Contemporary Mathematics

Lehmer's Problem, McKay's Correspondence, and $2, 3, 7$

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Dedicated to the memory of Ruth Michler

ABSTRACT. This paper analyses Lehmer's problem in the context of Coxeter systems. The point of view leads to a topological generalization of McKay's correspondence, and highlights some special properties of the triple $(2, 3, 7)$.

1. Introduction

This paper addresses a long-standing open problem due to Lehmer in which the triple 2,3,7 plays a notable role. Lehmer's problem asks whether there is a gap between 1 and the next largest algebraic integer with respect to Mahler measure. The question has been studied in a wide range of contexts including number theory, ergodic theory, hyperbolic geometry, and knot theory; and relates to basic questions such as describing the distribution of heights of algebraic integers, and of lengths of geodesics on arithmetic surfaces. See, for example, [**EW99**] and [**GH01**] for surveys and references. This paper focuses on Lehmer's problem applied to Coxeter systems. The analysis leads to a topological version of McKay's correspondence.

We review some properties of Coxeter systems in Section 1, and Coxeter links in Section 2. Section 3 describes Lehmer's problem, and Section 4 contains some remarks on McKay's correspondence.

2. Coxeter Systems

A *Coxeter system* consists of a vector space V with a distinguished ordered basis $e_1, \ldots, e_n$, and an inner product

$$\langle e_i, e_j \rangle = -2 \cos \frac{\pi}{m_{i,j}},$$

where $m_{i,i} = 1$, and if $i \neq j$, $m_{i,j} \in \{2, 3, \ldots, \infty\}$. Associated to a Coxeter system is the *Coxeter group* $G \subset \mathrm{GL}(V)$ generated by reflections $S = \{s_1, \ldots, s_n\}$ through hyperplanes perpendicular to $e_1, \ldots, e_n$ respectively. The action of $s_i \in S$ on the

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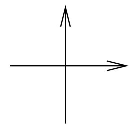

FIGURE 1. Vertical arrow crosses *positively* over horizontal arrow.

basis of V is given by

$$\begin{aligned} s_i(e_j) &= e_j - 2\,\mathrm{proj}_{e_i} e_j \\ &= e_j - \langle e_i, e_j \rangle e_i. \end{aligned}$$

The group G has presentation

$$G = \langle\ s_1, \dots, s_n\ :\ (s_i s_j)^{m_{i,j}} = 1\ \rangle.$$

The Coxeter system will be denoted by (G, S).

A Coxeter system is determined by its *Coxeter graph* Γ. This is the graph with vertices $\nu_1, \dots, \nu_n$ corresponding to the elements of S and edges labeled $m_{i,j}$ connecting distinct vertices ν_i and ν_j whenever $m_{i,j} > 2$.

The *Coxeter element* of (G, S) is the product of reflections

$$C = s_1 \dots s_n,$$

and is an important invariant of the system (see, for example, [**Hum90**], Chapter 3.16).

2.1. Coxeter Links. In this section, we show how to associate fibered links to simply-laced Coxeter systems whose Coxeter graphs are realizable by positive chord systems. We begin with some preliminary definitions.

A Coxeter system is *simply-laced* if $m_{i,j}$ is either 2 or 3 whenever $i \neq j$. Since in this case all edges on the Coxeter graph Γ are labeled 3, we drop the labeling.

By a *chord*, we will mean a line segment on a 2-disk D connecting 2 distinct points on the boundary of D. A chord is *oriented* once the initial- and end-point of the chord are specified. Given two distinct oriented chords ℓ and ℓ', ℓ crosses ℓ' *positively* (resp., *negatively*) if ℓ and ℓ' intersect, and the angle θ between the vectors defined by ℓ' and ℓ is in the open interval $(0, \pi)$ (resp., in $(\pi, 2\pi)$). Figure 1 gives an example of a positive crossing. The *intersection number* of ℓ with ℓ' is defined to be

$$I(\ell, \ell') = \begin{cases} 0 & \text{if } \ell \text{ and } \ell' \text{ are disjoint;} \\ 1 & \text{if } \ell \text{ crosses } \ell' \text{ positively; and} \\ -1 & \text{if } \ell \text{ crosses } \ell' \text{ negatively} \end{cases}$$

A *chord diagram* $\mathcal{L}$ is a collection of chords whose endpoints are pairwise disjoint. If the chords are oriented, we call $\mathcal{L}$ a *chord system*. A chord diagram or system $\mathcal{L}$ is *ordered* if its chords are ordered. An ordered chord system $\mathcal{L} = \{\ell_1, \dots, \ell_n\}$ is *positive* if

$$I(\ell_i, \ell_j) \geq 0$$

whenever $i > j$.

The *intersection matrix* for an ordered chord system $\mathcal{L}$ is the matrix of intersection numbers

$$A = [I(\ell_i, \ell_j)]\,.$$

From the definitions it follows that an ordered chord system $\mathcal{L}$ is positive if and only if the lower diagonal entries of A are non-negative.

The incidence graph of a chord system $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$ is the graph with vertices ν_i corresponding to each chord ℓ_i, and edges connecting ν_i and ν_j whenever ℓ_i and ℓ_j meet. An ordered chord system thus gives rise to an ordered graph. We will call a simply-laced Coxeter system *realizable* if its Coxeter graph is the incidence graph of a positive chord system. More will be said about realizable Coxeter systems in Section 2.2.

We now recall some definitions and properties of fibered links. A link K in S^3 is *fibered with fiber* Σ if Σ is a surface embedded in S^3, K is the boundary of Σ, and there is a fibration

$$\begin{array}{c} S^3 \setminus K \\ \Big\downarrow \\ S^1 \end{array}$$

with general fiber Σ. The fibration is equivalent to a continuous map

$$H : \Sigma \times [0,1] \to S^3,$$

where for some surface homeomorphism

$$h : \Sigma \to \Sigma$$

we have

1. $H(x,t) = x$ for all $x \in K$;
2. $H(x,1) = H(h(x),0)$; and
3. H is otherwise one-to-one.

The surface Σ is called a *Seifert surface* for K, and the map h is called the *monodromy* of the fibration. (See, for example, [**BZ85**] Ch. 5, and [**Lin85**] for more information on fibered links.)

Define the *positive push-off* map

$$\sigma_+ : \Sigma \to S^3 \setminus \Sigma$$

by

$$\sigma_+(x) = H(x, \frac{1}{2}).$$

Given a basis $\alpha_1, \ldots, \alpha_n$ of $\mathrm{H}_1(\Sigma; \mathbb{R})$, let V be the matrix defined by

$$V = [\mathrm{link}(\alpha, \sigma_+\beta)]$$

where $\mathrm{link}(\eta, \gamma)$ is the usual linking number in S^3. The matrix V is called a *Seifert matrix* for the fibered link K. Then the restriction map of the monodromy h,

$$h_* : \mathrm{H}_1(\Sigma; \mathbb{R}) \to \mathrm{H}_1(\Sigma; \mathbb{R}),$$

is represented by the matrix

$$h_* = V^{-1}V^t \tag{1}$$

in terms of the basis $\alpha_1, \ldots, \alpha_n$. The *Alexander polynomial* of a fibered link is the characteristic polynomial of h_*. (See [**Rol76**], Ch. 8, for more information on Seifert matrices and Alexander polynomials.)

From an ordered chord diagram $\mathcal{L} = \{\ell_1, \ldots, \ell_n\}$, we define a fibered link $K_{\mathcal{L}}$ with fiber $\Sigma_{\mathcal{L}}$ by starting with a disk in S^3 and doing successive Murasugi sums [**MM82**] of Hopf bands as in Figure 2. That is, we embed the disk D in R^3 as

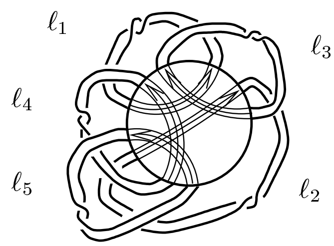

FIGURE 2. Murasugi sum.

the unit disk on the x, y-plane. Then we successively attach positively twisted bands $\eta_1, \dots, \eta_n$ to D, along the thickened arcs $\ell_1, \dots, \ell_n$ so that η_i passes over η_j whenever $i > j$.

The surface $\Sigma_{\mathcal{L}} \subset \mathbb{R}^3$ and its link boundary $K_{\mathcal{L}} = \partial\Sigma_{\mathcal{L}}$ determine a link in S^3 considered as the one point compactification

$$S^3 = \mathbb{R}^3 \cup \{\infty\}.$$

Stallings shows (see [**Sta75**]) that the Murasugi sum of two fibered links is fibered. Since the Hopf link and the unknot are fibered links, the links $K_{\mathcal{L}}$ are fibered with fiber $\Sigma_{\mathcal{L}}$.

If the chords $\ell_1, \dots, \ell_n$ in $\mathcal{L}$ are oriented, then we can associate a basis $\alpha_1, \dots, \alpha_n$ for $\mathrm{H}_1(\Sigma_{\mathcal{L}}; \mathbb{R})$ by extending each ℓ_i to a closed loop running along the band η_i. Let Γ be the incidence graph of $\mathcal{L}$, and let A be the intersection matrix of $\mathcal{L}$. One can easily verify that the Seifert matrix for $\mathcal{L}$ with respect to this basis is given by $M = I + A^+$, where A^+ is the upper triangular part of A, and hence equation (1) implies that h_* is presented by

$$h_* = M^{-1}M^t, \tag{2}$$

with respect to the basis $\alpha_1, \dots, \alpha_n$.

If the chord system $\mathcal{L}$ is positive, with ordered incidence graph Γ, then the bilinear form of the simply-laced Coxeter system (G, S) associated to Γ can be written as

$$B = M + M^t$$

and the Coxeter element of (G, S) equals

$$C = -M^{-1}M^t = -h_*$$

by [**How82**] Theorem 2.1, and (2). We will call the pair $(K_{\mathcal{L}}, \Sigma_{\mathcal{L}})$ a *Coxeter link* associated to (G, S).

The discussion above is completely separate from another study of chord diagrams and knots (see, for example, [**Fen94**]), which has been studied in relation to Vassiliev invariants.

2.2. Realizable graphs. In this section, we give a brief discussion of some combinatorics of chord diagrams.

A graph Γ is *realizable* if it is the incidence graph of a chord diagram. It is not hard to check that the following graphs are realizable.

1. Complete graphs;

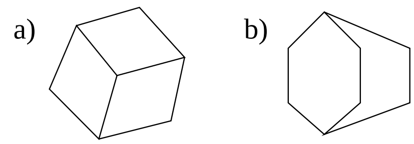

FIGURE 3. Non-realizable graphs.

2. cycles;
3. joins of two realizable graphs at one vertex; and
4. trees.

There are, however, obstructions to realizability, as shown in the propositions below.

Let Γ be a graph with vertices S. A subgraph $\Gamma' \subset \Gamma$ is an *induced subgraph* if for some $S' \subset S$, Γ' is the subgraph containing all edges on Γ whose endpoints are in S'. We say Γ' is *induced* by the vertex set S'. An *induced cycle* in Γ is a cycle which is an induced subgraph. A subset S' of the vertices S is *disjoint* if there is no edge on Γ connecting any pair of vertices in S'.

To the author's knowledge, a complete set of sufficient conditions for a graph to be realizable is not known. The following two propositions provide obstructions to realization. Their proofs follow easily from the observation that a cycle has, up to orientation, only one realization, and will be left to the reader.

PROPOSITION 2.1. *A graph Γ is not realizable if there is a subset $S' \subset S$ such that*

1. *S' contains at least three vertices;*
2. *S' is disjoint;*
3. *there is an $s \in S$ so that s is joined by an edge in Γ to every vertex in S'; and*
4. *there is an induced cycle in Γ containing S'.*

Figure 3 a) is an example of a graph satisfying Proposition 2.1. One can also check that the graph in Figure 3 b) is not realizable, as implied by the proposition below.

PROPOSITION 2.2. *A graph Γ is not realizable if there are two subsets $S', S'' \subset S$ such that*

1. *S' and S'' are two induced cycles;*
2. *the induced subgraph of Γ generated by the vertices in $S' \cap S''$ is connected (and hence the induced subgraphs generated by $S' \setminus S''$ and $S'' \setminus S'$ are also connected);*
3. *$S' \cap S''$ contains at least 4 vertices; and*
4. *$S' \setminus S''$ and $S'' \setminus S'$ each contain at least two vertices.*

2.3. Ordering and positivity. In Section 2.1, we saw that a Coxeter graph Γ has a corresponding Coxeter link if the graph is realizable as a chord diagram, and its ordering is compatible with a positive orientation on the chord diagram.

Not all orderings on a realizable graph, however, are realizable by a positive chord system. For example, one can easily check that for the ordered graph in Figure 4 any orientation of ℓ_1 determines orientations on ℓ_2 and ℓ_4 which are not compatible with any orientation of ℓ_3.

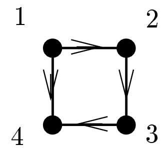

FIGURE 4. Ordered graph which cannot be realized by a positive chord system

FIGURE 5. Coxeter links of trees.

LEMMA 2.3. *Any chord diagram admits an ordering and orientation which is positive.*

PROOF. Choose a direction vector v from the center of the disk and orient the chords so that their direction vectors have positive inner product with v with respect to the usual Euclidean metric on $\mathbb{R}^2$. Now order the chords counter-clockwise starting with the chord pointing furthest to the right of v. □

Given an ordered graph, there is an associated directed graph, where edges are directed so that they point to the vertex with larger index. As can be seen from the construction, we have the following.

PROPOSITION 2.4. *If two orderings on a chord diagram $\mathcal{L}$ have the same directed incidence graph then the resulting fibered links are the same.*

Proposition 2.4 is analogous to the following observation by Shi [**Shi97**].

PROPOSITION 2.5. *The Coxeter element of Coxeter system (G, S) depends only on the directed Coxeter graph Γ of (G, S).*

2.4. Examples. This section contains some examples of links associated to Coxeter systems.

The case when the incidence graph of $\mathcal{L}$ is a tree has been well studied, and the corresponding link has been called an *arborescent link* [**Con70**]. Arborescent links also appear as *slalom links* in [**A'C98**]. Since any tree is realizable, there exists a Coxeter link associated to any tree. While different realizations can lead to different Coxeter links, as in Figure 8, once the chord diagram is fixed, one can easily see that the link is independent of the ordering.

Some useful examples of trees are the star diagrams. In [**Hir02**], we show that the star diagram shown in Figure 6, which we will denote by $\mathrm{Star}(p_1, \dots, p_k)$, gives rise to the pretzel link $K_{p_1,\dots,p_k}$ shown in Figure 7, where the twists have orders $p_1, \dots, p_k$.

It is possible to cook up examples of non-equivalent links associated to the same ordered Coxeter system using star-diagrams. Take the two realizations of the

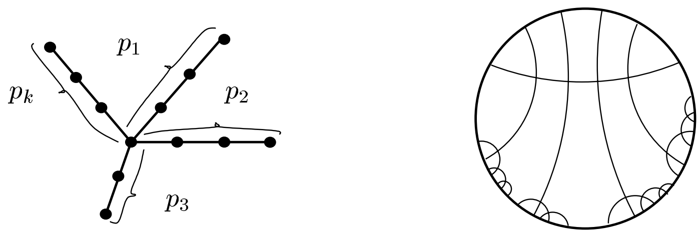

FIGURE 6. Star diagram and realization.

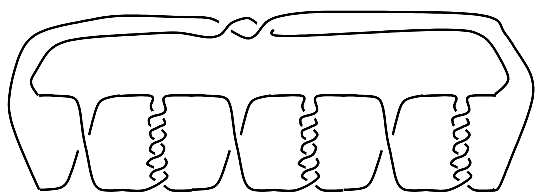

FIGURE 7. Coxeter Link for a star graph.

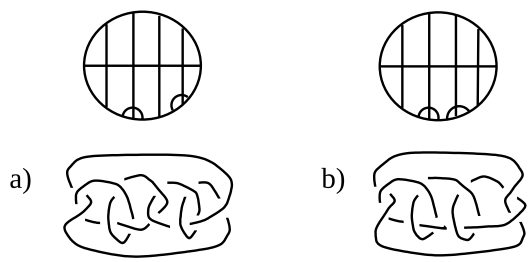

FIGURE 8. Two embeddings of the same tree.

same tree shown in Figure 8. One sees that the link in Figure 8 a) has two knotted components, while the link in Figure 8 b) has a component which is the unknot. Hence the links are not equivalent.

Different orderings on a chord system can give rise to different links when the graph contains cycles. Consider for example, the 5-cycle. Up to isotopy, there is only one chord diagram with this incidence graph, but there are two inequivalent positive orderings as shown in Figure 9. The Coxeter elements for the different orderings are:

$$a)\ 1 - t - t^4 + t^5 \qquad\qquad b)\ 1 - t^2 - t^3 + t^5.$$

This implies that the fibrations of the corresponding fibered links are non-equivalent; one can also easily observe from Figure 9 that the links themselves are distinct iterated torus links.

For the classical Dynkin diagrams it is not hard to check that the associated links don't depend on their realization as a chord system. The Dynkin diagrams and their corresponding Coxeter links are shown in Figure 10.

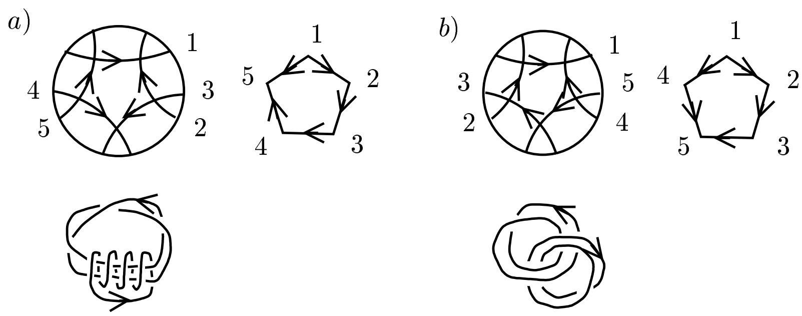

FIGURE 9. Two orderings on the 5-cycle and their resulting Coxeter links.

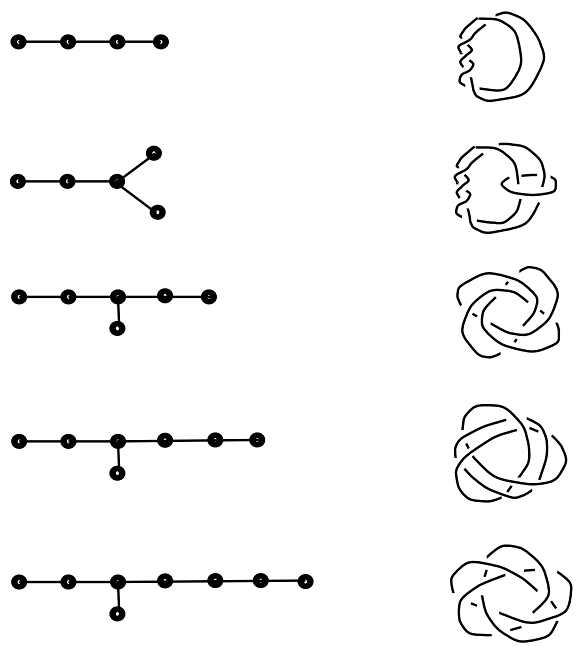

FIGURE 10. Links associated to Dynkin diagrams.

The simply-laced minimal hyperbolic Coxeter system of smallest dimension is a triangle with a tail. The Coxeter link (see Figure 11) is uniquely determined in this case by the requirement of positivity, and equals the mirror of the 10_{145}-knot in Rolfsen's table [**Rol76**], which is $(22, 3, 3-)$ in Conway's notation [**Con70**].

2.5. Remarks on the Geometry of Coxeter systems and Coxeter links. We will assume throughout this section that Coxeter systems are irreducible, or equivalently the Coxeter graph is connected.

A Coxeter system is finite if and only if its associated bilinear form is positive definite ([**Hum90**], Chapter 6.4). In this case the Coxeter group is a Euclidean reflection group and the Coxeter system is called *spherical*. A Coxeter group has

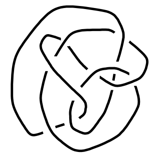

FIGURE 11. Coxeter link associated to smallest hyperbolic Coxeter system.

a natural representation as an affine reflection group if and only if the associated bilinear form is positive semi-definite ([**Hum90**], Chapter 6.5).

A'Campo [**A'C76**] and Howlett [**How82**] proved that whether a Coxeter system is spherical, affine, or neither can be detected by the eigenvalues of the Coxeter element.

THEOREM 2.6. *Let* Γ *be a Coxeter system and* C *a Coxeter element. Then*

1. *the eigenvalues of* C *lie on* $\mathbb{R} \cup S^1$*;*
2. Γ *is spherical if and only if* C *has finite order; and*
3. Γ *is neither affine or spherical if and only if* C *has at least one eigenvalue with norm greater than one.*

Fibered knots and links K in S^3 have an analogous classification as torus links, satellite links, and hyperbolic links [**Thu82**]. All iterated torus links have Alexander polynomials with Mahler measure equal to one. This implies the following.

COROLLARY 2.7. *The only Coxeter links, coming from irreducible Coxeter systems, which are iterated torus links are the ones corresponding to the spherical and affine Coxeter systems.*

Theorem 2.6 and Corollary 2.7 imply that the Mahler measure gives a kind of measure of hyperbolicity for Coxeter links.

QUESTION 2.8. *Can one find a more precise relation between the Mahler Measure and geometric invariants for Coxeter links, for example, the entropy?*

3. Lehmer's Problem

Let $p(x)$ be a monic integer polynomial, and define its *Mahler measure* to be

$$\|p(x)\| \quad = \quad \prod_{\beta} |\beta|, \tag{3}$$

where β runs through all (complex) roots of $p(x)$ outside the unit circle. It is well known that $\|p(x)\| = 1$ if and only if all roots of $p(x)$ are roots of unity.

In 1933, Lehmer [**Leh33**] asks whether for each $\delta > 0$, there exists a monic integer polynomial $p(x)$ such that

$$1 < \|p(x)\| < 1 + \delta.$$

It is an easy exercise to see that for each positive integer d, there is a gap between 1 and the Mahler measures of monic integer polynomials which are not products of cyclotomics. Lehmer found polynomials with smallest Mahler measure for small degrees and states in [**Leh33**] that the smallest he could find for degree 10 or higher is

$$p_L(x) = x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1,$$

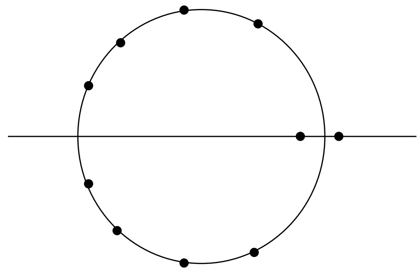

FIGURE 12. Roots of the Lehmer polynomial

which has Mahler measure

$$\|p_L(x)\| = 1.17628\ldots.$$

Boyd [**Boy89**] and Mossinghoff [**Mos98**] have done searches up to degree 40, but so far no one has found a monic noncyclotomic integer polynomial with smaller Mahler measure.

One observes immediately that $p_L(x)$ is reciprocal, that is

$$p_L(x) = x^d p_L(\frac{1}{x}),$$

where d is the degree of $p_L(x)$ (in this case $d = 10$).

Smyth [**Smy71**] shows

$$p_S(x) = x^3 - x + 1$$

solves Lehmer's problem for non-reciprocal polynomials, that is, if $p(x)$ is monic and non-reciprocal, then

$$\|p(x)\| \geq \|p_S(x)\| = 1.32472\ldots.$$

Since

$$\|p_S(x)\| > \|p_L(x)\|,$$

it remains to determine whether there is a similar minimum for Mahler measures of reciprocal monic integer polynomials.

A *Salem number* is an algebraic integer whose algebraic conjugates lie on or within the unit circle, with at least one conjugate on the unit circle (making the minimal polynomial necessarily reciprocal). As shown in Figure 12, $p_L(x)$ has only one root, $\alpha_L = 1.17628\ldots$, which we'll call *Lehmer's number*, outside the unit circle. Thus, the Mahler measure of $p_L(x)$ is α_L, and α_L is a Salem number. (Analogously, the Mahler measure of $p_S(x)$ is its largest root α_S, and Smyth's theorem implies that α_S is the smallest *P-V number*, that is, an algebraic integer all of whose conjugates lie strictly within the unit circle.) It is not known whether there exist Salem numbers smaller than α_L. Furthermore, it is not known whether Lehmer's question can be answered by resolving the minimization problem for Salem numbers.

It has been observed in various contexts that Lehmer's polynomial $p_L(x)$ is related to the triple $(2, 3, 7)$ and more abstractly to the notion of minimal hyperbolicity. Before going to examples, it is worth remarking that the triple $(2, 3, 7)$ has the simple distinguishing property that, among all k-tuples of positive integers

$(p_1, \ldots, p_k)$, $(2,3,7)$ gives the minimal positive value for

$$k - 2 - \sum_{i=1}^{k} \frac{1}{p_k}$$

(see, for example, [**Hir01**] Lemma 3.1.) This property comes into play in the minimality of Lehmer's number α_L among the series of Salem numbers and algebraic numbers which we describe below in this section.

3.1. Growth rates and the $(2,3,7)$-triangle group. Consider any pair (G, S), where G is a group and S is a set of generators. Let

$$w_n = \text{number of words of minimal word length } n \text{ in } S.$$

The *growth series* of (G, S) is the formal power series

$$f_{(G,S)} = \sum_{n=1}^{\infty} w_n t^n$$

and the *growth rate* α equals

$$\alpha = \frac{1}{\text{radius of convergence of } f_{(G,S)}}.$$

Another way to say this is that w_n grows like α^n as n gets large.

Let $G = T_{p_1,\ldots,p_k}$ be a polygonal reflection group acting on S^2, $\mathbb{E}^2$ or $\mathbb{H}^2$. The group G has presentation

$$G = \langle\ s_1, \ldots, s_k\ :\ (s_k s_1)^{p_k},\ \text{and}\ (s_i s_{i+1})^{p_i}, \quad \text{for } i = 1, \ldots, k-1\ \rangle.$$

Let $[p]$ denote the polynomial $[p] = 1 + x + \cdots + x^{p-1}$, and let

$$\Delta_{p_1,\ldots,p_k}(x) = (x - k + 1) \prod_{i=1}^{k} [p_i]\ +\ \sum_{i=1}^{k} [p_1] \cdots \widehat{[p_i]} \cdots [p_k].$$

For (G, S) as above Floyd and Plotnick [**FP88**] show the following. (See also, [**CW92**].)

THEOREM 3.1. *The growth series for (G, S) is the rational function*

$$f_{(G,S)}(x) = \frac{[2][p_1] \cdots [p_k]}{\Delta_{p_1,\ldots,p_k}(x)}.$$

Furthermore, $\Delta_{p_1,\ldots,p_k}(x)$ is a reciprocal monic integer polynomial with at most one root, necessarily a Salem number, outside the unit circle. This root occurs if and only if

$$\frac{1}{p_1} + \cdots + \frac{1}{p_k} < k - 2.$$

Thus, the growth rate $\alpha_{p_1,\ldots,p_k}$ of (G, S) is a Salem number if and only if the orbifold Euler characteristic

$$\chi = \frac{1}{p_1} + \cdots + \frac{1}{p_k} - (k - 2)$$

of the quotient space by G is negative, and otherwise $\alpha_{p_1,\ldots,p_k} = 1$. The polynomial $\Delta_{2,3,7}(x)$ equals Lehmer's polynomial $p_L(x)$, and hence the triangle group $T_{2,3,7}$ has growth rate equal to α_L.

For the family of polynomials $\Delta_{p_1,\ldots,p_k}(x)$, Lehmer's problem is decided by the following theorem.

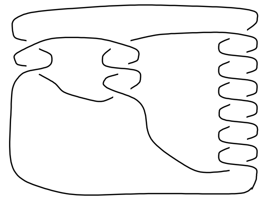

FIGURE 13. The pretzel link $K_{2,3,7}$.

THEOREM 3.2. *Among the polynomials*

$$\Delta_{p_1,\dots,p_k}(x),$$

the one with smallest Mahler measure is Lehmer's polynomial $\Delta_{2,3,7}(x)$.

PROOF. See [**Hir01**], Theorem 1.3. □

This result is suggestive since among hyperbolic orbifold spheres

$$(S^2; p_1, \dots, p_k),$$

the one with maximal orbifold Euler characteristic χ and minimal hyperbolic area is $(S^2; 2, 3, 7)$.

3.2. Alexander polynomials and the $(2,3,7,-1)$ pretzel knot. In his book, published around the time of Lehmer's question, Reidemeister remarks that the $(-2,3,7)$-pretzel knot shown in Figure 13 has Alexander polynomial $p_L(-x)$ [**Rei32**]. This fact is also noted in Kirby's collection of Problems [**Kir97**], and was pointed out to the author by D. Lind in private conversation.

One can easily observe that the $(-2,3,7)$ pretzel knot is equivalent to the $(2,3,7,-1)$ pretzel knot. Let $K_{p_1,\dots,p_k}$ be the $(p_1,\dots,p_k,-1,\dots,-1)$-pretzel link, where the number of "-1"s is $k-2$. The theorem below gives a relation between the Alexander polynomial of $K_{p_1,\dots,p_k}$ and the denominator of the growth series of $T_{p_1,\dots,p_k}$ defined in Section 3.1.

THEOREM 3.3. *The pretzel link* $K_{p_1,\dots,p_k}$ *is fibered and has Alexander polynomial*

$$\Delta_{p_1,\dots,p_k}(-x),$$

where $\Delta_{p_1,\dots,p_k}(x)$ *is the denominator for the growth series of the polygonal reflection group* $T_{p_1,\dots,p_k}$.

PROOF. The links $K_{p_1,\dots,p_k}$ are fibered since they are the Coxeter links associated to $\mathrm{Star}(p_1,\dots,p_k)$ (see Section 2.4). The rest follows from [**Hir01**] Theorem 1.2. □

Thus, Theorem 3.2 implies the following.

COROLLARY 3.4. *Among pretzel links* $K_{p_1,\dots,p_k}$, *the Mahler measure of the Alexander polynomial is minimized by* $K_{2,3,7}$.

We will describe a natural topological relation between the pretzel links and the polygonal reflection groups in Section 4.

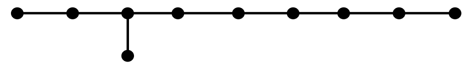

FIGURE 14. E_{10}-Coxeter graph

3.3. E_{10} **diagram.** One observes that the Coxeter E_{10} diagram can be thought of as the Star$(2, 3, 7)$ Coxeter graph by comparing Figure 14 with Figure 6. The link $K_{2,3,7}$ is thus the Coxeter link for the E_{10} diagram.

McMullen observes [**McM02**] that the characteristic polynomial of the Coxeter element is Lehmer's polynomial $p_L(x)$, and its leading eigenvalue is α_L. Furthermore, he shows the following.

THEOREM 3.5. *Let C be the Coxeter element of a Coxeter system, and let $\lambda(C)$ be the spectral radius of C. Then among non-spherical and non-affine Coxeter systems, $\lambda(C)$ achieves its minimum when (G, S) is the Coxeter system corresponding to the E_{10} diagram.*

This solves Lehmer's problem for Coxeter elements of Coxeter systems, and as a consequence also for the Alexander polynomials of Coxeter links, generalizing Corollary 3.4.

COROLLARY 3.6. *Let $p(x)$ be the characteristic polynomial of the Coxeter element of a non-spherical or non-affine Coxeter system. Then*

$$\|p(x)\| \geq \|p_L(x)\|.$$

COROLLARY 3.7. *Let $\Delta(x)$ be the Alexander polynomial of a non-algebraic Coxeter link. Then*

$$\|\Delta(x)\| \geq \|p_L(x)\|.$$

4. Correspondence between stars and polygons

In the previous section, we showed the equality of

1. the growth rate of the polygonal reflection group $T_{p_1,\ldots,p_k}$;
2. the Mahler measure of the Alexander polynomial of $K_{p_1,\ldots,p_k}$; and
3. the spectral radius of the Coxeter system Star$(p_1, \ldots, p_k)$.

Here we give a topological relation between the polygonal reflection groups and Star-Coxeter systems. Let P denote the sphere S^2, the Euclidean plane $\mathbb{E}^2$, or the hyperbolic plane $\mathbb{H}^2$. Let $T(P)$ be the unit tangent bundle of P. Let G be the $(p_1, \ldots, p_k)$-polygonal reflection group $T_{p_1,\ldots,p_k}$ acting on P. It is not hard to see that the induced action of G on $T(P)$ makes $T(P)$ a branched cover over S^3 with branching index 2 on a link K.

The covering $T(P) \to S^3$ factors through an unbranched covering $T(P) \to M$, induced by the action of the orientation preserving subgroup $G^{(2)} \subset G$, and a double branched cover $M \to S^3$ with branch locus K.

(4)
$$\begin{tikzcd} T(P) \arrow[r, "\tau"] \arrow[d, "G^{(2)}"'] & P \arrow[d, "G^{(2)}"'] \\ M \arrow[r, "f"] \arrow[d, "\mathbb{Z}_2"'] & S^2 \\ S^3. & \end{tikzcd}$$

The horizontal arrows are the natural S^1-bundle $\tau : T(P) \to P$, and a Seifert fibration $f : M \to S^2$ with singular fibers of orders $p_1, \ldots, p_k$. The following proposition is not hard to verify from the definitions.

PROPOSITION 4.1. *The branch locus K of the action of $T_{p_1,\ldots,p_k}$ on $T(P)$ is the* Star$(p_1, \ldots, p_k)$*-Coxeter link $K_{p_1,\ldots,p_k}$.*

In general, any Seifert fibration over S^2 admits a double covering of S^3 branched along a link, called a *Montesinos link*, and to each k-tuple, $p_1, \ldots, p_k$, there are several possible links with varying Alexander polynomials and Mahler measures. (For a survey of Montesinos links see [**BZ85**], Chapter 12, and references therein). In particular, given a k-tuple $p_1, \ldots, p_k$, there are several Montesinos links associated to $p_1, \ldots, p_k$ whose Alexander polynomial does not have Mahler measure equal to the growth rate of $T_{p_1,\ldots,p_k}$.

QUESTION 4.2. *What special properties distinguish the Coxeter links $K_{p_1,\ldots,p_k}$ from general Montesinos links?*

The above discussion generalizes a phenomenon appearing in the theory of isolated hypersurface singularities. For a beautiful exposition on Klein singularities and McKay's correspondence, the reader is referred to [**Slo83**].

Briefly the correspondence goes as follows. Let $\widetilde{G}$ be a finite subgroup of SU(2) acting in the usual way on $\mathbb{C}^2$. The quotient X is a hypersurface in $\mathbb{C}^3$ with isolated singularity called a Klein singularity. The resolution diagrams of these singularities give rise to all the Coxeter graphs (or Dynkin diagrams) of the simply-laced spherical Coxeter systems. The correspondence between finite subgroups of SU(2) and the Dynkin diagrams is known as McKay's correspondence [**McK80**].

Coxeter links can be used to give a topological description of the correspondence. Let $\widetilde{G}$ be a finite subgroup of SU(2). Then $\widetilde{G}$ is the binary extension of a finite subgroup $G^{(2)}$ of SO(3) which in turn (with the exceptional case of $G^{(2)} = \mathbb{Z}_n$) is an index 2 subgroup of a (p, q, r)-triangle group $G = T_{p,q,r}$. The group $G^{(2)} = \mathbb{Z}_n$ corresponds to the $(2, n)$-triangle group. Note, that all the Dynkin diagrams are star-diagrams. Relating each group $T_{p,q,r}$ with the corresponding Star(p, q, r) diagram gives the McKay correspondence shown in the table below.

$\mathbb{Z}_n$	$T_{2,n}$	$\mathrm{Star}(2, n) = A_{n+1}$
$\mathcal{D}_n$	$T_{2,2,n}$	$\mathrm{Star}(2, 2, n) = D_{n+2}$
S_3	$T_{2,3,3}$	$\mathrm{Star}(2, 3, 3) = E_6$
S_4	$T_{2,3,4}$	$\mathrm{Star}(2, 3, 4) = E_7$
A_5	$T_{2,3,5}$	$\mathrm{Star}(2, 3, 5) = E_8$

For the triples p, q, r in the above table, the (p, q, r)-star Coxeter link K is algebraic. The induced branched covering of the link M of the singularity X, $M \to S^3$, branched over K, is the restriction of a double branched covering $X \to \mathbb{C}^2$. This gives the following commutative diagram.

$$\begin{tikzcd}
\mathbb{RP}^3 = T(S^2) \arrow[d, "G^{(2)}"'] & S^3 \subset \mathbb{C}^2 \arrow[l, "\mathbb{Z}_2"'] \arrow[d, "\tilde{G}"'] \\
M \arrow[r, "\subset"] \arrow[d, "\mathbb{Z}_2"'] & X \arrow[d, "\rho"'] \\
S^3 \arrow[r, "\subset"] & \mathbb{C}^2.
\end{tikzcd}$$

For the general situation of Proposition 4.1, M is also the link of an isolated surface singularity, but the branched covering $M \to S^3$ is not induced by an algebraic map, since as noted in Corollary 2.7, in general the links $K_{p_1,\dots,p_k}$ are not algebraic links.

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