\mathcal{M}_{f, \iota}, m_0) \rightarrow L_f,$$

defined by the path-lifting theorem for coverings. That is, for each $\gamma \in \pi_1(\mathcal{M}_{f,\iota}, m_0)$, we lift γ to a path γ' on $\mathcal{T}_{(S^2, B)}$ based at d_0 , and let $\ell(\gamma)$ be the mapping class that takes d_0 to the end point of γ' .

PROPOSITION 2.3. *The restriction of $\rho : \mathcal{T}_f \rightarrow \mathcal{M}_f$ to $\mathcal{D}_{f,\iota}$ defines a covering map*

$$\rho_{\mathcal{D}} : \mathcal{D}_{f,\iota} \rightarrow \mathcal{V}_{f,\iota},$$

and the image E_{f,ι,d_0} of

$$\pi_1(\mathcal{V}_{f,\iota}, m_0) \rightarrow \pi_1(\mathcal{M}_{f,\iota}, m_0) \xrightarrow{\ell} L_f$$

is the stabilizer of the connected component of $\mathcal{D}_{f,\iota}$ that contains d_0 .

PROOF. We show that the projection $\rho_{\mathcal{D}_{f,\iota}}$ satisfies the path-lifting theorem. Let $d_0 = (\phi_0, \psi_0, F_0)$, and let (i_t, j_t, F_t) be a path in $\mathcal{V}_{f,\iota}$ with $m_0 = (j_0, i_0, F_0)$. Let ξ and η be representatives of the class of ϕ_0 and ψ_0 so that the diagram commutes

$$\begin{array}{ccc} S^2 & \xrightarrow{\eta} & \mathbb{P}^1 \\ f \downarrow & & \downarrow F \\ S^2 & \xrightarrow{\xi} & \mathbb{P}^1, \end{array}$$

$\eta|_A = \psi_0|_A$ and $\xi|_B = \psi|_B$. Let $p_t : (S^2, B) \rightarrow (S^2, B_t)$ be any continuous family of homeomorphisms. Then

$$(\xi \circ p_t|_{B_t}, \eta \circ p_t|_{A_t}, F_t) \sim (i_t, j_t, F_t)$$

and $(\xi \circ p_t, \eta \circ p_t, F_t)$ is a lift of (i_t, j_t, F_t) and lies in $\mathcal{D}_{f,\iota}$.

Thus, as a restriction of an unbranched covering map, $\rho_{\mathcal{D}}$ is itself is a covering map. The rest follows from basic covering space theory. \square

The lifting map $f^\# : L_f \rightarrow \text{Mod}(S^2, A)$ is uniquely defined and satisfies the commutative diagram

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{f^\# h} & (S^2, A) \\ f \downarrow & & \downarrow f \\ (S^2, B) & \xrightarrow{h} & (S^2, B). \end{array}$$

For a homeomorphism $h : (S^2, B) \rightarrow (S^2, B)$, let h_A be the element of $\text{Mod}(S^2, A)$ defined by ignoring the points of $B \setminus A$, that is h_A is the isotopy class of h defined up to homeomorphisms isotopic to the identity rel A . Define

$$\begin{aligned} \iota^\# : \text{Mod}(S^2, B) &\rightarrow \text{Mod}(S^2, A) \\ h &\mapsto h_A. \end{aligned}$$

Let $S_{f,\iota} \subset L_f$ be the equalizer

$$S_{f,\iota} = \text{Eq}(f^\#, \iota^\#) \subset L_f.$$

By the identification of L_f with $\pi_1(\mathcal{M}_{f,\iota}, m_0)$, $S_{f,\iota}$ can equivalently be defined as the equalizer of the homomorphisms

$$(\bar{p}_L)_*, (\bar{p}_U)_* : \pi_1(\mathcal{M}_{f,\iota}, m_0) \rightarrow \pi_1(\mathcal{M}_{(S^2, A)}, a_0)$$

where $a_0 = \bar{p}_L(m_0) = \bar{p}_U(m_0)$, where $m_0 \in \mathcal{V}_{f,\iota} = \text{Eq}(\bar{p}_L, \bar{p}_U)$.

PROPOSITION 2.4 ([11] Proposition 2.5). *The stabilizer in L_f of $\mathcal{D}_{f,\iota}$ equals $S_{f,\iota}$, and $S_{f,\iota}$ acts transitively on the fibers of the covering*

$$\mathcal{D}_{f,\iota} \rightarrow \mathcal{V}_{f,\iota}.$$

Thus the covering is regular and $S_{f,\iota}$ is the group of covering automorphisms.

COROLLARY 2.5. *The deformation space $\mathcal{D}_{f,\iota}$ is connected if and only if $\mathcal{V}_{f,\iota}$ is connected and $S_{f,\iota} = E_{f,\iota,d_0}$.*

In the case when (f,ι) is associated to an element of $\text{Per}_4(0)^*$, E_{f,ι,d_0} has infinite index in $S_{f,\iota}$ ([11], Proposition 2.11).

2.4. Augmented spaces. By a *rational curve* \mathcal{C} we mean a connected nodal curve with the following properties

- (a) the irreducible components of \mathcal{C} are isomorphic to \mathbb{P}^1 , and
- (b) the fundamental group of \mathcal{C} is trivial.

A *pre-stable rational curve* $(\mathcal{C}, \mathcal{A})$ is a rational curve \mathcal{C} together with a finite set \mathcal{A} contained in the complement of the nodes of \mathcal{C} . The set of nodes of \mathcal{C} union the points of \mathcal{A} form the *distinguished points* of \mathcal{C} . A *stable rational curve* is a pre-stable rational curve with the following additional property:

- (c) the number of distinguished points on each irreducible component of \mathcal{C} is greater than or equal to 3.

For each component C of a pre-stable rational curve $(\mathcal{C}, \mathcal{A})$ there are three possibilities:

- (1) C is stable, i.e., it contains at least 3 distinguished points;
- (2) C is unstable, and contains two nodes; or
- (3) C is unstable, and contains one node and zero or one point in \mathcal{A} .

Let $\Sigma_{(S^2, \mathcal{A})}^{\text{pre}}$ be the set of pre-stable rational curves $(\mathcal{C}, \mathcal{A})$ with a bijection $A \rightarrow \mathcal{A}$, and let $\Sigma_{(S^2, \mathcal{A})} \subset \Sigma_{(S^2, \mathcal{A})}^{\text{pre}}$ be the space of stable rational curves. Define a map

$$\mathfrak{s} : \Sigma_{(S^2, \mathcal{A})}^{\text{pre}} \rightarrow \Sigma_{(S^2, \mathcal{A})}$$

sending $(\mathcal{C}, \mathcal{A})$ to the result of contracting components of \mathcal{C} using the following rule: in case (1) leave C alone; and in case (2) and (3) contract C to a point. If C contains a point of \mathcal{A} , then mark the image of the contraction by that point. This map is well-defined since in case (3) there is at most one point of \mathcal{A} in C . We call \mathfrak{s} the *stabilization map*.

A *marking* of a pre-stable rational curve $(\mathcal{C}, \mathcal{A}) \in \Sigma_{(S^2, \mathcal{A})}^{\text{pre}}$ is a quotient map

$$\phi : (S^2, A) \rightarrow (\mathcal{C}, \mathcal{A}),$$

such that ϕ is a homeomorphism when restricted to $S^2 \setminus \gamma$ for some multi-curve $\gamma \subset S^2 \setminus A$, ϕ restricts to a bijection $A \rightarrow \mathcal{A}$, and the components of γ are in one-to-one correspondence with the nodes of \mathcal{C} . The curve γ is called the *contracting curve* for ϕ . We consider two markings *equivalent* if they are the same up to post-composition by automorphisms of \mathcal{C} and pre-composition by homeomorphisms $(S^2, A) \rightarrow (S^2, A)$ that are isotopic to the identity $\text{rel } A$.

The collection of markings of pre-stable and stable rational curves by (S^2, A) is denoted by $\mathcal{AT}_{(S^2, \mathcal{A})}^{\text{pre}}$ and $\mathcal{AT}_{(S^2, \mathcal{A})}$, respectively. Post-composition by \mathfrak{s} defines a surjection

$$\mathcal{AT}_{(S^2, \mathcal{A})}^{\text{pre}} \rightarrow \mathcal{AT}_{(S^2, \mathcal{A})}.$$

The space $\mathcal{AT}_{(S^2, A)}$ is called the *augmented Teichmüller space* of (S^2, A) . There is a natural topology on $\mathcal{AT}_{(S^2, A)}$ such that points on $\mathcal{AT}_{(S^2, A)} \setminus \mathcal{T}_{(S^2, A)}$ are the limits of sequences points on $\mathcal{T}_{(S^2, A)}$ for which the length of γ tends to zero (see [2] for more precise definitions).

REMARK 2.6. The mapping class group $\text{Mod}(S^2, A)$ extends to actions on $\mathcal{AT}_{(S^2, A)}^{\text{pre}}$ and $\mathcal{AT}_{(S^2, A)}$. Given a point in $\mathcal{AT}_{(S^2, A)}$, the stabilizer is the free abelian group of mapping classes generated by Dehn twists along the components of the contracting curve.

The action of $\text{Mod}(S^2, A)$ on $\mathcal{AT}_{(S^2, A)}$ defines a branched covering

$$\mathcal{AT}_{(S^2, A)} \rightarrow \mathcal{AM}_{(S^2, A)}$$

where $\mathcal{AM}_{(S^2, A)}$ is the space of inclusions

$$A \hookrightarrow \mathcal{C}$$

of A into a rational curve \mathcal{C} up to holomorphic automorphism of \mathcal{C} that do not permute components (cf. [5]).

We now define the augmented Teichmüller and moduli spaces of f . A rational map $F : (\mathcal{C}_U, \mathcal{A}) \rightarrow (\mathcal{C}_L, \mathcal{B})$ is *pre-admissible* if

- (a) F defines a surjective map from \mathcal{C}_U to \mathcal{C}_L of generically constant degree that maps nodes to nodes;
- (b) locally near each node of \mathcal{C}_U , F has generically constant degree; and
- (c) $(\mathcal{C}_L, \mathcal{B})$ is a stable rational curve and $(\mathcal{C}_U, \mathcal{A})$ is pre-stable.

We say that F is *admissible* if in addition

- (b) $(\mathcal{C}_U, \mathcal{A})$ is stable.

The *augmented Teichmüller space* \mathcal{AT}_f for f is the collection of holomorphic markings

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\mathcal{C}_U, \mathcal{A}) \\ f \downarrow & & \downarrow F \\ (S^2, B) & \xrightarrow{\phi} & (\mathcal{C}_L, \mathcal{B}) \end{array}$$

where the horizontal maps are markings in $\mathcal{AT}_{(S^2, A)}$ and $\mathcal{AT}_{(S^2, B)}$ respectively, and F is a pre-admissible covering. Here, as in the definition of \mathcal{T}_f , we take (ϕ, ψ, F) up to the natural equivalences.

With this definition, the projection $\tilde{q} : \mathcal{AT}_f \rightarrow \mathcal{AT}_{(S^2, B)}$ defines an isomorphism. Let $\tilde{p}_U, \tilde{p}_L : \mathcal{AT}_f \rightarrow \mathcal{AT}_{(S^2, A)}$ be defined by

$$\begin{aligned} \tilde{p}_L(\phi, \psi, F) &= \mathfrak{s}(\phi) \\ \tilde{p}_U(\phi, \psi, F) &= \mathfrak{s}(\psi) \end{aligned}$$

PROPOSITION 2.7. *The subgroup of liftables $L_f \subset \text{Mod}(S^2, B)$ extends to an action on \mathcal{AT}_f .*

PROOF. Let $\phi : (S^2, B) \rightarrow (\mathcal{C}_L, \mathcal{B})$ be an element of $\mathcal{AT}_{(S^2, B)}$, and let γ be the contracting multi-curve. Since f is unbranched outside of B , it follows that $f^{-1}(\gamma)$ is a multi-curve on $S^2 \setminus A$, and since h is liftable

$$f^\# h(f^{-1}(\gamma)) = f^{-1}(h(\gamma)).$$

Thus $h \in L_f$ takes (ϕ, ψ, F) to $(\phi \circ h, \psi \circ f^\#h, F)$ where $\phi \circ h$ and $\psi \circ f^\#h$ contract the curves $h(\gamma)$ and $f^{-1}(h(\gamma))$ respectively. \square

Let $\mathcal{AM}_f = \mathcal{AT}_f/L_f$. Then the points of \mathcal{AM}_f are defined by diagrams

$$\begin{array}{ccc} A & \xrightarrow{i} & C_U \\ f|_A \downarrow & & \downarrow F \\ B & \xrightarrow{j} & C_L \end{array}$$

where (C_L, B) is stable, (C_U, A) is pre-stable, and the inclusions i, j are defined up to holomorphic automorphisms of C_U and C_L . We denote an element by (i, j, F) . Let $\bar{p}_U, \bar{p}_L : \mathcal{AM}_f \rightarrow \mathcal{AM}(S^2, A)$ be defined by

$$\begin{aligned} \bar{p}_L([\phi, \psi, F]) &= \mathfrak{s} \circ j \\ \bar{p}_U([\phi, \psi, F]) &= \mathfrak{s} \circ i \end{aligned}$$

While the action of L_f on \mathcal{T}_f has no fixed points and the quotient map $\mathcal{T}_f \rightarrow \mathcal{M}_f$ is a covering, the quotient map $\mathcal{AT}_f \rightarrow \mathcal{AM}_f$ can have branch points.

Summarizing, we have a commutative diagram of augmented spaces:

$$\begin{array}{ccccc} & & \mathcal{AT}_f & & \\ & \swarrow \tilde{q} & \downarrow \tilde{\rho} & \searrow \tilde{p}_U & \\ \mathcal{AT}(S^2, B) & & & & \mathcal{AT}(S^2, A) \\ & \searrow /L_f & \downarrow & & \downarrow /Mod(S^2, A) \\ /Mod(S^2, B) & & \mathcal{AM}_f & & \\ & \swarrow \bar{q} & & \searrow \bar{p}_U & \\ & & \mathcal{AM}(S^2, B) & & \mathcal{AM}(S^2, A) \end{array}$$

The augmented deformation space is defined to be the equalizer

$$\mathcal{AD}_{f,\iota} = \text{Eq}(\tilde{p}_U, \tilde{p}_L)$$

and contains the deformation space $\mathcal{D}_{f,\iota}$. Let $\mathcal{AV}_{f,\iota} = \tilde{\rho}(\mathcal{AD}_{f,\iota})$. Then we have

$$\mathcal{AV}_{f,\iota} = \text{Eq}(\bar{p}_U, \bar{p}_L),$$

2.5. Stabilizer of augmented deformation space. In this section, we give a necessary and sufficient condition for \mathcal{AD}_f to be connected. Unlike in the case for Corollary 2.5 $\mathcal{AD}_{f,\iota} \rightarrow \mathcal{AV}_{f,\iota}$ is not an unbranched covering. To get around this we use the notion of regular neighborhoods. Recall that, for a simplicial subcomplex V embedded in a manifold X , a *regular neighborhood* $N(V)$ of V is an open subset of X containing V that has a deformation retract to V .

PROPOSITION 2.8. *The stabilizer in L_f of $\mathcal{AD}_{f,\iota}$ equals $S_{f,\iota}$, that is, if $g \in L_f$ is such that $g(\alpha) = \alpha$ for all $\alpha \in \mathcal{AD}_{f,\iota}$, then $g \in S_{f,\iota}$.*

PROOF. The stabilizer in L_f of $\mathcal{AD}_{f,\iota}$ must be contained in $S_{f,\iota}$, since $\mathcal{D}_{f,\iota} \subset \mathcal{AD}_{f,\iota}$. Let $h \in S_{f,\iota}$ and let $(\phi, \psi, F) \in \mathcal{AD}_{f,\iota}$. Then by definition $f^\#h = i^\#h$. Thus,

we have a commutative diagram

$$\begin{array}{ccccc} (S^2, A) & \xrightarrow{f^\sharp h} & (S^2, A) & \xrightarrow{\psi} & (\mathcal{C}_U, \mathcal{A}) \\ f \downarrow & & \downarrow f & & \downarrow F \\ (S^2, B) & \xrightarrow{h} & (S^2, B) & \xrightarrow{\phi} & (\mathcal{C}_L, \mathcal{B}) \end{array}$$

and we have

$$\tilde{f}^*(h([\phi])) = [\psi \circ f^\sharp h] = [\psi \circ \iota^\sharp h] = \tilde{\iota}^*(h([\phi])).$$

Thus h stabilizes $\mathcal{AD}_{f,\iota}$. \square

REMARK 2.9. Conversely, one can ask whether if $g \in L_f$ and $g(\alpha) = \alpha$ for some $\alpha \in \mathcal{AD}_{f,\iota}$, then does it follow that $g \in S_{f,\iota}$? This is not true in general, since points in the boundary of $\mathcal{AD}_{f,\iota}$ have extra automorphisms that need not be in $S_{f,\iota}$.

PROPOSITION 2.10. *Suppose there is a connected quasi-projective variety X with $\mathcal{V}_{f,\iota} \subset X \subset \mathcal{AV}_{f,\iota}$ such that X has a regular neighborhood $N(X) \subset \mathcal{AM}_f$ with the properties*

- (1) $N(X) \cap \mathcal{M}_f$ is connected, and
- (2) the image of the homomorphism induced by inclusion

$$\pi_1(N(X) \cap \mathcal{M}_f, m_0) \rightarrow \pi_1(\mathcal{M}_f, m_0)$$

contains $S_{f,\iota}$.

Then $\mathcal{D}_{f,\iota}$ is contained in a connected component of $\mathcal{AD}_{f,\iota}$.

PROOF. Let $Y_0 \subset \mathcal{AT}_f$ be the connected component of the preimage of X in \mathcal{AT}_f containing d_0 . Let U be the connected component of the preimage of $N(X)$ in \mathcal{AT}_f containing d_0 . Then since $N(X)$ has a retract to X , the set U has a corresponding retraction to a component of $\tilde{\rho}^{-1}(X)$. This component is necessarily Y_0 , since $Y_0 \cap U \neq \emptyset$. Since the image of $\pi_1(N(X) \cap \mathcal{M}_f, m_0)$ in $L_f = \pi_1(\mathcal{M}_f, m_0)$ contains $S_{f,\iota}$, it follows that the action of $S_{f,\iota}$ on \mathcal{AT}_f preserves U and hence Y_0 . Thus, we have

$$\mathcal{D}_{f,\iota} \subset Y_0 \subset \mathcal{AD}_{f,\iota}.$$

\square

3. Blowups and topology of curve complements

In this section we study the topology of surface/curve pairs and the effect of blowups (see for example [8] or [9] for a review of elementary blowup theory for surfaces).

3.1. Regular neighborhoods of algebraic curves on surfaces. Let X be a smooth complex projective surface, and let $V \subset X$ be a pure codimensional one subvariety. We first observe that we may assume that V is a finite union of smooth curves with normal crossings. Let Q be the set of singular points on V . Then $V \setminus Q$ is a finite union of smoothly embedded punctured Riemann surfaces in $X \setminus Q$. By successively blowing up X at the points of Q (and at points of the preimages of Q), it is possible to obtain a new projective surface \tilde{X} and a surjective morphism $\sigma : \tilde{X} \rightarrow X$ such that the preimage (or *total transform*) $\tilde{V} = \sigma^{-1}(V)$ is a union of

smooth curves with normal crossings. That is (\tilde{X}, \tilde{V}) is locally isomorphic near a point of intersection on \tilde{V} to $(\mathbb{C}^2, \{x = 0\} \cup \{y = 0\})$.

Hereafter in this section we assume that V has smooth components intersecting in normal crossing. As before, let Q be the set of intersections of V , and let V_1, \dots, V_k be the irreducible components of V . Since each V_i is smooth there is an embedded tubular neighborhood $T(V_i) \subset X$ so that for $i \neq j$, $T(V_i)$ only intersects $T(V_j)$ near points in Q where V_i and V_j intersect, and if V_i and V_j intersect at q , then

$$N(q) = T(V_i) \cap T(V_j)$$

is a neighborhood of q so that $(N(q), V, q)$ is homeomorphic to

$$(\{|x| < 1\} \times \{|y| < 1\}, \{x = 0\} \cup \{y = 0\}, (0, 0)).$$

For $i = 1, \dots, k$, let

$$V_i^c = V_i \setminus \bigcup_{q \in Q} N(q)$$

and let $T(V_i^c)$ be the tubular neighborhood given by

$$T(V_i^c) = T(V_i) \setminus \bigcup_{q \in Q} N(q).$$

Let $T(V) = \bigcup_{i=1}^k T(V_i^c)$, called the *tubular neighborhood* of V .

Let $S(V_i^c)$ be the circle bundle over V_i^c contained in the boundary of $T(V_i^c)$. Then $S(V_i^c)$ is an oriented 3-manifold with torus boundary components corresponding to the intersections of V_i with other components of V . In particular, if V_i is isomorphic to \mathbb{P}^1 , then $S(V_i^c)$ is the complement of thickened Hopf links in the 3-sphere. The *boundary manifold* of V is given by

$$S(V) = \bigcup_{i=1}^k S(V_i^c).$$

This manifold and its embedding in $X \setminus V$ is uniquely determined up to homeomorphisms of X that are isotopic to the identity rel V .

LEMMA 3.1. *The punctured tubular neighborhood $T(V) \setminus V$ has a deformation retraction to $S(V)$.*

PROOF. For each i , $T(V_i^c) \setminus V_i^c$ has a deformation retract to $S(V_i^c)$ corresponding from the retraction of a punctured disk to its boundary circle. Thus, we have only to consider what happens near the intersection points $q \in Q$. In $N(q)$ it is enough to show that

$$\{|x| < 1\} \times \{|y| < 1\} \setminus \{x = 0\} \cup \{y = 0\}$$

has a deformation retract to $\{|x| = 1\} \times \{|y| = 1\}$. Such a retraction is defined by the map

$$((x, y), t) \mapsto \left(\frac{x}{1 + t(|x| - 1)}, \frac{y}{1 + t(|y| - 1)} \right).$$

□

REMARK 3.2. By its construction, $S(V)$ is naturally homeomorphic to a graph manifold (see, for example, [10](#) for definitions) over the incidence graph Γ of the components of V ; this is the bipartite graph with vertices

$$\{v_i \mid i = 1, \dots, k\}$$

corresponding to the components of V , and edges between v_i and v_j for each $q \in Q$ such that V_i and V_j intersect at q . To each vertex v_i associate the manifold $S(V_i^c)$ and to each edge of Γ between v_i and v_j associate the common torus boundary component of their associated vertex manifolds.

3.2. Regular neighborhoods and fundamental groups. In this section we prove an easy variation of the Lefschetz hyperplane theorem [\[1\]](#) and a useful corollary.

LEMMA 3.3. *Let $\mathfrak{p} : X \rightarrow \mathbb{P}^1$ be a smooth projective surface fibered over the complex projective line, and let V be a fiber. Let $C \subset X$ be a pure codimension one algebraic subset none of whose components are fibers of \mathfrak{p} . Then V has a regular neighborhood $N(V)$ so that*

$$\pi_1(N(V) \setminus C) \rightarrow (X \setminus C)$$

induced by inclusion is surjective.

PROOF. Let $P \subset \mathbb{P}^1$ be the set of points p where \mathfrak{p} restricted to C drops in cardinality, i.e., at least one intersection point of C and the fiber above p has higher multiplicity. Let \mathfrak{p}° be the restriction of \mathfrak{p} to $X^\circ = \mathfrak{p}^{-1}(\mathbb{P}^1 \setminus P) \setminus C$. Then

$$\mathfrak{p}^\circ : X^\circ \rightarrow \mathbb{P}^1 \setminus P$$

is a fiber bundle, and the Zariski-Van Kampen theorem [\[18\]](#) [\[4\]](#) implies that for a general fiber V' of \mathfrak{p}°

$$\pi_1(X \setminus C) \simeq \pi_1(V')/K$$

where K is the subgroup of $\pi_1(V')$ generated by the relations $\gamma^{-1}\beta(\gamma)$, where β ranges over automorphisms of $\pi_1(V')$ determined by the action of $\pi_1(\mathbb{P}^1 \setminus P)$ on V' . In particular, the homomorphism

$$\pi_1(V') \rightarrow \pi_1(X \setminus C)$$

is surjective. Let $N(V) = (\mathfrak{p}^\circ)^{-1}(U)$ where U is a small neighborhood of $\mathfrak{p}(V)$. Then $N(\tilde{V}) \setminus C$ contains a general fiber of \mathfrak{p}° , and the claim follows. \square

COROLLARY 3.4. *Let $C \subset \mathbb{C}^2$ be an algebraic curve, and let $V \subset \mathbb{C}^2$ be a line not contained in C . Then we can include \mathbb{C}^2 as a Zariski open subset of a smooth projective surface X , and find a sequence of blowups $\sigma : \tilde{X} \rightarrow X$ such that*

- (1) σ is an isomorphism over $\mathbb{C}^2 \setminus C$; and
- (2) the total transform \tilde{V} of the closure of V in X has a regular neighborhood $N(\tilde{V})$ in \tilde{X} such that the map on fundamental groups induced by inclusion

$$\pi_1(N(\tilde{V}) \cap \mathbb{C}^2 \setminus C) \rightarrow \pi_1(\mathbb{C}^2 \setminus C)$$

is surjective.

PROOF. Let $\mathfrak{p} : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a linear projection so that V is a fiber. Then we can define completions $\mathbb{C}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{C} \subset \mathbb{P}^1$, so that \mathfrak{p} extends to a projection

$$\bar{\mathfrak{p}} : \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^1$$

so that the closure \bar{V} of V in $\mathbb{P}^1 \times \mathbb{P}^1$ is a fiber. Let \bar{C} be the union of the closure of C in $\mathbb{P}^1 \times \mathbb{P}^1$, and the two lines in $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathbb{C}^2$.

Let $\sigma : \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ be sequence of blowups over points of $\bar{C} \cap \bar{V}$ such that the total transform $\tilde{V} = \sigma^{-1}(\bar{V})$ and the proper transform \hat{C} over \bar{C} meet in normal

crossings. Let $N(\tilde{V})$ be a regular neighborhood of \tilde{V} as in Lemma 3.3. Then the homomorphism

$$\pi_1(N(\tilde{V}) \setminus \hat{C}) \rightarrow \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \setminus \hat{C})$$

induced by inclusion is surjective.

Since σ is an isomorphism outside the preimage of $\overline{C} \cap \overline{V}$, we have a commutative diagram

$$\begin{array}{ccc} \tilde{V} & \longrightarrow & N(\tilde{V}) \\ \downarrow & & \downarrow \\ \overline{V} & \longrightarrow & N(\overline{V}) \end{array}$$

where the vertical arrows are quotient maps that contract the exceptional curves (or 1-dimensional fibers) over points on $\overline{V} \cap \overline{C}$. It follows that the retraction of $N(\tilde{V}) \setminus \hat{C}$ to $\tilde{V} \setminus \hat{C}$ descends to a retraction of $N(\overline{V}) \setminus C$ to $\overline{V} \setminus C$, and hence $N(\overline{V}) \setminus C$ is a regular neighborhood of $\overline{V} \setminus C$. Finally, $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \overline{C} = \mathbb{C}^2 \setminus C$, so setting $N(V) = N(\overline{V}) \cap \mathbb{C}^2$, it follows that $N(V)$ is a regular neighborhood of V and the homomorphism

$$\pi_1(N(V) \setminus C) \rightarrow \pi_1(\mathbb{C}^2 \setminus C)$$

defined by inclusion is surjective. \square

Corollary 3.4 is used in our discussion of connectivity of augmented deformation space in Section 4.4.

REMARK 3.5. The boundary manifold $S(\tilde{V})$ associated to $N(\tilde{V})$ in the previous proof has the structure of a boundary manifold over the incidence graph of the irreducible components of \tilde{V} (cf. Remark 3.2). Furthermore, each component of \tilde{V} is isomorphic to a line, each vertex manifold is an S^1 fiber bundle over S^2 with a finite set of thickened fibers removed.

4. Application to the Main Example

Let $F \in \text{Per}_4(0)^*$, and let $f, \iota : (S^2, A) \rightarrow (S^2, B)$ be the underlying branched covering and identification of domain and range so that A is the periodic 4 cycle, and $B = A \cup \{v\}$ where v is the extra critical point. We study the inclusion $\mathcal{D}_{f,\iota} \subset \mathcal{AD}_{f,\iota}$ by looking at their images $\mathcal{V}_{f,\iota}$ and $\mathcal{AV}_{f,\iota}$ in \mathcal{M}_f and \mathcal{AM}_f .

4.1. Parameterization of moduli space. We begin by embedding \mathcal{M}_f in $\mathbb{P}^1 \times \mathbb{P}^1$ as follows. Consider an element of \mathcal{M}_f represented as a commutative diagram

$$\begin{array}{ccc} A & \xrightarrow{i} & \mathbb{P}^1 \\ f|_A \downarrow & & \downarrow F \\ B & \xrightarrow{j} & \mathbb{P}^1. \end{array}$$

By applying automorphisms of \mathbb{P}^1 on the right side, we can assume that

$$i(B) = \{0, 1, \infty, y, z\} \quad j(A) = \{0, 1, \infty, x\},$$

where

- (i) ∞ and z are the critical values of f ;
- (ii) 0 is a critical point in A with $f(0) = \infty$; and

(iii) $f(\infty) = 1$, $f(x) = 0$.

The above data completely determines $F : \mathbb{P}^1 \times \mathbb{P}^1$ as a rational function in the variable t :

$$F(t) = \frac{(t-x)(t-r)}{t^2} \quad r = \frac{x+y-1}{x-1}.$$

It follows that in this example z is determined by x and y :

$$(4.1) \quad z = -\frac{(1-2x+x^2-y)^2}{4x(x-1)(x+y-1)}.$$

We have the following (see also, [\[11\]](#)).

LEMMA 4.1. *There is an identification*

$$\mathcal{M}_f = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathcal{L} \cup \mathcal{Z},$$

assigning (i, j, F) to (x, y) , where

$$\mathcal{L} = \{x = 0\} \cup \{y = 0\} \cup \{x = 1\} \cup \{y = 1\} \cup \{x = \infty\} \cup \{y = \infty\}$$

and \mathcal{Z} is the closure in $\mathbb{P}^1 \times \mathbb{P}^1$ of the affine union of curves

$$\{1 - 2x + x^2 - y = 0\} \cup \{x^2 + y = 1\} \cup \{x + y = 1\} \cup \{2xy + x^2 - y - 2x + 1 = 0\}$$

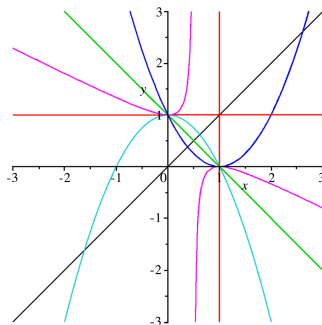


FIGURE 1. Picture of $\mathcal{L} \cup \mathcal{Z}$ in the affine plane. The lines in \mathcal{L} are shown here as the two horizontal lines $y = 0$ and $y = 1$, and two vertical lines $x = 0$ and $x = 1$.

By this parameterization, the image \mathcal{V} of $\mathcal{D}_{f,\ell}$ in \mathcal{M}_f equals the diagonal

$$\mathcal{V} = \{(x, y) \in \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathcal{L} \cup \mathcal{Z} \mid x = y\}.$$

Figure [1](#) gives a picture [2](#) of the real part of $\mathcal{L} \cup \mathcal{Z}$ in the affine open subset $\mathbb{C}^2 \subset \mathbb{P}^1 \times \mathbb{P}^1$, and Figure [2](#) gives a picture near (∞, ∞) .

²This figure was provided courtesy of Sarah Koch.

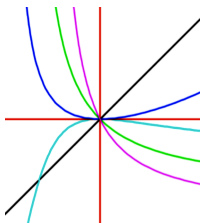


FIGURE 2. Picture of $\mathcal{L} \cup \mathcal{Z}$ in $\mathbb{P}^1 \times \mathbb{P}^1$ near (∞, ∞) . The lines $x = \infty$ and $y = \infty$ are the horizontal and vertical lines, respectively, and \mathcal{V} is the diagonal line $x = y$.

4.2. The quotient of augmented deformation space. Let \mathcal{AM}_f be the quotient of \mathcal{AT}_f by the action of L_f , and let $\mathcal{AV}_{f,\iota}$ be the image of $\mathcal{AD}_{f,\iota}$ in \mathcal{AM}_f . Our goal in this section is to concretely describe a subspace $X \subset \mathcal{AV}_{f,\iota}$ satisfying the properties in Proposition 2.10.

First we recall that the elements of $\mathcal{AV}_{f,\iota}$ are the elements of \mathcal{AM}_f that equalize the two maps

$$p_L, p_U : \mathcal{AM}_f \rightarrow \mathcal{AM}_{(S^2, A)}.$$

Each stratum of \mathcal{AM}_f is described by a partition of $\{0, 1, \infty, y, z\}$ into two or three sets by an admissible multi-curve γ , as in Figure 3. Here, the empty multi-curve corresponds to the principal stratum $\mathcal{M}_f \subset \mathcal{AM}_f$. The corresponding stable curves

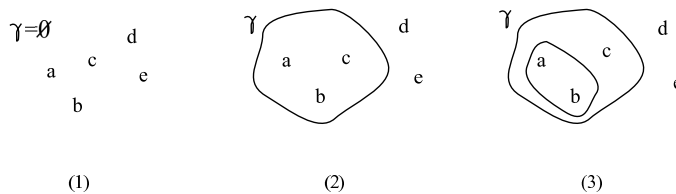


FIGURE 3. Possible partitions of five points by an admissible multi-curve.

are shown in Figure 4, with each \mathbb{P}^1 , homeomorphic to S^2 , is drawn as a line.

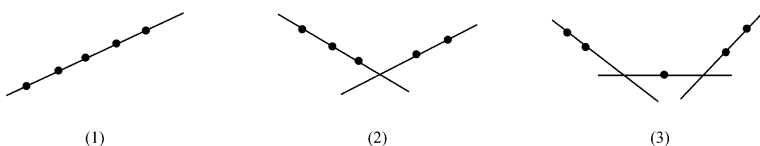


FIGURE 4. The three types of stable curves for (S^2, B) where B has five elements. The two left define one-dimensional strata, and the right defines a point stratum.

For $\mathcal{AM}_{(S^2, A)}$ there are only two isomorphism types (shown in Figure 5). The left picture depicts points belonging to the main component $\mathcal{M}_{(S^2, A)} \subset \mathcal{AM}_{(S^2, A)}$, which is isomorphic to a thrice punctured sphere, while the right picture depicts one of the three single point boundary points.

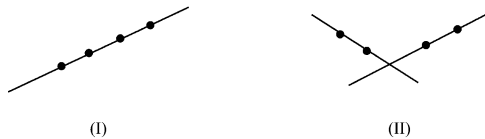


FIGURE 5. The two topological homeomorphism types of stable curves for (S^2, A) where A has four elements.

The elements $\alpha \in \mathcal{AM}_f$ that lie in positive dimensional strata must be of the form (1) or (2) in Figure 4. Those of type (1) lie in $\mathcal{M}_f = \mathbb{P}^1 \times \mathbb{P}^1 \setminus \mathcal{L} \cup \mathcal{Z}$ and map under both p_U and p_L to elements of $\mathcal{AM}_{(S^2, A)}$ of type (I). Those of type (2) divide into four subtypes: those that correspond to a partition of the form

- (2a) $\{a, b, z\} \cup \{c, \infty\}$;
- (2b) $\{a, \infty, z\} \cup \{b, c\}$;
- (2c) $\{a, b, c\} \cup \{\infty, z\}$; or
- (2d) $\{a, b, \infty\} \cup \{c, z\}$.

For types (2a) and (2b), p_L maps α to an element in $\mathcal{AM}_{(S^2, A)}$ of type (II), while for types (2c) and (2d), p_L maps α to one of type (I). For types (2a) and (2d) p_U maps α to an element of type (II), while for type (2b) and (2c) α could a priori map to an element of type (I) or (II). This is because the critical values ∞ and z lie in the same component. Thus the rational map $F : \mathcal{C}_U \rightarrow \mathcal{C}_L$ must have two isomorphic irreducible components in \mathcal{C}_U lying over the unramified component in \mathcal{C}_L upon which the distinguished points will be distributed.

From this we can reduce the types that can be in \mathcal{AV} to (2a), (2d), as well as (2b) and (2c) under the condition that the preimage under F of the distinguished points in the unramified component lie on the same component of \mathcal{C}_U . In the allowable case of types (2a) and (2b), we see that for $p_L(\alpha)$ and $p_U(\alpha)$ to be equal the images of the two maps in $\mathcal{AM}_{(S^2, A)}$ must give the partition

$$\{0, 1\} \cup \{x, \infty\}.$$

Thus, the partition given in (2a) can only be

$$\{0, 1, z\} \cup \{y, \infty\}$$

and for (2b) it can only be

$$\{\infty, y, z\} \cup \{0, 1\}.$$

For type (2c), $p_L(\alpha)$ and $p_U(\alpha)$ must be equal, and F defines the isomorphism on stable curves. Taking into account the combinatorics of f , this implies equality of the cross ratios: $(0, \infty; 1, x)$ and $(\infty, 1; x, 0)$, which is false under the assumption that $x \notin \{0, 1, \infty\}$.

Let $\mathcal{A}_1, \mathcal{A}_2 \subset \mathcal{AM}_f$ be the subsets corresponding to the partitions

$$\mathcal{A}_1 : \{\infty, y, z\} \cup \{0, 1\},$$

and

$$\mathcal{A}_2 : \{0, 1, z\} \cup \{y, \infty\}.$$

Then each of these is isomorphic to $\text{Mod}_{0,4} \times \text{Mod}_{(0,3)}$. Furthermore, the closures of \mathcal{A}_1 and \mathcal{A}_2 in \mathcal{AM}_f intersect at the point corresponding to the partition

$$\{0, 1\} \cup \{z\} \cup \{y, \infty\}.$$

We have shown the following.

PROPOSITION 4.2. *The pure 1-dimensional algebraic set $\mathcal{V} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathcal{AM}_f$ is contained in $\mathcal{AV}_{f,\iota}$, and its complement is a finite set of points (possibly empty).*

REMARK 4.3. We leave the question of whether $\mathcal{AV}_{f,\iota}$ is connected (in this case, and in general) to future study.

4.3. Blowups. In this section, we find a connected pure 1-dimensional algebraic subset $X \subset \mathcal{AV}$ that contains \mathcal{V} , and whose complement in \mathcal{AV} is finite.

Let

$$\sigma : \widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

be sequence of blowups defined as follows. First blowup the points $(0, 0)$, $(1, 1)$ and (∞, ∞) to get the exceptional curves E_0, E_1, E_∞ . Next blowup the point of intersection $q \in E_\infty \cap \widehat{L}_y$, where \widehat{L}_y is the proper transform of $\{y = \infty\}$. Let E_q be the exceptional divisor. The union of curves is drawn in Figure 6 (compare Figure 1).

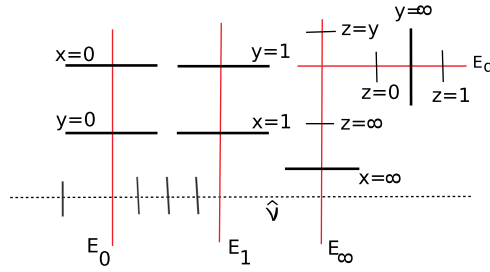


FIGURE 6. The proper transform of \mathcal{V} is drawn as a dotted line.

LEMMA 4.4. *The map σ has the following properties:*

- (1) σ restricts to an isomorphism on $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1} \setminus \sigma^{-1}(Q)$; and
- (2) the total transform $\widetilde{\mathcal{V}}$ and the proper transform $\widehat{\mathcal{L} \cup \mathcal{Z}}$ meet in normal crossing singularities.
- (3) inclusion induces a surjection on fundamental groups

$$\pi_1(\mathcal{N}(\widetilde{\mathcal{V}}) \cap \mathcal{M}_f) \rightarrow \pi_1(\mathcal{M}_f).$$

PROOF. Properties (1) and (2) follow from the definitions, and property (3) follows from Lemma 3.3. \square

Let

$$X = (\widehat{\mathcal{V}} \cup E_\infty \cup E_q) \setminus \widehat{\mathcal{L} \cup \mathcal{Z}}.$$

Then X is connected since the punctures of $\widetilde{\mathcal{V}}$ at intersections with $\widehat{\mathcal{L} \cup \mathcal{Z}}$ occur only in smooth points of $\widetilde{\mathcal{V}}$. Since σ is an isomorphism over \mathcal{V} , there is a natural inclusion $\nu : \mathcal{V} \hookrightarrow X$.

LEMMA 4.5. *The inclusion $\mathcal{V} \hookrightarrow \mathcal{AV}$ induced by $\mathcal{D}_{f,\iota} \hookrightarrow \mathcal{AD}_{f,\iota}$ factors as $\xi \circ \nu$ for some embedding*

$$\xi : X \hookrightarrow \mathcal{AV}.$$

PROOF. Recall from Proposition 4.2 the subset $\mathcal{V} \cup \mathcal{A}_1 \cup \mathcal{A}_2 \subset \mathcal{AV}$. In what follows we define embeddings

$$\begin{aligned}\kappa_\infty : E_\infty \setminus \widehat{\mathcal{L} \cup \mathcal{Z}} &\rightarrow \mathcal{A}_1 \\ \kappa_q : E_q \setminus \widehat{\mathcal{L} \cup \mathcal{Z}} &\rightarrow \mathcal{A}_2\end{aligned}$$

that together with the projection $\widehat{V} \rightarrow \mathcal{V}$ extend to define ξ .

When $\alpha \in \mathcal{M}_f$ approaches a general point of $\mathcal{L} \cup \mathcal{Z}$, it corresponds to two points in $\{0, 1, \infty, y, z\}$ coming together. For the lines ($y = 0$, $y = 1$ and $y = \infty$, the pairs $\{y, 0\}$, $\{y, 1\}$ and $\{y, \infty\}$ approach each other, while simultaneously $\{x, 1\}$, $\{1, \infty\}$ and $\{1, 0\}$ approach each other. Near the lines $x = 0$, $x = 1$, $x = \infty$ the pairs $\{x, 0\}$ (and $\{0, \infty\}$), $\{x, 1\}$ (and $\{0, y\}$) and $\{x, \infty\}$ (and $\{0, 1\}$) approach each other. For z approaching $0, 1, \infty$ or y , we have

$$\begin{aligned}\{z, 0\} &\text{ near } 1 - 2x + x^2 - y = 0; x \neq 0, 1; x + y \neq 1 \\ \{z, 1\} &\text{ near } x^2 + y = 1; x \neq 0; x + y \neq 1 \\ \{z, \infty\} &\text{ near } x = 0, x = 1, \text{ or } x + y = 1 \\ \{z, y\} &\text{ near } 2xy + x^2 - y - 2x + 1 = 0\end{aligned}$$

We choose local coordinates for a neighborhood of E_∞ as follows. Let $\bar{x} = 1/x$ and $\bar{y} = 1/y$ be coordinates for an open neighborhood of $p_\infty = (\infty, \infty)$ in $\mathbb{P}^1 \times \mathbb{P}^1$, so that in the coordinates (\bar{x}, \bar{y}) we have $p_\infty = (0, 0)$. Let u (and $\bar{u} = \frac{1}{u}$) be coordinates for E_∞ . Then a neighborhood of E_∞ in $\mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to the algebraic subset of $\mathbb{C}^2 \times \mathbb{P}^1$ defined by

$$\{(\bar{x}, \bar{y}) \times u \in \mathbb{C}^2 \times \mathbb{P}^1 \mid \bar{x} = u\bar{y}\},$$

and

$$E_\infty = \{(\bar{x}, \bar{y}) \times u \in \mathbb{C}^2 \times \mathbb{P}^1 \mid \bar{x} = \bar{y} = 0\}.$$

With respect to the \bar{x}, \bar{y} coordinates, we have

$$\begin{aligned}z &= -\frac{\bar{x}^4 \bar{y}^2 (1 - 2x + x^2 - y)^2}{4\bar{x}^4 \bar{y}^2 x(x-1)(x+y-1)} \\ &= -\frac{(\bar{x}^2 \bar{y} - 2\bar{x}\bar{y} + \bar{y} - \bar{x}^2)^2}{4\bar{x}\bar{y}(1-\bar{x})(\bar{y} + \bar{x} - \bar{x}\bar{y})}\end{aligned}$$

Using the identity $\bar{x} = u\bar{y}$, we have

$$z = -\frac{(u^2 \bar{y}^2 - 2u\bar{y} + 1 - u^2 \bar{y})^2}{4u\bar{y}(1-u\bar{y})(1+u(1-\bar{y}))}.$$

To each (u, \bar{y}) , associate the point in $\mathcal{AM}_{(S^2, B)}$ defined by

$$([u, \infty; y, z], *)$$

where the first component is the cross ratio of the points u, ∞, y, z and the second component is the unique triple of points in \mathbb{P}^1 up to automorphism. This defines a point in $\mathcal{AM}_{(S^2, B)}$.

Since cross ratio is preserved under automorphisms of \mathbb{P}^1 we have

$$[u, \infty; y, z] = [u\bar{y}, \infty, 1, z\bar{y}].$$

As \bar{y} approaches 0, the cross ratio approaches

$$\left[0, \infty; 1, -\frac{1}{4u(1+u)}\right]$$

and is degenerate only when $u = 0$ ($x = \infty$), $u = -1$ ($z = \infty$), $u = -\frac{1}{2}$ ($z = y$), or $u = \infty$ ($y = \infty$).

We thus have a well-defined embedding:

$$\begin{aligned} \kappa_\infty : E_\infty \setminus \{\widehat{\mathcal{L} \cup \mathcal{Z}}\} &\rightarrow \mathcal{A}_1 \\ u &\mapsto ([u, \infty; y, z], *). \end{aligned}$$

Here u corresponds to a line through (∞, ∞) in $\mathbb{P}^1 \times \mathbb{P}^1$, and we can think of the contracting curve γ as a small loop on this line around (∞, ∞) .

The intersection of E_∞ with E_q occurs at the point on E_∞ corresponding to $(\bar{y}, \bar{u}) = (0, 0)$, and E_q has a neighborhood parameterized by $((\bar{y}, \bar{u}), v) \in \mathbb{C}^2 \times \mathbb{P}^1$ where $\bar{y} = v\bar{u}$. Then $\bar{x} = u\bar{y} = v$ and

$$\begin{aligned} z &= -\frac{(1-2x+x^2-y)^2}{4x(x-1)(x+y-1)} \\ &= -\frac{(\bar{u}(1-2\bar{v}+\bar{v}^2)-\bar{v})^2}{4\bar{v}(\bar{v}-1)(\bar{v}+\bar{u}\bar{v}-1)} \end{aligned}$$

and as \bar{u} goes to 0,

$$z = -\frac{\bar{v}}{4(\bar{v}-1)^2} = -\frac{v}{4(1-v)^2}.$$

Then we have

$$[v, 0; 1, z] = [1, 0; \bar{v}, \bar{v}z] = [1, 0; \bar{v}, -\frac{1}{4(1-v)^2}].$$

The cross ratio is degenerate when $v = \infty$ (where $\{y = \infty\}$ meets E_q), $v = 1$ (where E_∞ and E_q intersect), and where $z = 0$ and $z = 1$ (corresponding to two distinct values of v not equal to 1 or ∞). The points correspond to $v \in E_q \setminus \widehat{\mathcal{L} \cup \mathcal{Z}}$ except at the point where $v = 1$.

For this point, consider the 2-component multicurve γ on $S^2 \setminus B$ determined by two loops on the planes defined by E_∞ and $\widehat{y = \infty}$ around $(\bar{y}, \bar{u}) = (0, 0)$. The corresponding point $\alpha_0 \in \mathcal{AM}_f$ corresponds to the partition

$$\{y, \infty\} \cup \{z\} \cup \{0, 1\}.$$

Define

$$\begin{aligned} \kappa_q : E_q \setminus \{\widehat{\mathcal{L} \cup \mathcal{Z}}\} &\rightarrow \mathcal{A}_2 \\ v &\mapsto \begin{cases} ([0, 1; \infty, \bar{v}], *) & \text{if } v \neq 1 \\ \alpha_0 & \text{if } v = 1 \end{cases} \end{aligned}$$

Then κ_∞ and κ_q extend to a morphism $\xi : X \rightarrow \mathcal{AV}$. \square

PROOF OF THEOREM 1.3 Fix $d_0 \in \mathcal{D}_{f,\iota}$, and let \mathcal{D}_0 be the connected component of $\mathcal{D}_{f,\iota}$ that contains d_0 . Let $\overline{\mathcal{D}_0}$ be the closure of \mathcal{D}_0 in $\mathcal{AD}_{f,\iota}$. Then $\overline{\mathcal{D}_0}$ maps to $\widehat{\mathcal{V}}$ under the projection from $\mathcal{AT}_f \rightarrow \mathcal{AM}_f$. Let $\widehat{\mathcal{D}_0} \subset \mathcal{AT}_f$ be the connected component of the preimage of $\widehat{\mathcal{V}}$ that intersects \mathcal{D}_0 . Then, since $\widehat{\mathcal{V}}$ is closed in X and X is connected, we can find a connected component Y_0 of the preimage of X in \mathcal{AT}_f that contains $\widehat{\mathcal{D}_0}$, and $\widehat{\mathcal{D}_0}$ is the closure in Y_0 of \mathcal{D}_0 . That is, $\widehat{\mathcal{D}_0} = \overline{\mathcal{D}_0}$.

We claim that there is an element

$$g \in S_{f,\iota} \cap \text{Image}(\pi_1(N(X) \cap \mathcal{M}_f) \rightarrow \pi_1(\mathcal{M}_f)) \setminus \text{Image}(\pi_1(\widehat{V}) \rightarrow \pi_1(\mathcal{M}_f)).$$

Let v_0 be the image of d_0 in \mathcal{V} , and let γ be a close path starting at v_0 and passing along X to a point near an intersection of \widehat{Z} with $E_\infty \cup E_q$, forming a small loop around that intersection point, and returning along the original path back to v_0 . Then γ is not homotopic in \mathcal{AM}_f to a closed path on \widehat{V} since \widehat{Z} and \widehat{V} do not intersect. Let g be the image of γ in $\pi_1(\mathcal{M}_f)$ after pushing off the boundary of \mathcal{AM}_f into \mathcal{M}_f .

Since $g \notin \text{Image}(\pi_1(\widehat{V}) \rightarrow \pi_1(\mathcal{M}_f))$, g does not preserve \widehat{D}_0 , and hence $g(\widehat{D}_0)$ and \widehat{D}_0 are disjoint. These are the closures of \mathcal{D}_0 and $g(\mathcal{D}_0)$ in $\mathcal{AD}_{f,\iota}$ and the claim follows. \square

4.4. Connectivity of augmented deformation space. We finish this paper by giving a sufficient condition for $\mathcal{AD}_{f,\iota}$ to be connected in the $\text{Per}_4(0)^*$ case.

By Corollary 3.4 we know that \widetilde{V} has a regular neighborhood U in $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ so that

$$\pi_1(U \cap \mathcal{M}_f) \rightarrow \pi_1(\mathcal{M}_f)$$

is surjective.

As a closure of \mathcal{M}_f , $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ is birationally equivalent to \mathcal{AM}_f (but not isomorphic). By the birational theory of complex projective surfaces, there is a minimal smooth surface Z with birational morphisms to \mathcal{AM}_f and $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$

$$\begin{array}{ccc} & Z & \\ & \swarrow & \searrow \\ \mathcal{AM}_f & & \widetilde{\mathbb{P}^1 \times \mathbb{P}^1}. \end{array}$$

Lifting U to Z and projecting to \mathcal{AM}_f gives a regular neighborhood U' of $\overline{V} \cup \mathcal{K}$, where \mathcal{K} is a union of boundary curves in \mathcal{AM}_f . Since \mathcal{M}_f is a smooth subset of both \mathcal{AM}_f and $\widetilde{\mathbb{P}^1 \times \mathbb{P}^1}$ the two projections of Z are isomorphisms over \mathcal{M}_f . Thus

$$U' \cap \mathcal{M}_f = U \cap \mathcal{M}_f$$

and hence

$$\pi_1(U') \rightarrow \pi_1(\mathcal{M}_f)$$

is surjective. It follows that U' has a connected preimage in \mathcal{AT}_f and that the preimage of $\overline{V} \cup \mathcal{K}$ in \mathcal{AT}_f is connected.

We have thus shown the following sufficient condition for $\mathcal{AD}_{f,\iota}$ to be connected.

PROPOSITION 4.6. *If \mathcal{AV} is connected and is Zariski dense in $\overline{V} \cup \mathcal{K}$, then $\mathcal{AD}_{f,\iota}$ is connected.*

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