

A DISCONNECTED DEFORMATION SPACE OF RATIONAL MAPS

ERIKO HIRONAKA AND SARAH KOCH

for Tan Lei

ABSTRACT. The deformation space of a branched cover $f : (S^2, A) \rightarrow (S^2, B)$ is a complex submanifold of a certain Teichmüller space, which consists of classes of marked rational maps $F : (\mathbb{P}^1, A') \rightarrow (\mathbb{P}^1, B')$ that are combinatorially equivalent to f . In the case $A = B$, under a mild assumption on f , William Thurston gave a topological criterion for which the deformation space of $f : (S^2, A) \rightarrow (S^2, B)$ is nonempty, and he proved that it is always connected. We show that if $A \subsetneq B$, then the deformation space need not be connected. We exhibit a family of quadratic rational maps for which the associated deformation spaces are disconnected; in fact, each has infinitely many components.

1. INTRODUCTION

Let $f : (S^2, A) \rightarrow (S^2, B)$ denote a map of pairs where $f : S^2 \rightarrow S^2$ is an orientation-preserving branched covering, A and B are finite sets containing at least 3 points, $f(A) \subseteq B$, and B contains the critical values of f . Following [K], we refer to such a map f as an *admissible branched cover*. In this paper, we study spaces of rational maps equivalent to f from two different perspectives. The first perspective is ‘nondynamical’: we let \mathcal{U}_f be the space of *marked rational maps* that are Hurwitz equivalent to f , where A and B are not related. The second perspective is ‘dynamical’, where we identify domain and range and assume $A \subseteq B$. This determines a subspace $\mathcal{D}_f \subseteq \mathcal{U}_f$ of rational maps combinatorially equivalent to f called the *deformation space* of f . In the purely dynamical setting where $A = B$, W. Thurston gave a criterion for \mathcal{D}_f to be nonempty and established the following result.

Theorem 1.1 (W. Thurston [DH], [BCT]). *Suppose that $A = B$. If \mathcal{D}_f is nonempty and does not contain an element of Lattès type¹, then \mathcal{D}_f is a single point.*

More generally, when $A \subseteq B$, A. Epstein proved the following theorem.

Theorem 1.2 (A. Epstein [E]). *If \mathcal{D}_f is nonempty and does not contain an element of Lattès type, then \mathcal{D}_f is a smooth complex submanifold of \mathcal{U}_f of dimension $|B - A|$.*

The proof of Theorem 1.2 is purely local and reveals nothing about the global topology of \mathcal{D}_f . We are interested in the following question:

Question 1.3. *Is \mathcal{D}_f always connected?*

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¹We refer the reader to [M1] for the definition of a Lattès map.

The main result in this paper is that \mathcal{D}_f need not be connected in general. Assume \mathcal{D}_f is nonempty, and let $u_0 \in \mathcal{D}_f$. There is a covering map $\omega : (\mathcal{U}_f, u_0) \rightarrow (\mathcal{W}_f, v_0)$, defined by forgetting the markings, and it restricts to a covering map $\nu : (\mathcal{D}_f, u_0) \rightarrow (\mathcal{V}_f, v_0)$:

$$\begin{array}{ccc} (\mathcal{D}_f, u_0) & \hookrightarrow & (\mathcal{U}_f, u_0) \\ \nu \downarrow & & \downarrow \omega \\ (\mathcal{V}_f, v_0) & \hookrightarrow & (\mathcal{W}_f, v_0) \end{array}$$

We study the connected components of \mathcal{D}_f using the covering map ν . To this end, we consider a natural subgroup of the pure mapping class group of (S^2, B) , denoted S_f , which preserves \mathcal{D}_f , and we prove that the covering map $\nu : \mathcal{D}_f \rightarrow \mathcal{V}_f$ is the quotient map associated to the action of S_f on \mathcal{D}_f . Let $E_{u_0} \subseteq S_f$ be the setwise stabilizer of the connected component of \mathcal{D}_f containing u_0 . It follows that there is a bijection between the connected components of \mathcal{D}_f and the set of cosets S_f/E_{u_0} .

We prove that \mathcal{D}_f is disconnected when f belongs to a particular subspace of the space of quadratic rational maps. Let M_2^{cm} be the moduli space of critically-marked quadratic rational maps. Let $\text{Per}_4(0) \subseteq M_2^{\text{cm}}$ be the subspace of maps with a marked superattracting 4-cycle (see [M2]), and define $\text{Per}_4(0)^* \subseteq \text{Per}_4(0)$ to be the subspace for which the superattracting 4-cycle contains only one critical point. Let f represent an element of $\text{Per}_4(0)^*$, let A be the set of points in the marked superattracting 4-cycle, and let $B := A \cup \{b\}$ where $b \notin A$ is the other critical value of f . Then $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ is an admissible branched cover and \mathcal{D}_f has a canonical basepoint.

Theorem 1.4. *If $\langle f \rangle \in \text{Per}_4(0)^*$, then \mathcal{D}_f has infinitely many connected components.*

Remark 1.5. Firsova, Kahn, and Selinger have given a different and independent proof that \mathcal{D}_f is disconnected for $\langle f \rangle \in \text{Per}_4(0)^*$. Rees also has a substantial body of work related to the topology of deformation spaces [R].

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2. SPACES OF RATIONAL MAPS AND THEIR MODULAR GROUPS

Let $f : (S^2, A) \rightarrow (S^2, B)$ be an admissible branched cover. We first define the spaces of rational maps \mathcal{U}_f and \mathcal{W}_f associated to f , and we define the regular covering map $\omega : \mathcal{W}_f \rightarrow \mathcal{U}_f$. We then suppose $A \subseteq B$, define the deformation space \mathcal{D}_f , and study the regular covering map $\nu : \mathcal{D}_f \rightarrow \mathcal{V}_f$ induced by ω . The language of covering maps allows us to translate the problem of comparing subgroups of covering automorphisms to subgroups of the fundamental groups of spaces. We then describe how equalizers play an important role in this discussion.

2.1. Rational maps marked by a branched covering. Fix an admissible branched cover $f : (S^2, A) \rightarrow (S^2, B)$. A rational map $F : (\mathbb{P}^1, A') \rightarrow (\mathbb{P}^1, B')$ is *Hurwitz equivalent* to f if

there is a commutative diagram

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, A') \\ f \downarrow & & \downarrow F \\ (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, B') \end{array}$$

where ϕ and ψ are orientation-preserving homeomorphisms of pairs. The commutative diagram is called an *f*-*marking* of F and is denoted (ψ, ϕ, F) . Two *f*-markings (ψ_1, ϕ_1, F_1) and (ψ_2, ϕ_2, F_2) are *equivalent* if there is a commutative diagram

$$\begin{array}{ccccc} (S^2, A) & & \xrightarrow{\psi_2} & & (\mathbb{P}^1, A'_2) \\ & \searrow e' & & \swarrow \alpha & \\ & (S^2, A) & \xrightarrow{\psi_1} & (\mathbb{P}^1, A'_1) & \\ & f \downarrow & & \downarrow F_1 & \\ & (S^2, B) & \xrightarrow{\phi_1} & (\mathbb{P}^1, B'_1) & \\ & \nearrow e & & \nwarrow \beta & \\ (S^2, B) & & \xrightarrow{\phi_2} & & (\mathbb{P}^1, B'_2) \end{array}$$

where e' is isotopic to the identity rel A , e is isotopic to the identity rel B , and α and β are Möbius transformations. The set of equivalence classes $[\psi, \phi, F]$ of *f*-markings (ψ, ϕ, F) forms a space we denote by \mathcal{U}_f .

The space \mathcal{U}_f may be thought of as the *Teichmüller space* of *f*-marked rational maps. In fact, it can be canonically identified with the Teichmüller space \mathcal{T}_B of (S^2, B) which we now define.

2.2. Teichmüller and moduli spaces. Recall that the *Teichmüller space* of (S^2, A) , denoted \mathcal{T}_A , is the space of orientation-preserving homeomorphisms or *markings*

$$\phi : (S^2, A) \rightarrow (\mathbb{P}^1, A')$$

up to pre-composition by isotopy equivalence rel A and post-composition by Möbius transformations.

The branched covering $f : (S^2, A) \rightarrow (S^2, B)$ defines a pullback map $\sigma_f : \mathcal{T}_B \rightarrow \mathcal{T}_A$ (see [DH]).

Proposition 2.1. *The space \mathcal{U}_f is naturally isomorphic to the graph of σ_f .*

The Teichmüller space \mathcal{T}_A has a natural quotient space \mathcal{M}_A , the *moduli space* of (S^2, A) ; that is, the set of all injective maps $i : A \rightarrow \mathbb{P}^1$ up to post-composition by Möbius transformations. The map $\mathcal{T}_A \rightarrow \mathcal{M}_A$ which sends the equivalence class of ϕ to the equivalence class of $\phi|_A$ is a universal covering map. The *pure mapping class group* of (S^2, A) is the group of isotopy classes rel A of orientation-preserving homeomorphisms $(S^2, A) \rightarrow (S^2, A)$ that fix A pointwise. This group acts freely and properly discontinuously on \mathcal{T}_A by pre-composition and is isomorphic to the *modular group* Mod_A , the group of covering automorphisms of $\mathcal{T}_A \rightarrow \mathcal{M}_A$.

2.3. Moduli space of rational maps Hurwitz equivalent to f . We define a moduli space \mathcal{W}_f , a natural quotient map $\omega : \mathcal{U}_f \rightarrow \mathcal{W}_f$, and an associated modular group, which we denote by L_f .

Consider the space of triples (i, j, F) where $i : A \rightarrow \mathbb{P}^1$ and $j : B \rightarrow \mathbb{P}^1$ are injective maps, and $F : (\mathbb{P}^1, i(A)) \rightarrow (\mathbb{P}^1, j(B))$ is a rational map of pairs. Two such triples (i_1, j_1, F_1) and (i_2, j_2, F_2) are *equivalent* if there is a commutative diagram

$$\begin{array}{ccccc} & & i_2 & & \\ & \nearrow & & \searrow & \\ A & \xrightarrow{i_1} & \mathbb{P}^1 & \xrightarrow{\alpha} & \mathbb{P}^1 \\ \downarrow f|_A & & \downarrow F_1 & & \downarrow F_2 \\ B & \xrightarrow{j_1} & \mathbb{P}^1 & \xrightarrow{\beta} & \mathbb{P}^1 \\ & \nwarrow & & \nearrow & \\ & & j_2 & & \end{array}$$

where α and β are Möbius transformations. Let $[i, j, F]$ denote the equivalence class of (i, j, F) . Consider the map which sends $[\psi, \phi, F] \in \mathcal{U}_f$ to the equivalence class $[\psi|_A, \phi|_B, F]$. Let \mathcal{W}_f be the image of \mathcal{U}_f , and define ω to be

$$\begin{aligned} \omega : \mathcal{U}_f &\rightarrow \mathcal{W}_f \\ [\psi, \phi, F] &\mapsto [\psi|_A, \phi|_B, F]. \end{aligned}$$

The space \mathcal{W}_f can be thought of as the *moduli space* of f -marked rational maps. The map $\omega : \mathcal{U}_f \rightarrow \mathcal{W}_f$ is a regular covering map whose group of covering automorphisms is the group L_f , which we now define.

2.4. The liftables. Let $h : (S^2, B) \rightarrow (S^2, B)$ be an orientation-preserving homeomorphism that fixes B pointwise. We say that h is *liftable* if there is an orientation-preserving homeomorphism $h' : (S^2, A) \rightarrow (S^2, A)$ fixing A pointwise so that the following diagram commutes.

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{h'} & (S^2, A) \\ \downarrow f & & \downarrow f \\ (S^2, B) & \xrightarrow{h} & (S^2, B). \end{array}$$

Any lift h' is unique up to covering automorphisms. The condition that h' fixes A makes h' unique (cf. [K]). By the homotopy-lifting property, the group of liftable homeomorphisms descends to the corresponding group of *liftables* associated to f

$$L_f := \{[h] \in \text{Mod}_B \mid h \text{ is liftable}\}.$$

This defines the *lifting homomorphism* (see [KPS]).

$$\begin{aligned} \Phi_f : L_f &\rightarrow \text{Mod}_A \\ [h] &\mapsto [h']. \end{aligned}$$

There is a natural action of L_f on \mathcal{U}_f given by

$$[h] \cdot [\psi, \phi, F] \mapsto [\psi \circ h', \phi \circ h, F].$$

See [K] for a proof of the following proposition.

Proposition 2.2. *The map $\omega : \mathcal{U}_f \rightarrow \mathcal{W}_f$ is a regular covering map with group of covering automorphisms isomorphic to L_f .*

2.5. Equalizers. Much of the discussion in this paper use the language of equalizers. Let $f, g : X \rightarrow Y$ be two maps between sets X and Y . The *equalizer* of f and g is

$$\mathcal{E}q(f, g) := \{x \in X \mid f(x) = g(x)\}.$$

Note that the set $\mathcal{E}q(f, g)$ may be empty.

2.6. The deformation space. We now identify domain and range of $f : (S^2, A) \rightarrow (S^2, B)$ and assume that $A \subseteq B$. There are two natural maps

$$\tau_1, \tau_2 : \mathcal{U}_f \rightarrow \mathcal{T}_A$$

given by

$$\tau_1 : [\psi, \phi, F] \mapsto [\psi] \quad \text{and} \quad \tau_2 : [\psi, \phi, F] \mapsto [\phi]_A$$

where $[\phi]_A$ denotes the equivalence class of ϕ in \mathcal{T}_A .

An f -marking (ψ, ϕ, F) is a *combinatorial equivalence (relative to A)* if $[\phi]_A = [\psi]$ and $\psi|_A = \phi|_A$. Then $[\psi, \phi, F]$ satisfies $\tau_1([\psi, \phi, F]) = \tau_2([\psi, \phi, F])$ if and only if $[\psi, \phi, F]$ contains a combinatorial equivalence. We define the *deformation space* \mathcal{D}_f to be

$$\mathcal{D}_f := \mathcal{E}q(\tau_1, \tau_2).$$

We can think of \mathcal{D}_f as the *dynamical Teichmüller space* defined by f .

2.7. Moduli space of rational maps combinatorially equivalent to f . We denote the image of \mathcal{D}_f in \mathcal{W}_f as $\mathcal{V}_f := \omega(\mathcal{D}_f)$. The space \mathcal{V}_f can be thought of as the *dynamical moduli space* defined by f , or as the *moduli space* of rational maps which are combinatorially equivalent to f . We define the associated modular group in the next section.

There are two natural maps

$$\rho_1, \rho_2 : \mathcal{W}_f \rightarrow \mathcal{M}_A$$

given by

$$\rho_1 : [\psi|_A, \phi|_B, F] \mapsto [\psi|_A] \quad \text{and} \quad \rho_2 : [\psi|_A, \phi|_B, F] \mapsto [\phi|_A].$$

Proposition 2.3. *The space \mathcal{V}_f is a union of connected components of $\mathcal{E}q(\rho_1, \rho_2)$.*

Proof. Suppose that C is a connected component of $\mathcal{E}q(\rho_1, \rho_2)$ so that $C \cap \mathcal{V}_f \neq \emptyset$. Fix $v \in \mathcal{V}_f \cap C$, and fix $c \in C$. Let $\gamma : [0, 1] \rightarrow C$ be a path with $\gamma(0) = v$ and $\gamma(1) = c$. Let $\tilde{\gamma} : [0, 1] \rightarrow \mathcal{U}_f$ be the lift of γ with initial point at some $u \in \mathcal{D}_f$. We show that $\tilde{\gamma}(t) \in \mathcal{D}_f$ for all $t \in [0, 1]$, establishing the proposition.

For each t , represent $\tilde{\gamma}(t)$ with the triple (ϕ_t, ψ_t, F_t) . Because $\gamma(t) \subseteq \mathcal{E}q(\rho_1, \rho_2)$, it follows that

$$\phi_t^{-1} \circ \psi_t|_A = \text{id}|_A$$

for all t . Since $u = \tilde{\gamma}(0) \in \mathcal{D}_f$, the map $\phi_0^{-1} \circ \psi_0$ is isotopic to the identity rel A . The map

$$\begin{aligned} H : [0, 1] \times S^2 &\rightarrow S^2 \\ (t, x) &\mapsto \phi_t^{-1} \circ \psi_t(x) \end{aligned}$$

is a homotopy from $\phi_0^{-1} \circ \psi_0$ to $\phi_t^{-1} \circ \psi_t$, fixing A pointwise throughout. Therefore, ϕ_t is isotopic to ψ_t rel A , so $\tilde{\gamma}(t) \in \mathcal{D}_f$. \square

2.8. The special liftables. The condition $A \subseteq B$ determines a subgroup of L_f which preserves \mathcal{D}_f . Given $g \in \text{Mod}_B$, we denote by g_A the mapping class in Mod_A defined by g . We define the *special liftables* to be

$$S_f := \{\ell \in L_f \mid \Phi_f(\ell) = \ell_A\}.$$

Suppose $\mathcal{D}_f \neq \emptyset$, and consider the map $\nu : \mathcal{D}_f \rightarrow \mathcal{V}_f$ given by restricting $\omega : \mathcal{U}_f \rightarrow \mathcal{W}_f$. As established in Proposition 2.6 below, ν is a regular covering map corresponding to the quotient of the action of S_f on \mathcal{D}_f . We can therefore think of S_f as the *dynamical modular group* of f .

Lemma 2.4. *The group S_f preserves \mathcal{D}_f .*

Proof. An f -marking (ψ, ϕ, F) represents an element in \mathcal{D}_f if and only if $[\psi] = [\phi]_A$ in \mathcal{T}_A . Because $[h] \in S_f$, $[\phi \circ h]_A = [h]_A \cdot [\phi]_A = [h'] \cdot [\psi] = [\psi \circ h']$, so $[h] \cdot [\psi, \phi, F] \in \mathcal{D}_f$. \square

Lemma 2.5. *For $[h] \in L_f$, if $[h] \cdot \mathcal{D}_f \cap \mathcal{D}_f \neq \emptyset$, then $[h] \in S_f$.*

Proof. Take $[h] \in L_f$, and let $[\psi, \phi, F] \in \mathcal{D}_f$. Assume that $[h] \cdot [\psi, \phi, F] \in \mathcal{D}_f$. Then we have

$$[\psi \circ h'] = [\phi \circ h]_A \quad \text{equivalently} \quad [h'] \cdot [\psi] = [h]_A \cdot [\phi]_A$$

in \mathcal{T}_A . Because $[\psi] = [\phi]_A$ and because the action of Mod_A on \mathcal{T}_A is free, we have $[h'] = [h]_A$, so $[h] \in S_f$. \square

Proposition 2.6. *The map $\nu : \mathcal{D}_f \rightarrow \mathcal{V}_f$ is a regular covering map and S_f is a subgroup of the group of covering automorphisms that acts transitively on fibers.*

Proof. Lemmas 2.4 and 2.5 imply that \mathcal{V}_f is the quotient of \mathcal{D}_f by the action of S_f . \square

2.9. Connected components of \mathcal{D}_f . Assume $\mathcal{D}_f \neq \emptyset$, and let $u_0 \in \mathcal{D}_f$ be a basepoint. Let $v_0 := \nu(u_0)$, let \mathcal{V}_0 be the connected component of \mathcal{V}_f containing v_0 , let $\mathcal{D}_0 := \nu^{-1}(\mathcal{V}_0)$ and let $\nu_0 := \nu|_{\mathcal{D}_0}$. Note that \mathcal{D}_0 may be disconnected. By Proposition 2.6

$$\nu_0 : (\mathcal{D}_0, u_0) \rightarrow (\mathcal{V}_0, v_0)$$

is a regular covering map. Since S_f acts transitively on fibers of ν_0 , path-lifting at u_0 defines a homomorphism

$$\mathfrak{h}_0 : \pi_1(\mathcal{V}_0, v_0) \rightarrow S_f.$$

Let \mathcal{D}_{u_0} be the connected component of \mathcal{D}_0 containing u_0 . Then the image E_{u_0} of \mathfrak{h}_0 is the group of covering automorphisms of the covering

$$\nu_0|_{\mathcal{D}_{u_0}} : (\mathcal{D}_{u_0}, u_0) \rightarrow (\mathcal{V}_0, v_0).$$

Lemma 2.7. *There is a bijection between the cosets of E_{u_0} in S_f and the connected components of \mathcal{D}_0 given by*

$$gE_{u_0} \mapsto g(\mathcal{D}_{u_0}).$$

Proof. Since S_f acts transitively on the points in $\nu_0^{-1}(v_0)$ we have a bijection

$$\begin{aligned} S_f &\rightarrow \nu_0^{-1}(v_0) \\ g &\mapsto g(u_0). \end{aligned}$$

It suffices to note that we have equality of cosets $g_1 E_{u_0} = g_2 E_{u_0}$ if and only if $g_1(u_0)$ and $g_2(u_0)$ are in the same connected component of \mathcal{D}_0 . Indeed $g_1 E_{u_0} = g_2 E_{u_0}$ if and only if $g_1^{-1} g_2 \in E_{u_0}$ or equivalently $g_1^{-1} g_2(\mathcal{D}_{u_0}) = \mathcal{D}_{u_0}$. Multiplying by g_1 gives the equivalent statement

$$g_1(\mathcal{D}_{u_0}) = g_2(\mathcal{D}_{u_0}).$$

□

Corollary 2.8. *If \mathcal{V}_f is connected, there is a bijection between the cosets of E_{u_0} in S_f and the connected components of \mathcal{D}_f .*

2.10. Equalizers and fundamental groups. We study the groups S_f and E_{u_0} using the language of equalizers. In general, given two maps

$$\xi_1, \xi_2 : (\mathcal{X}, x_0) \rightarrow (\mathcal{Y}, y_0)$$

between topological spaces with basepoints, we define the equalizer of the induced maps on fundamental group:

$$S(\xi_1, \xi_2) := \{\gamma \in \pi_1(\mathcal{X}, x_0) \mid (\xi_1)_*(\gamma) = (\xi_2)_*(\gamma)\}.$$

Let $\iota : (\mathcal{E}q(\xi_1, \xi_2), x_0) \hookrightarrow (\mathcal{X}, x_0)$ be the inclusion, and define the image of the fundamental group of the equalizer:

$$E(\xi_1, \xi_2) := \iota_*(\pi_1(\mathcal{E}q(\xi_1, \xi_2), x_0)).$$

We have $E(\xi_1, \xi_2) \subseteq S(\xi_1, \xi_2)$.

2.11. The subgroups S_f and E_{u_0} . We apply the discussion in Section 2.10 to study the subgroups S_f and E_{u_0} . Recall the natural maps $\rho_1, \rho_2 : (\mathcal{W}_f, v_0) \rightarrow (\mathcal{M}_A, m_0)$, and consider the groups $S(\rho_1, \rho_2)$ and $E(\rho_1, \rho_2)$. These are related to S_f and E_{u_0} via the defining map

$$\mathfrak{h} : \pi_1(\mathcal{W}_f, v_0) \rightarrow L_f$$

for the covering $\omega : (\mathcal{U}_f, u_0) \rightarrow (\mathcal{W}_f, v_0)$. Recall that \mathfrak{h} is defined by $\mathfrak{h}(\gamma) := [h_\gamma]$, where $[h_\gamma]$ is the unique covering automorphism taking u_0 to the endpoint of the lift of γ to \mathcal{U}_f based at u_0 .

Let \mathfrak{h}_S and \mathfrak{h}_E be the restrictions of \mathfrak{h} to the subgroups $S(\rho_1, \rho_2)$ and $E(\rho_1, \rho_2)$ in $\pi_1(\mathcal{W}_f, v_0)$. Then we have the following.

Proposition 2.9. *The maps \mathfrak{h}_E and \mathfrak{h}_S define isomorphisms*

$$\mathfrak{h}_E : E(\rho_1, \rho_2) \rightarrow E_{u_0},$$

and

$$\mathfrak{h}_S : S(\rho_1, \rho_2) \rightarrow S_f.$$

Lemma 2.7 gives the following result.

Corollary 2.10. *There is a bijection between the connected components of \mathcal{D}_0 and the cosets $S(\rho_1, \rho_2)/E(\rho_1, \rho_2)$.*

Remark 2.11. Since the spaces \mathcal{U}_f and \mathcal{W}_f are connected, $S(\rho_1, \rho_2)$ only depends on f and not on the choice of basepoint u_0 . It is currently unknown, at least to the authors, whether \mathcal{V}_f is always connected, or whether $E(\rho_1, \rho_2)$ is independent of the choice of basepoint.

2.12. Example. We give an example where $E(\xi_1, \xi_2)$ has infinite index in $S(\xi_1, \xi_2)$ that will be useful in the next section.

Let

$$\mathcal{X} := \{(x, y) \in \mathbb{C}^2 \mid xy(x-1)(y-1)(x+y-1) \neq 0\},$$

and let ξ_1, ξ_2 be the projections onto the x and y coordinates. Let $\delta : \mathcal{X} \rightarrow \mathcal{X}$ be given by $\delta(x, y) := (y, x)$. Then $\xi_1 = \xi_2 \circ \delta$ and $\xi_2 = \xi_1 \circ \delta$.

Let a satisfy $1/2 < a < 1$, and let $x_0 := (a, a) \in \mathcal{E} := \mathcal{E}q(\xi_1, \xi_2)$.

Let $\mathcal{K}_1 \subseteq \mathcal{X}$ be

$$\mathcal{K}_1 := \xi_1^{-1}(a) = \{(a, y) \in \mathcal{X} \mid y \notin \{0, 1, 1-a\}\},$$

and let $\mathcal{K}_2 := \delta(\mathcal{K}_1)$ (see Figure 1). Then $x_0 \in \mathcal{K}_1 \cap \mathcal{K}_2$. For $i = 1, 2$, let $k_i : \mathcal{K}_i \rightarrow \mathcal{X}$ be the inclusion map; since ξ_i is a fibration with fiber \mathcal{K}_i and base $\mathbb{C} - \{0, 1\}$, we have the exact sequence

$$1 \rightarrow \pi_1(\mathcal{K}_i, x_0) \xrightarrow{(k_i)_*} \pi_1(\mathcal{X}, x_0) \xrightarrow{(\xi_i)_*} \pi_1(\mathbb{C} - \{0, 1\}, a) \rightarrow 1,$$

and we have

$$(k_i)_*(\pi_1(\mathcal{K}_i, x_0)) = \ker((\xi_i)_*).$$

Lemma 2.12. *We have $\gamma \in S(\xi_1, \xi_2)$ if and only if one or both of the following hold:*

- (1) $\gamma \in E(\xi_1, \xi_2)$, or
- (2) $\gamma = \gamma_1 \cdot \gamma_2$ where γ_i is in $(k_i)_*(\pi_1(\mathcal{K}_i), x_0)$, and $(\xi_1)_*(\gamma_2) = (\xi_2)_*(\gamma_1)$.

Proof. Suppose that $\gamma \notin E(\xi_1, \xi_2)$. Consider $(\xi_2)_*(\gamma) \in \pi_1(\mathbb{C} - \{0, 1\}, a)$. Because $(\xi_2)_*$ is surjective when restricted to $(k_1)_*(\pi_1(\mathcal{K}_1, x_0))$, there is $\gamma_1 \in (k_1)_*(\pi_1(\mathcal{K}_1, x_0))$ so that $(\xi_2)_*(\gamma_1) = (\xi_2)_*(\gamma)$. Then $\gamma_2 := \gamma_1^{-1} \cdot \gamma$ is in $(k_2)_*(\pi_1(\mathcal{K}_2, x_0))$.

The converse is immediate. □

In order to fully exploit the symmetry of \mathcal{X} , we define a quotient as follows. Let

$$\mathcal{Q} := \{(z, w) \in \mathbb{C}^2 \mid w(z-1)(w-z+1) \neq 0\}.$$

Let \mathfrak{s} be the quotient map

$$\begin{aligned} \mathfrak{s} : (\mathcal{X}, x_0) &\rightarrow (\mathcal{Q}, q_0) \\ (x, y) &\mapsto (x+y, xy), \end{aligned}$$

where two points are identified if and only if they are related by δ . The map \mathfrak{s} is a degree 2 branched covering map, branched along \mathcal{E} .

Consider the map

$$\begin{aligned} \sigma : \mathcal{Q} &\rightarrow \mathbb{C} - \{0, 1\} \\ (z, w) &\mapsto \frac{w}{z-1}. \end{aligned}$$

This is a trivial fibration with fiber $\sigma^{-1}(m)$ equal to the complex line $w = m(z-1)$ in \mathbb{C}^2 punctured at $(1, 0)$. Note that $\sigma(q_0) = a^2/(2a-1)$. Let $\mathcal{F}_0 := \sigma^{-1}(\sigma(q_0))$ be the fiber that contains q_0 . Let $i : \mathcal{F}_0 \hookrightarrow \mathcal{Q}$ be the inclusion. Because σ is trivial, for any lift

$$j : \pi_1(\mathbb{C} - \{0, 1\}, \sigma(q_0)) \rightarrow \pi_1(\mathcal{Q}, q_0),$$

we have

$$\pi_1(\mathcal{Q}, q_0) = i_*(\pi_1(\mathcal{F}_0, q_0)) \times j_*(\pi_1(\mathbb{C} - \{0, 1\}, \sigma(q_0))) \cong \mathbb{Z} \times \mathbb{F}_2,$$

and $i_*(\pi_1(\mathcal{F}_0, q_0))$ is the center of $\pi_1(\mathcal{Q}, q_0)$, which is the \mathbb{Z} -factor.

Lemma 2.13. *The intersection of $\mathfrak{s}_*(E(\xi_1, \xi_2))$ with the center of $\pi_1(\mathcal{Q}, q_0)$ is trivial.*

Proof. Consider the inclusion $i : \mathfrak{s}(\mathcal{E}) \hookrightarrow \mathcal{Q}$. The map $\sigma \circ i$ is double covering. Indeed, all fibers of σ intersect $\mathfrak{s}(\mathcal{E})$ in two distinct points. It follows that $(\sigma \circ i)_*$ is injective on fundamental groups. Thus $i_*(\pi_1(\mathfrak{s}(\mathcal{E}), q_0))$ intersects the kernel of σ_* trivially. Since the kernel of σ_* is the center of $\pi_1(\mathcal{Q}, q_0)$, the claim follows. \square

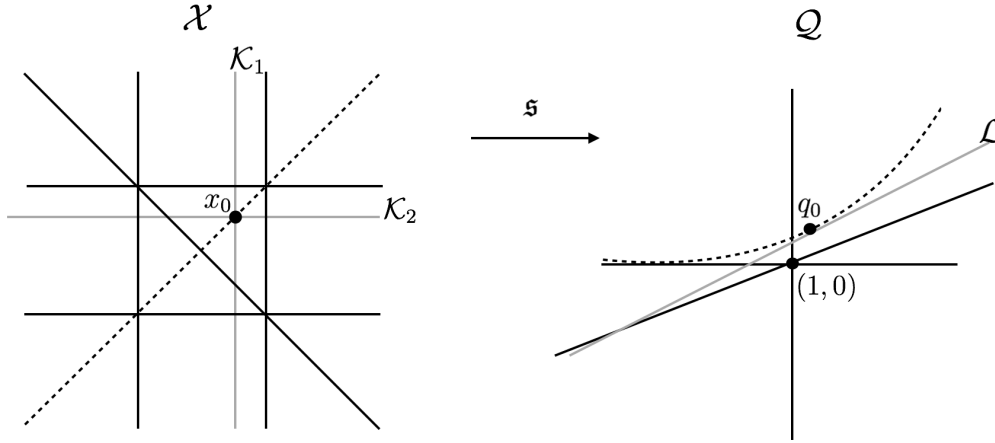


FIGURE 1. The space \mathcal{X} (on the left) is the complement in \mathbb{C}^2 of the five solid black lines. The dashed diagonal line is \mathcal{E} . The space \mathcal{Q} (on the right) is the complement in \mathbb{C}^2 of the three solid black lines intersecting at $(1,0)$. The line \mathcal{L} is the image of \mathcal{K}_1 and \mathcal{K}_2 under the map \mathfrak{s} ; it is tangent to the dashed conic $\mathfrak{s}(\mathcal{E})$ at q_0 .

The next proposition will be a key step in our proof of Theorem 1.4.

Proposition 2.14. *There is a $\gamma \in S(\xi_1, \xi_2)$ so that $\gamma^n \notin E(\xi_1, \xi_2)$ for all $n \neq 0$, and hence*

$$[S(\xi_1, \xi_2) : E(\xi_1, \xi_2)] = \infty.$$

Proof. We find a $\gamma \in S(\xi_1, \xi_2)$, which satisfies Condition (2) of Lemma 2.12, and maps by \mathfrak{s}_* to a nontrivial element of the center of $\pi_1(\mathcal{Q}, q_0)$. By Lemma 2.13 this implies that γ and all its nonzero powers lie in $S(\xi_1, \xi_2) - E(\xi_1, \xi_2)$, proving the claim.

Let $\mathcal{L}' := \sigma^{-1}(a)$, and let $i' : \mathcal{L}' \rightarrow \mathcal{Q}$ be the inclusion map. Since σ is trivial, for any $\ell \in \mathcal{L}'$, and any path τ from q_0 to ℓ in \mathcal{Q} , there is a commutative diagram

$$(1) \quad \begin{array}{ccc} \pi_1(\mathcal{F}_0, q_0) & & \\ \simeq \downarrow & \searrow i_* & \\ \pi_1(\mathcal{L}', \ell) & \xrightarrow{(i')_*} & \pi_1(\mathcal{Q}, q_0) \end{array}$$

where the vertical arrow is defined via conjugation by τ . Since the image $i_*(\pi_1(\mathcal{F}_0, q_0))$ is central in $\pi_1(\mathcal{Q}, q_0)$, the isomorphism is independent of the choice of path τ .

Consider the lines

$$\mathcal{L}_t := \{(z, w) \in \mathbb{C}^2 \mid w = a(z - 1) + (a - a^2)t\}.$$

For $t \in [0, 1]$, $\mathcal{L}_t \cap \mathcal{Q}$ is the complement in \mathcal{L}_t of the points

$$P_t := \{(1 + t(a - 1), 0), \quad (1, (a - a^2)t), \quad (at + 1, at)\}$$

Since $1/2 < a < 1$, the z -coordinates of the points in P_t lie within the disk $|z| < 2$. Let

$$\Delta := \{z \in \mathbb{C} \mid |z| < 2\} \quad \text{and} \quad \mathcal{N} := \{(z, w) \in \mathbb{C}^2 \mid |z| < 2\}.$$

Then

$$\bigcup_{t \in [0, 1]} P_t \subseteq \mathcal{N}.$$

Hence we have a well-defined isotopy

$$\begin{aligned} H : (\mathbb{C} - \Delta) \times [0, 1] &\rightarrow \mathcal{Q} - \mathcal{N} \\ (z, t) &\mapsto (z, a(z - 1) + (a - a^2)t) \in \mathcal{L}_t \end{aligned}$$

between the maps

$$H_0 : (\mathbb{C} - \Delta) \rightarrow \mathcal{L}' - \mathcal{N} \quad \text{and} \quad H_1 : (\mathbb{C} - \Delta) \rightarrow \mathcal{L} - \mathcal{N}$$

where \mathcal{L} is equal to

$$\mathcal{L} := \mathfrak{s}(\mathcal{K}_1) = \mathfrak{s}(\mathcal{K}_2) = \{(z, w) \in \mathcal{Q} \mid w = az - a^2\}$$

(see Figure 1). Note that $\mathcal{L}' = \mathcal{L}_0 \cap \mathcal{Q}$ and $\mathcal{L} = \mathcal{L}_1 \cap \mathcal{Q}$.

Let $H_t : (\mathbb{C} - \Delta) \rightarrow \mathcal{L}_t - \mathcal{N}$ be defined by $H_t(z) = H(z, t)$. Pick any $c \in \mathbb{C} - \Delta$, and let $\ell_t = H_t(c)$. At $t = 0$, the punctures in P_t collide, and $\mathcal{L}' = \mathcal{L}_0 \cap \mathcal{Q}$ has just one puncture at $(1, 0)$, so $\mathcal{L}_0 - \mathcal{N}$ and \mathcal{L}' have the same homotopy type. Thus, the inclusion

$$(\mathcal{L}_0 - \mathcal{N}) \hookrightarrow \mathcal{L}'$$

defines an isomorphism

$$(2) \quad \pi_1(\mathcal{L}_0 - \mathcal{N}, \ell_0) \rightarrow \pi_1(\mathcal{L}', \ell_0).$$

Since H defines an ambient isotopy between the $\mathcal{L}_t - \mathcal{N}$ for $t \in [0, 1]$ in $\mathcal{Q} - \mathcal{N}$, we have a commutative diagram

$$(3) \quad \begin{array}{ccccc} & & \pi_1(\mathcal{L}_0 - \mathcal{N}, \ell_0) & \xrightarrow{\simeq} & \pi_1(\mathcal{L}', \ell_0) & \xrightarrow{\simeq} & \pi_1(\mathcal{F}_0, q_0) \\ & \nearrow (H_0)_* & \downarrow \simeq & & & & \searrow \\ \pi_1(\mathbb{C} - \Delta, c) & & & & & & \pi_1(\mathcal{Q}, q_0) \\ & \searrow (H_1)_* & \downarrow & \longrightarrow & \pi_1(\mathcal{L}, \ell_1) & \longrightarrow & \pi_1(\mathcal{L}, q_0) \end{array}$$

where the top left isomorphism comes from Equation (2), the top right isomorphism comes from Diagram (1), the left vertical isomorphism is induced by H . The bottom left arrow is the map induced by inclusion, and the bottom right arrow is a base change defined by composing the path from ℓ_1 to ℓ_0 defined by $t \mapsto H_{1-t}(c)$, and the path τ from ℓ_0 to q_0 defining the top right arrow.

Let $\gamma'_1 \in \pi_1(\mathcal{L}, q_0)$ be the image of a generator of $\pi_1(\mathbb{C} - \Delta, c)$ under the composition of maps in Diagram (3).

Let $\gamma_1 \in (k_1)_*(\pi_1(\mathcal{K}_1, x_0))$ be such that $\mathfrak{s}_*(\gamma_1) = \gamma'_1$. Let $\gamma = \gamma_1 \cdot \delta_*(\gamma_1)$. Then $\gamma \in S_f$ by Lemma 2.12, and $\mathfrak{s}_*(\gamma)$ is the square of the generator for the central \mathbb{Z} -factor of $\pi_1(\mathcal{Q}, q_0)$, as desired. \square

3. APPLICATION TO QUADRATIC RATIONAL MAPS

In this section, we complete the proof of Theorem 1.4. Let f represent an element of $\text{Per}_4(0)^*$. By conjugating with a Möbius transformation, we may suppose that f has a superattracting cycle of the form

$$(4) \quad 0 \xrightarrow{2} \infty \longrightarrow 1 \longrightarrow a$$


where 0 is the periodic critical point.

Let $A := \{0, 1, \infty, a\}$, let $B := A \cup \{b\}$ where $b \notin A$ is the other critical value of f , and let $u_0 \in \mathcal{D}_f$ be the basepoint associated to the rational map f .

Let (α, β, F) represent a point in \mathcal{W}_f . By post-composing with Möbius transformations, we may suppose that

$$\alpha|_{\{0,1,\infty\}} = \text{id}|_{\{0,1,\infty\}} \quad \text{and} \quad \beta|_{\{0,1,\infty\}} = \text{id}|_{\{0,1,\infty\}}.$$

Then the point $[\alpha, \beta, F] \in \mathcal{W}_f$ is determined by the complex numbers $x := \alpha(a)$, $y := \beta(a)$, $z := \beta(b)$, and a quadratic rational map

$$F : (\mathbb{P}^1, \{0, 1, \infty, x\}) \rightarrow (\mathbb{P}^1, \{0, 1, \infty, y, z\})$$

satisfying

$$\begin{array}{ccccc} 0 & \infty & 1 & x & c \\ \downarrow 2 & \downarrow & \downarrow & \downarrow & \downarrow 2 \\ \infty & 1 & y & 0 & z \end{array}$$

where 0 and c are the two critical points of F . The map F must be of the form

$$F(t) = \frac{(t-x)(t-r)}{t^2}$$

where $r = \frac{y}{x-1} + 1$. Thus, F and the critical value z , are determined by x and y , as long as x, y satisfy certain algebraic conditions. In these coordinates, $\mathcal{M}_A \approx \mathbb{C} - \{0, 1\}$, the map

$$\rho := (\rho_1, \rho_2) : \mathcal{W}_f \rightarrow \mathcal{M}_A \times \mathcal{M}_A \quad \text{is given by} \quad \rho : [x, (y, z), F] \mapsto (x, y),$$

and the image of ρ is equal to $\mathcal{M}_A \times \mathcal{M}_A - \mathcal{C}$ for an algebraic set \mathcal{C} . A computation shows that \mathcal{C} is given explicitly by

$$\mathcal{C} = \{x + y = 1\} \cup \{x^2 - y - 2x + 1 = 0\} \cup \{x^2 + y = 1\} \cup \{2xy + x^2 - y - 2x + 1 = 0\}.$$

See Figure 2.

To summarize, we have the following proposition.

Proposition 3.1. *The map $\rho : \mathcal{W}_f \rightarrow \mathcal{M}_A \times \mathcal{M}_A - \mathcal{C}$ is an isomorphism, and \mathcal{V}_f maps isomorphically to $\rho(\mathcal{W}_f) \cap \mathcal{E}\mathfrak{q}(\rho_1, \rho_2)$; in particular, \mathcal{V}_f is connected.*

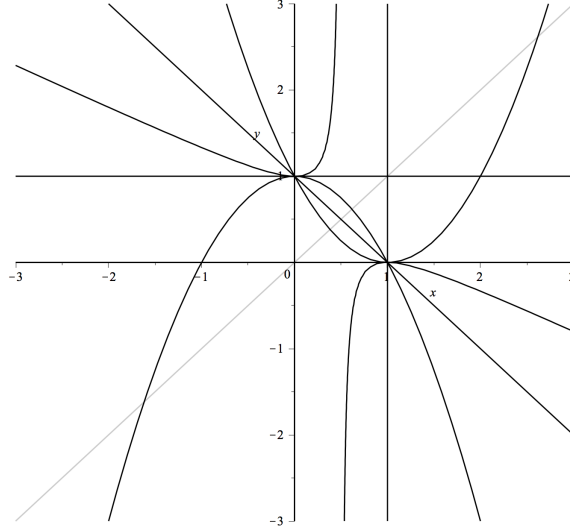


FIGURE 2. The space \mathcal{W}_f is the complement of the black curves in \mathbb{C}^2 (including the axes). The space $\mathcal{V}_f \subseteq \mathcal{W}_f$ is represented by the grey diagonal line.

Proof. By Proposition 2.3, $\rho(\mathcal{V}_f)$ is a union of connected components of $\mathcal{E}q(\rho_1, \rho_2)$. In this case, $\mathcal{E}q(\rho_1, \rho_2)$ is equal to the diagonal in $\mathcal{M}_A \times \mathcal{M}_A - \mathcal{C}$. The space $\mathcal{E}q(\rho_1, \rho_2)$ (and therefore \mathcal{V}_f) is isomorphic to \mathbb{P}^1 with 10 punctures, 7 of which are visible in Figure 2 (the intersection of the grey line with the black curves). \square

Since \mathcal{V}_f is connected, there is a bijection between the cosets of $E(\rho_1, \rho_2)$ in $S(\rho_1, \rho_2)$, and the connected components of \mathcal{D}_f by Corollary 2.8.

We are now ready to complete the proof of Theorem 1.4.

Proof of Theorem 1.4. By Corollary 2.10, to prove Theorem 1.4 it suffices to show that there is an element $\gamma \in S(\rho_1, \rho_2)$ such that no nonzero power of γ lies in $E(\rho_1, \rho_2)$. We do this by reducing to the setting of the example in Section 2.12.

The space \mathcal{W}_f embeds into subset in $\mathcal{X} := \mathcal{M}_A \times \mathcal{M}_A - \{x + y = 1\}$. Let $\iota : \mathcal{W}_f \rightarrow \mathcal{X}$ be the embedding, and let $x_0 := \iota(v_0)$. Then the induced map

$$\iota_* : \pi_1(\mathcal{W}_f, v_0) \rightarrow \pi_1(\mathcal{X}, x_0)$$

is surjective since $\mathcal{X} - \iota(\mathcal{W}_f)$ is a submanifold of \mathcal{X} of (real) codimension 2. Furthermore, the maps $\rho_1, \rho_2 : \mathcal{W}_f \rightarrow \mathcal{M}_A$ factor as

$$\rho_1 = \xi_1 \circ \iota \quad \text{and} \quad \rho_2 = \xi_2 \circ \iota$$

where $\xi_1, \xi_2 : \mathcal{X} \rightarrow \mathcal{M}_A$ are projections onto the x and y coordinates. It follows that $\iota_*(S(\rho_1, \rho_2)) = S(\xi_1, \xi_2)$ and $\iota_*(E(\rho_1, \rho_2)) = E(\xi_1, \xi_2)$.

By Proposition 2.14 the index $[S(\xi_1, \xi_2) : E(\xi_1, \xi_2)]$ is infinite. It follows that the index $[S(\rho_1, \rho_2) : E(\rho_1, \rho_2)]$ is also infinite. By Lemma 2.7 this implies that \mathcal{D}_f has infinitely many components. \square

We finish with a constructive description of an element in S_f whose action on connected components of \mathcal{D}_f has an infinite orbit. The Birman exact sequence [B] for $A = \{0, 1, \infty, a\}$

is

$$1 \rightarrow \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, a) \xrightarrow{\eta} \text{Mod}_A \longrightarrow \text{Mod}_{\{0,1,\infty\}} \rightarrow 1,$$

where η takes a loop ℓ based at a to the point-push map associated to ℓ . Since $\text{Mod}_{\{0,1,\infty\}}$ is trivial, η is an isomorphism. Our choice of basepoint $a = m_0$ identifies $\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, a)$ with $\pi_1(\mathcal{M}_A, m_0)$ and η becomes the defining map from $\pi_1(\mathcal{M}_A, m_0)$ to Mod_A that determines the regular covering $\mathcal{T}_A \rightarrow \mathcal{M}_A$.

Proposition 3.2. *Let \mathcal{D}_{u_0} be the connected component of \mathcal{D}_f containing u_0 . Let $\kappa \in \pi_1(\mathbb{P}^1 - \{0, 1, \infty\}, a)$ be represented by a simple closed loop based at a , separating $\{0, 1\}$ from $\{\infty\}$. Then there is an element $s \in S_f$ such that*

$$\Phi_f(s) = s_A = \eta(\kappa),$$

and the map

$$n \mapsto s^n(\mathcal{D}_{u_0})$$

defines a bijection from \mathbb{Z} to a subset of the connected components of \mathcal{D}_f .

Proof. Let $\gamma'_1 \in \pi_1(\mathcal{X}, x_0)$ be such that $(\xi_1)_*(\gamma'_1) = \kappa$. As in our proof of Proposition 2.14, we can assume that γ'_1 has a representative ℓ contained in the generic fiber $\mathcal{K}_1 := \xi_1^{-1}(a)$, such that ℓ avoids the set

$$\tau := \{(a, y) \in \mathbb{C}^2 \mid 0 \leq y \leq 1\}$$

in \mathcal{K}_1 . Let $\gamma_1 \in \pi_1(\mathcal{W}_f, v_0)$ be such that the map induced by inclusion $\iota : \mathcal{W}_f \rightarrow \mathcal{X}$ gives

$$\iota_*(\gamma_1) = \gamma'_1,$$

and let $\gamma_2 \in \pi_1(\mathcal{W}_f, v_0)$ be such that

$$\iota_*(\gamma_2) = \delta_*(\gamma'_1).$$

The elements γ_1 and γ_2 exist because ι_* is surjective. Then $\gamma = \gamma_1 \cdot \gamma_2$ defines an element of $S(\rho_1, \rho_2)$, and $s := \mathfrak{h}(\gamma) \in S_f$ satisfies $\Phi_f(s) = s_A = \eta(\kappa)$. \square

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E-mail address: `hironaka@math.fsu.edu`

DEPARTMENT OF MATHEMATICS, FLORIDA STATE UNIVERSITY, 1017 ACADEMIC WAY, 208 LOV,
TALLAHASSEE, FL 32306-4510

E-mail address: `kochsc@umich.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MICHIGAN, EAST HALL, 530 CHURCH STREET, ANN
ARBOR, MICHIGAN 48109