INTERSECTION THEORY ON BRANCHED COVERING SURFACES AND POLYNOMIAL PERIODICITY

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1. Introduction

Let Y be a smooth complex projective surface and let $\mathcal{B} \subset Y$ be a finite union of irreducible algebraic curves on Y. Let $\rho_n^u : X_n^u \to Y \setminus \mathcal{B}$ be the topological abelian unbranched covering corresponding to the natural map

$$\pi_1(Y \setminus \mathcal{B}) \to \mathrm{H}_1(Y \setminus \mathcal{B}; \mathbb{Z}/n\mathbb{Z}).$$

By the Grauert-Remmert-Enriques Theorem [GR], there is a canonical algebraic completion $\rho_n : X_n \to Y$ of the unbranched covering, where X_n is a normal surface and ρ_n is a finite surjective morphism.

The first Betti number b_1 is a birational invariant of smooth surfaces. Thus, for fixed Y and $\mathcal{B}$, $b_1(\widetilde{X}_n)$ is a well-defined function on the the natural numbers $\mathbb{N}$, where $\widetilde{X}_n$ is any desingularization of X_n.

Definition 1.1. A function $f : \mathbb{N} \to \mathbb{Z}$ is *polynomial periodic* if there are polynomials $p_1, \dots, p_N \in \mathbb{Z}[x]$ so that $f(n) = p_i(n)$ whenever $i \equiv n \pmod N$.

The main result of this paper is the following theorem.

Theorem 1.2. *For any smooth complex projective surface Y and finite union of algebraic curves $\mathcal{B}$, $b_1(\widetilde{X}_n)$ is polynomial periodic.*

The general approach of the proof is the same as that found in [H2], where we prove the same statement under more stringent restrictions on the branch locus.

The outline of the argument is as follows. We use the following general result about unbranched coverings of finite CW complexes due to Sarnak [Sr].

Theorem 1.3. *Let Y be a finite CW complex and let $\rho_n : X_n \to Y$ be the abelian unbranched covering defined by the map*

$$\pi_1(Y) \to \mathrm{H}_1(Y; \mathbb{Z}/n\mathbb{Z}).$$

Then $b_1(X_n)$ is polynomial periodic.

Using this theorem, it suffices to show that the difference $b_1(\widetilde{X}_n) - b_1(X_n^u)$ is polynomial periodic.

There are several ways to find this difference. We will use intersection theory of curves on the covering surfaces.

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Definition. Let X be a smooth complex projective surface. Let $\mathcal{C}$ be a finite union of irreducible curves $C_1, \dots, C_k$ on X. Define $\mathrm{Null}(\mathcal{C})$ to be the nullity of the intersection matrix

$$[C_i.C_j]$$

for $i, j = 1, \dots, k$, where $C_i.C_j$ is the intersection number of C_i and C_j.

The following lemma (see [H1], Proposition I.5.3 or [H2], Proposition 4.1) shows how to relate the difference in Betti numbers and the nullity of an intersection matrix.

Lemma 1.4. *Let X be any smooth complex projective surface and $\mathcal{C}$ a finite union of curves supporting an ample divisor. Then*

$$b_1(X \setminus \mathcal{C}) - b_1(X) = \mathrm{Null}(\mathcal{C}).$$

It follows that the difference $b_1(X_n^u) - b_1(\widetilde{X}_n)$ equals the nullity of the intersection matrix for curves in $\widetilde{X}_n$ lying above $\mathcal{B}$. (Here we need to assume that $\mathcal{B}$ supports an ample divisor. If this condition is not satisfied we can look at curves above $\mathcal{C} = \mathcal{B} \cup H$ where H is any ample curve on Y.)

In [H2] basic formulas for intersections on the desingularized branched coverings are given in the case when the normal covering surfaces X_n have simple desingularizations for all n. One can, however, find similar formulas by doing the intersection theory directly on the normal surfaces X_n using Mumford's theory [M]. This has led to a proof of the more general statement.

The paper is organised as follows. In §2 we review the theory of intersections on normal surfaces and, in §3, put this in the context of branched coverings. In §4, we outline the techniques for the proof of polynomial periodicity found in [H2] and prove the main theorem.

Throughout this paper *curves* and *surfaces* are complex projective irreducible varieties. A *divisor* on a surface (not necessarily smooth) is a formal sum of curves on the surface with integer coefficients. The group of divisors is denoted $\mathrm{Div}(X)$. For any two divisors C and D, $C.D$ denotes the intersection number of C and D. If two curve C and D intersect properly at p, we denote by $(C.D)_p$ the intersection multiplicity of C and D at p. If S is a finite set then $|S|$ denotes the number of elements in S.

I am indebted to W. Veys for helpful conversations.

Added in proofs:. Recently, M. Sakuma has independently generalized the results of [H2] using different methods (see [Sk]).

2. Intersection theory on normal surfaces

Recall the following property of intersections of curves on surfaces and blowups. Let $\sigma : Z \to X$ be a blowup of a smooth surface X at a point $p \in X$, with exceptional set E. The pullback map

$$\sigma^* : \mathrm{Div}(X) \to \mathrm{Div}(Z)$$

is defined by

$$\sigma^* C = \widetilde{C} + mE,$$

where m is the multiplicity of the point p on C and $\widetilde{C}$ is the closure of $\sigma^{-1}(C \setminus \{p\})$ in Z, or the *proper transform* of C. If σ is a composition of blowups $\sigma = \sigma_1 \circ \cdots \circ \sigma_n$, then $\sigma^* = \sigma_n^* \circ \cdots \circ \sigma_1^*$. By the standard projection formula for proper maps and the fact that σ is generically degree one, we have the following.

Lemma 2.1. *Let $\sigma : Z \to X$ be a composition of blowups. The pullback map σ^* has the following properties:*

(1)
$$(\sigma^* C).E = 0 \qquad \text{if } \sigma(E) \text{ is a point;}$$

(2)
$$(\sigma^* C).(\sigma^* D) = C.D.$$

As Mumford notes [M], there is a unique way to define intersection numbers for curves on a normal surface with these properties. To do this one must allow rational intersection numbers.

Let X be a normal surface and let $\sigma : Z \to X$ be a birational morphism from a smooth surface Z to X. It is a standard fact that such a desingularization always exists.

Let $\mathcal{E}$ be the union of curves on Z mapping to points on X, i.e., the exceptional set, and let $S = \sigma(\mathcal{E})$. Then σ induces an isomorphism $Z \setminus \mathcal{E} \to X \setminus S$. A fundamental result ([M], p. 6) states that the intersection matrix for $\mathcal{E}$ is negative definite, so, in particular, nondegenerate. As a consequence, for any curve C in X there is a unique rational divisor $\sigma^* C$ supported on the proper transform $\widetilde{C}$ of C and the irreducible components of $\mathcal{E}$, such that $(\sigma^* C).E = 0$ for all irreducible components $E \subset \mathcal{E}$ and the coefficient of $\widetilde{C}$ in $\sigma^* C$ is 1.

Let $q \in X$ be any point and let $\mathcal{E}_q = \sigma^{-1}(q)$. By the same reasoning as above, there is a unique rational divisor supported on $\widetilde{C} \cup \mathcal{E}_q$, which we will denote $(\sigma^* C)_q$, such that $(\sigma^* C)_q.E = 0$ for all exceptional curves $E \subset \mathcal{E}_q$. If $\mathcal{E}_q$ is a single point, then $(\sigma^* C)_q$ equals $\widetilde{C}$. If C and D have no common components, set

$$(\sigma^* C.\sigma^* D)_q = (\sigma^* C)_q.(\sigma^* D)_q - \sum_{p \notin \mathcal{E}_q} (\widetilde{C}.\widetilde{D})_p.$$

Since any birational map between smooth surfaces can be written as a composition of blowups and blowdowns (the inverse of blowups), property (1) in Lemma 2.1 implies that the rational numbers $(\sigma^* C).(\sigma^* D)$, for any pair of curves C and D on X, and the local intersection numbers $(\sigma^* C.\sigma^* D)_q$, for $q \in C \cap D$, do not depend on the desingularization $\sigma : Z \to X$.

The definitions and properties of $\sigma^* C$ and $(\sigma^* C)_q$ above, extend to all divisors in $\mathrm{Div}(X)$.

Definition 2.3. Let C and D be two divisors on a normal surface X and let $\sigma : Z \to X$ be any desingularization of X. For any pair of divisors $C, D \in \mathrm{Div}(X)$, define the *rational intersection number* of C and D to be

$$C.D = (\sigma^* C).(\sigma^* D)$$

and, for $q \in C \cap D$, the *rational intersection multiplicity* of C and D at q to be

$$(C.D)_q = (\sigma^* C.\sigma^* D)_q.$$

The projection formula holds for this definition of intersection multiplicity, since it holds for the smooth models.

3. Intersection theory on abelian coverings of surfaces

Let $\rho : X \to Y$ be an abelian branched covering with branch locus $\mathcal{B}$ and Galois group G. Let $\mathcal{C}$ be a finite union of curves on X. By the formulas of §2, in order to find intersections on X it is enough to find them on $\widetilde{X}$, for any desingularization of X. By doing an embedded resolution on $\mathcal{C} \cup \mathcal{B}$ sitting inside Y (i.e., blowing up until the preimage of $\mathcal{C} \cup \mathcal{B}$ is smooth with normal crossings) and pulling back the covering, the preimage of $\mathcal{C} \cup \mathcal{B}$ will be a finite union of smooth curves with normal crossings. The new covering X' will have a natural birational morphism to X and be a covering of the blowup of Y with branch locus the preimage of $\mathcal{B}$. Any desingularization of X' will be a desingularization of X. Thus, to simplify, we will assume throughout this section that $\mathcal{C} \cup \mathcal{B}$ is a finite union of smooth curves with normal crossings.

We will find intersection formulas for curves in $\rho^{-1}(\mathcal{C})$ in terms of the inertia I_C and stabilizer H_C subgroups associated to irreducible components $C \subset \mathcal{C}$. For any irreducible subvariety $S \subset Y$, define

$$\begin{aligned} I_S &= \{g \in G \quad | \quad g(x) = x, \quad \text{for all} \quad x \in \rho^{-1}(S)\}, \\ H_S &= \{g \in G \quad | \quad g(S') = S' \quad \text{for all} \quad S' \overset{\text{irred. comp.}}{\subset} \rho^{-1}(S)\}. \end{aligned}$$

Note that I_S is nontrivial if and only if $S \subset \mathcal{B}$.

Since the branch curves are smooth and intersect in normal crossings any point $p \in Y$ can lie on at most two curves in $\mathcal{B}$. Thus,

$$I_p = \begin{cases} (0) & \text{if } p \notin \mathcal{B} \\ I_C & \text{if } p \in C \setminus \mathcal{P} \subset \mathcal{B}. \\ I_C + I_D & \text{if } p \in \mathcal{P} \cap C \cap D \text{ for } C, D \subset \mathcal{B}, C \neq D \end{cases}$$

Note that the irreducible components above any curve $C \subset Y$ are in one-to-one correspondence with cosets of H_C in G. Thus, if we choose curves $C_0 \subset \rho^{-1}(C)$ for each $C \subset \mathcal{B}$, then we can label the curves in $\rho^{-1}(\mathcal{C})$ as

$$\alpha C_0$$

where C ranges in $\mathcal{C}$ and α ranges in G/H_C.

Let $\mathcal{P}$ be the set of intersection points on $\mathcal{C}$. Assume choices of C_0 for each $C \subset \mathcal{C}$ are fixed. For each triple (C, D, p), where $C, D \subset \mathcal{C}$ and $p \in \mathcal{P} \cap C \cap D$, choose $\mathrm{tw}(C, D, p) \in G$ such that

$$C_0 \cap \mathrm{tw}(C, D, p) D_0 \cap \rho^{-1}(p) \neq \emptyset.$$

The main goal of this section is to prove the following result.

Proposition 3.1. *For each $p \in S$ and $C, D \subset \mathcal{C}$, with $p \in C \cap D$,*

$$\alpha C_0.\beta D_0 = \sum_{p\in\mathcal{P}\cap C\cap D} \frac{|H_C \cap \operatorname{tw}(C,D,p)^{-1}H_D|}{|I_C||I_D|} c(C,D,p),$$

where

$$c(C,D,p) = \begin{cases} \dfrac{C.C}{|\mathcal{P}\cap C|} & \text{if } C = D \\ 1 & \text{otherwise.} \end{cases}$$

A particular case of this statement was proved in [H2]. The method we use here is similar.

The following lemma lists some easy to verify facts about the combinatorics of abelian coverings (see also [H2], p. 308.)

Lemma 3.2. *Let $C, D \subset \mathcal{C}$ be distinct curves and $a, b \in G$. The number of points in the intersection of aC_0 and bD_0 is*

$$|aC_0 \cap bD_0| = \sum_{p\in C\cap D\cap\mathcal{P}} \frac{|aH_C \cap \operatorname{tw}(C,D,p)^{-1}bH_D|}{|I_p|}. \tag{1}$$

For $p \in \mathcal{P}$ and $C, D \subset \mathcal{C}$ such that $p \in C \cap D$,

$$|I_p| = |I_C + I_D|. \tag{2}$$

For any $C \subset \mathcal{C}$, we have

$$|aC_0 \cap bC_0| \neq 0 \qquad \text{if and only if} \qquad aH_C = bH_D. \tag{3}$$

Furthermore, with the intersection numbers defined on X as in §2, we have the following immediate property.

Lemma 3.3. *Let $C \subset \mathcal{C}$. Then, for any $a \in G$,*

$$aC_0.aC_0 = C_0.C_0 = \frac{|H_C|}{|I_C|^2}C^2.$$

Proof. The first equality comes from transitivity of the action of G. Let $\sigma : Z \to X$ be a desingularization and let $\widetilde{\rho} = \sigma \circ \rho$. By the projection formula

$$\begin{aligned}(\widetilde{\rho}^* C).(\widetilde{\rho}^* C) &= (\deg \widetilde{\rho})C^2 \\ &= |G|C^2.\end{aligned} \tag{1}$$

Since as a divisor

$$\begin{aligned}\widetilde{\rho}^* C &= \sigma^*\Big(\sum_{a\in G/H_C} |I_C| aC_0\Big) \\ &= \sum_{a\in G/H_C} |I_C|\sigma^*(aC_0),\end{aligned}$$

this gives

$$(\widetilde{\rho}^* C).(\widetilde{\rho}^* C) = \frac{|G|}{|H_C|}|I_C|^2\sigma^*(C_0)^2 \tag{2}$$

Since $\sigma^*(C_0)^2 = C_0^2$, putting (1) and (2) together gives the desired formula. $\square$

To prove Proposition 3.1, it now suffices to show the following fact.

Lemma 3.4. *Let $C, D \subset \mathcal{C}$ be distinct curves and $q \in C_1 \cap D_1$, for any $C_1 \subset \rho^{-1}(C)$ and $D_1 \subset \rho^{-1}(D)$. Then*

$$(C_1.D_1)_q = \frac{1}{|I_C \cap I_D|}.$$

To calculate $(C_1.D_1)_q$ we need to find the pullbacks $\sigma^*(C_1)$ and $\sigma^*(D_1)$ in a desingularization $\sigma : Z \to X$. Here we will use the fact that $\mathcal{C} \cup \mathcal{B}$ is smooth with normal crossings.

As shown in [H1], Section I.3, if

$$\mathrm{H}_1(Y \setminus \mathcal{B}; \mathbb{Z}) \to G$$

is the group homomorphism defining the covering ρ and g_C is the image of a positive meridian loop around C, then I_C is generated by g_C.

Let $p \in C \cap D$ where $C, D \subset \mathcal{C}$ are distinct curves and let $q \in \rho^{-1}(p)$. Since the singularities of X must lie above singularities in $\mathcal{B}$, q is a singular point only if C and D are in $\mathcal{B}$. In [H1], Section I.4, it is shown that q is singular if and only if $I_C \cap I_D \neq (0)$.

Now suppose q is singular. Let $C_1 \subset \rho^{-1}(C)$ and $D_1 \subset \rho^{-1}(D)$ be the curves containing q. Note that this pair is unique. Let

$$\begin{aligned} rg_C + sg_D &= 0 \\ tg_D &= 0 \end{aligned}$$

generate the relations in $I_C + I_D$, where r, s, t are all positive.

Lemma 3.5. *The number $|I_C \cap I_D|$ can be computed from the numbers r, s, t as follows:*

$$|I_C \cap I_D| = \frac{t}{\gcd(s,t)}.$$

Proof. The subgroup $I_C \cap I_D$ is cyclic and generated by $sg_D = -rg_C$. Since t is the order of g_D in G, $\frac{t}{\gcd(s,t)}$ is the order of sg_D in G.

Let

$$\frac{t}{s} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \cfrac{\cdots}{m_{k-1} - \cfrac{1}{m_k}}}}$$

be a continued fraction expansion of $\frac{t}{s}$. Then as explained in [L] (see also [H2], pp. 309-310), X can be desingularized over q by replacing q with exceptional curves $E_1, \ldots, E_k$, where

$$E_i^2 = -m_i;$$

$$E_i.E_j = \begin{cases} 1 & \text{if } |i-j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and if $\widetilde{C}_1$ and $\widetilde{D}_1$ are the proper transforms of C_1 and D_1, then we have

$$E_i.\widetilde{C}_1 = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise;} \end{cases}$$

$$E_i.\widetilde{D}_1 = \begin{cases} 1 & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

Before going further with our analysis of the desingularization, we recall some properties of continued fraction expansions. For $i = 1, 2, 3, \ldots$, let $f_i \in \mathbb{Z}[x_1, \ldots, x_i]$ be polynomials so that if

$$\frac{b}{a} = m_1 - \cfrac{1}{m_2 - \cfrac{1}{m_3 - \cfrac{\cdots}{m_{k-1} - \frac{1}{m_k}}}},$$

then

$$\frac{b}{a} = \frac{f_k(m_1, \ldots, m_k)}{f_{k-1}(m_2, \ldots, m_k)}.$$

Clearly, the f_i can be defined recursively as follows.

$$\begin{aligned} f_0 &= 1; \\ f_1(x_1) &= x_1; \\ f_k(x_1, \ldots, x_k) &= x_1 f_{k-1}(x_2, \ldots, x_k) - f_{k-2}(x_3, \ldots, x_k). \end{aligned}$$

The following are some less obvious properties.

Lemma 3.6. *The polynomials f_k defined above satisfy, for $k \geq 2$,*

(1) $f_k(x_1, \ldots, x_k) = x_k f_{k-1}(x_1, \ldots, x_{k-1}) - f_{k-2}(x_1, \ldots, x_{k-2})$ *and*
(2) $f_k(x_1, \ldots, x_k) = f_k(x_k, \ldots, x_1)$.

Proof. Note that (2) holds for f_0 and f_1. Suppose (2) holds for f_{k-1} and f_{k-2}. We have

$$\begin{aligned} f_k(x_1, \ldots, x_k) &= x_1 f_{k-1}(x_2, \ldots, x_k) - f_{k-2}(x_3, \ldots, x_k) \\ &= x_1 f_{k-1}(x_k, \ldots, x_2) - f_{k-2}(x_k, \ldots, x_3) \\ &= x_1[x_k f_{k-2}(x_{k-1}, \ldots, x_2) - f_{k-3}(x_{k-2}, \ldots, x_2)] \\ &\quad - [x_k f_{k-3}(x_{k-1}, \ldots, x_3) - f_{k-4}(x_{k-2}, \ldots, x_3)] \\ &= x_k[x_1 f_{k-2}(x_2, \ldots, x_{k-1}) - f_{k-3}(x_3, \ldots, x_{k-1})] \\ &\quad - [x_1 f_{k-3}(x_2, \ldots, x_{k-2}) - f_{k-4}(x_3, \ldots, x_{k-2})] \\ &= x_k f_{k-1}(x_1, \ldots, x_{k-1}) - f_{k-2}(x_1, \ldots, x_{k-2}). \end{aligned}$$

This proves (1). Continuing on from the last line, we get

$$\begin{aligned} f_k(x_1, \ldots, x_k) &= x_k f_{k-1}(x_{k-1}, \ldots, x_1) - f_{k-2}(x_{k-2}, \ldots, x_1) \\ &= f_k(x_k, \ldots, x_1). \end{aligned}$$

□

Now let $\sigma : Z \to X$ be the desingularization formed by replacing each singularity $q \in X$ by exceptional sets as above.

If we write $\sigma^*(C_0)$ and $\sigma^*(D_0)$ as

$$\sigma^*(C_0) = \widetilde{C}_0 + \sum_{i=1}^{k} c_i E_i$$

$$\sigma^*(D_0) = \widetilde{D}_0 + \sum_{i=1}^{k} d_i E_i,$$

then we have

$$\begin{aligned}(C_0.D_0)_q &= \widetilde{C}_0.(\widetilde{D}_0 + \sum_{i=1}^{k} d_i E_i)\\ &= d_1\end{aligned}$$

and

$$\begin{aligned}(C_0.D_0)_q &= (\widetilde{C}_0 + \sum_{i=1}^{k} c_i E_i).\widetilde{D}_0\\ &= c_k.\end{aligned}$$

Thus, we get the equality $d_1 = c_k$. This number is computed in the following lemma.

Lemma 3.7. *With notation as above*

$$d_1 = c_k = \frac{1}{f_k(m_1,\ldots,m_k)} = \frac{\gcd(s,t)}{t}$$

Proof. To find the coefficients of $\sigma^*(C_0)$ we need to solve the matrix equation

$$(*)\qquad \begin{bmatrix} -m_1 & 1 & 0 & \ldots & 0\\ 1 & -m_2 & 1 & \ldots & 0\\ & & \ldots & & \\ 0 & \ldots & 0 & 1 & -m_k \end{bmatrix}\begin{bmatrix} c_1\\ c_2\\ \ldots\\ c_k\end{bmatrix} = \begin{bmatrix} -1\\ 0\\ 0\\ \ldots\\ 0\end{bmatrix}.$$

To find the coefficients of $\sigma^*(D_0)$ the matrix equation becomes

$$(**)\qquad \begin{bmatrix} -m_1 & 1 & 0 & \ldots & 0\\ 1 & -m_2 & 1 & \ldots & 0\\ & & \ldots & & \\ 0 & \ldots & 0 & 1 & -m_k \end{bmatrix}\begin{bmatrix} d_1\\ d_2\\ \ldots\\ d_k\end{bmatrix} = \begin{bmatrix} 0\\ 0\\ \ldots\\ 0\\ -1\end{bmatrix}.$$

Note that once we show that $d_1 = \frac{1}{f_k(m_1,\ldots,m_k)}$, property (2) of Lemma 3.6 implies that $c_k = d_1$. This is because solving $(*)$ for c_k is the same as solving $(**)$ for d_1 except that you flip the matrix in $(**)$ along the anti-diagonal. This is the same as replacing $(m_1,\ldots,m_k)$ by $(m_k,\ldots,m_1)$.

To find d_1 note that the first $k-1$ rows of the matrix equation $(**)$ give the following relations:

$$\begin{aligned}d_2 &= m_1 d_1\\ d_3 &= m_2 d_2 - d_1\\ &= (m_1 m_2 - 1)d_1\\ &= f_2(m_1,m_2)d_1\\ &\ldots\\ d_k &= m_{k-1}d_{k-1} - d_{k-2}\\ &= (m_{k-1}f_{k-2}(m_1,\ldots,m_{k-2}) - f_{k-3}(m_1,\ldots,m_{k-3}))d_1\\ &= f_{k-1}(m_1,\ldots,m_{k-1})d_1.\end{aligned}$$

The final row gives

$$d_{k-1} - m_k d_k = -1$$

or

$$\begin{aligned} 1 &= (m_k f_{k-1}(m_1, \dots, m_{k-1}) - f_{k-2}(m_1, \dots, m_{k-2}))d_1 \\ &= f_k(m_1, \dots, m_k) d_1. \end{aligned}$$

□

Putting together Lemma 3.5 and Lemma 3.7, we have proved Lemma 3.4 and hence Proposition 3.1.

4. Proof of main theorem

Let Y be a smooth surface and let $\mathcal{B}$ be a finite union of smooth curves on Y with normal crossings. Let $\mathcal{C}$ be a finite union of smooth curves with normal crossings containing $\mathcal{B}$ and supporting an ample divisor on Y. For each positive integer n, let $\rho_n : X_n \to Y$ be the abelian branched covering with branch locus $\mathcal{B}$ corresponding to the natural map

$$\mathrm{H}_1(Y \setminus \mathcal{B}; \mathbb{Z}) \to \mathrm{H}_1(Y \setminus \mathcal{B}; \mathbb{Z}/n\mathbb{Z}).$$

In [H2], it was shown, among other cases, that if X_n is smooth for all n, or equivalently, if $I_C \cap I_D$ is trivial whenever $C \cap D \neq \emptyset$, for all coverings ρ_n, then $b_1(X_n)$ is polynomial periodic. The outline of the proof goes as follows.

Let G be any finitely generated abelian group and let $G(n) = G \otimes \mathbb{Z}/n\mathbb{Z}$. Let $H(i), i = 1, \dots, k$ be a finite collection of subgroups of G and $H(i,n)$ their images in $G(n)$. Let $C(i,n)$ be the quotients groups $G(n)/H(i,n)$. We will denote elements of $C(i,n)$ by $g(i,n)$.

The following key lemma for the argument was proven in [H2], Proposition 3.2.

Proposition 4.1. *Let $T(n)$ be a bilinear form on the vector space*

$$V(n) = \oplus_{i=1}^{n} \mathbb{C}[C(i,n)].$$

For $i, j = 1, \dots, n$, let $\mathcal{P}_{i,j}$ be finite sets, and for $p \in \mathcal{P}_{i,j}$, let $c(i,j,p)$ be complex constants and let $\mathrm{tw}(i,j,p,n) \in G(n)$. Suppose that, for $g(i,n) \in C(i,n)$ and $h(j,n) \in C(j,n)$, $T(n)$ satisfies the formula

$$T(n)(g(i,n), h(j,n)) = \sum_{p \in \mathcal{P}_{i,j}} |g(i,n) \cap \mathrm{tw}(i,j,p,n)^{-1} h(i,n)| c(i,j,p),$$

where, the $g(i,n)$ and $h(j,n)$ between the absolute value signs are considered as subsets of $G(n)$. Suppose in addition that there is a $\mathrm{tw}(i,j,p) \in G$ such that $\mathrm{tw}(i,j,p,n)$ are all images of $\mathrm{tw}(i,j,p)$ under the quotient map from G to $G(n)$. Then the nullity of this form is polynomial periodic.

Now set $G = \mathrm{H}_1(Y \setminus \mathcal{B}; \mathbb{Z})$ and $G(n) = \mathrm{H}_1(Y \setminus \mathcal{B}; \mathbb{Z}/n\mathbb{Z})$. Let $\mathcal{P}$ be the points of intersection on $\mathcal{C}$, and for each distinct pair $C, D \in \mathcal{C}$ let $\mathcal{P}_{C,D}$ be equal to $\mathcal{P} \cap C \cap D$. In [H2], Lemma 6.1, we show that one can choose curves $C_{n,0} \subset \rho_n^{-1}(C)$, for each curve C in $\mathcal{C}$, and elements $\mathrm{tw}(C, D, p) \in G$, for all triples (C, D, p) where $C, D \subset \mathcal{C}$

are distinct and $p \in C \cap D$, such that the following holds. Let $\mathrm{tw}(C, D, p, n)$ be the image of $\mathrm{tw}(C, D, p)$ in $G(n)$ under the quotient map. Then

$$C_{n,0} \cap \mathrm{tw}(C, D, p, n) D_{n,0} \cap \rho_n^{-1}(p)$$

is nonempty for all n.

Using these $\mathrm{tw}(C, D, p, n)$ in the formula in Proposition 3.1 and multiplying rows and columns by the orders of the inertia subgroups, the intersection matrix has the form specified in Proposition 4.1. This finishes the proof of Theorem 1.2.

References

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