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by

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QUOTIENT FAMILIES OF MAPPING CLASSES

ERIKO HIRONAKA

ABSTRACT. We define quotient families of mapping classes parameterized by rational points on an interval, generalizing an example of R. C. Penner. This gives an explicit construction of families of mapping classes in a single flow-equivalence class of monodromies of a fibered 3-manifold M. The special structure of quotient families helps to compute global invariants of the mapping torus such as the Alexander polynomial, and (in the case when M is hyperbolic) the Teichmüller polynomial of the associated fibered face. These in turn give useful information about the homological and geometric dilatations of the mapping classes in the quotient family.

1. INTRODUCTION

In [10], R. C. Penner constructs a sequence of pseudo-Anosov mapping classes, sometimes called *Penner wheels*, with asymptotically small dilatations. In this paper, we define a generalization of Penner wheels called *quotient families*, and put them in the framework of the fibered face theory as discussed in [4], [8], and [11]. Specifically, we show that each quotient family corresponds naturally to a linear segment of a fibered face of a 3-manifold. Putting quotient families in the fibered face context helps to determine their Nielsen–Thurston classification, and, in the pseudo-Anosov case, makes it possible to compute dilatations via the Teichmüller polynomial.

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1.1. Pseudo-Anosov mapping classes, dilatations, and fibered faces.

Let S be a connected oriented surface of finite type with negative Euler characteristic $\chi(S)$. A *mapping class* $\phi : S \to S$ is an orientation preserving homeomorphism modulo isotopy (both the mapping class and the isotopies are assumed to fix the boundary if S has boundary). The Nielsen–Thurston classification states that mapping classes are either periodic, reducible, or pseudo-Anosov, where (S, ϕ) is *pseudo-Anosov* if ϕ preserves a pair of transverse measured singular *stable* and *unstable* foliations $(\mathcal{F}^{\pm}, \nu^{\pm})$ that satisfy $\phi^*(\nu^{\pm}) = \lambda^{\pm 1}\mu^{\pm}$ for some $\lambda > 0$ [12]. The constant

$$\lambda(\phi) = \lambda$$

is uniquely determined by (S, ϕ) and is called the *dilatation* of ϕ. The singularities of $\mathcal{F}^{\pm}$ are called the *singularities of* ϕ. See, for example, [3] for more details.

In [10], Penner constructs a sequence of pseudo-Anosov mapping classes (R_g, ψ_g), for $g \geq 3$, where R_g is a closed surface of genus $g \geq 2$ and $\lambda(\psi_g)^g \leq 11$. Using this, he shows that the minimum expansion factor l_g for a pseudo-Anosov mapping class on a genus g surface behaves asymptotically like $\log(l_g) \asymp \frac{1}{g}$ as a function of g; that is, there is a constant $C \geq 1$ so that

$$\frac{1}{Cg} \ \leq \ \log(l_g) \ \leq \ \frac{C}{g}.$$

Since then the Thurston–Fried–McMullen fibered face theory (recalled below) has been used to show that pseudo-Anosov mapping classes with bounded *normalized dilatation* $L(S, \phi) = \lambda(\phi)^{|\chi(S)|}$ are naturally grouped together into dynamical families: These arise as subcollections of monodromies of hyperbolic fibered 3-manifolds that have first Betti number greater than or equal to 2 [8] and are parameterized by rational points on compact subsets of fibered faces. Furthermore, the Farb–Leininger–Margalit universal finiteness theorem implies that any family of pseudo-Anosov mapping classes with bounded normalized dilatations is contained in the set of monodromies of a finite set of fibered 3-manifolds up to fiberwise Dehn fillings [2].

1.2. Penner wheels.

Penner wheels are defined as follows. Consider the genus g surface R_g as a surface with rotational symmetry of order g fixing two points, as drawn in Figure 1, and let ζ_g be the counterclockwise rotation by the angle $\frac{2\pi}{g}$. For a simple closed curve γ on a surface, let δ_γ be the right Dehn twist centered at γ. Let $\eta_g = \delta_{c_g}\delta_{b_g}^{-1}\delta_{a_g}$ be the product of Dehn twists centered along the labeled curves a_g, b_g, and c_g in Figure 1. Then Penner's sequence consists of the pairs (R_g, ψ_g), where $\psi_g = \zeta_g \eta_g$.

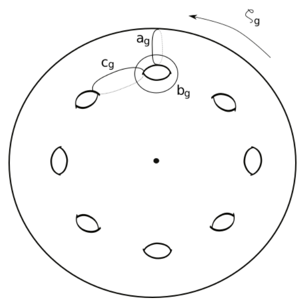

FIGURE 1. Penner wheel on a surface of genus g with rotational symmetry fixing two points. The central dot in the figure indicates one of the two points, and the other is hidden behind it.

1.3. Quotient families.

To define quotient families, consider a triple $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$, where $\widetilde{S}$ is an oriented surface of infinite type and

$$\widetilde{\zeta}, \widetilde{\eta} : \widetilde{S} \to \widetilde{S}$$

are homeomorphisms satisfying the following:

(1) $\widetilde{\zeta}$ generates a properly discontinuous, orientation-preserving, fixed-point free, infinite cyclic action on $\widetilde{S}$;
(2) $\widetilde{S}/\widetilde{\zeta}$ is a surface of finite type;
(3) the action of $\widetilde{\zeta}$ has a fundamental domain Σ_0, a compact, connected, oriented surface of finite type with boundary $\partial\Sigma_0$, satisfying
  (i) $\widetilde{\zeta}(\Sigma_0) \cap \Sigma_0 \subset \partial\Sigma_0$, and
  (ii) $\widetilde{\zeta}^2(\Sigma_0) \cap \Sigma_0 = \emptyset$;
  and
(4) the support of $\widetilde{\eta}$ is strictly contained in

$$\Sigma_0 \cup \widetilde{\zeta}\Sigma_0 \cup \cdots \cup \widetilde{\zeta}^{m_0}\Sigma_0$$

for some finite m_0.

We say that the triple $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ forms a *template* of *width* m_0. See Figure 2.

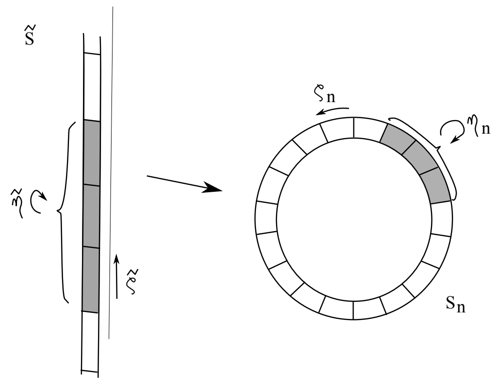

FIGURE 2. An illustration of a template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ of width $m_0 = 2$ and the quotient mapping class $(S_n, \zeta_n^k \circ \eta_n)$. The supports of $\widetilde{\eta}$ on $\widetilde{S}$ and η_n on S_n are shaded.

Let $I_{m_0}(\mathbb{Q})$ be the rational points on the open interval $I_{m_0} = (0, \frac{1}{m_0})$. From a template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ with width m_0, we define an associated *quotient family* $Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ parameterized by $I_{m_0}(\mathbb{Q})$. For $\mathfrak{c} \in I_{m_0}(\mathbb{Q})$, where $\mathfrak{c} = \frac{k}{n}$ is in reduced form, define a mapping class $\widetilde{\phi}_{\mathfrak{c}} : \widetilde{S} \to \widetilde{S}$ as follows. Let $\widetilde{\eta}_n$ be the composition

$$\widetilde{\eta}_n = \underset{r \in \mathbb{Z}}{\circ} \widetilde{\zeta}^{rn} \widetilde{\eta} \widetilde{\zeta}^{-rn}.$$

(See Figure 2). This is well defined since $n > m_0$ implies that the supports of $\widetilde{\zeta}^{rn} \widetilde{\eta} \widetilde{\zeta}^{-rn}$ are disjoint for distinct r.

Let

$$\widetilde{\phi}_{\mathfrak{c}} = \widetilde{\zeta}^{\overline{k}_n} \circ \widetilde{\eta}_n,$$

where $\overline{k}_n k = 1(\text{mod } n)$. Since $\widetilde{\eta}_n$ is invariant under conjugation by $\widetilde{\zeta}^n$, it defines a well-defined homeomorphism η_n on the quotient space $S_{\mathfrak{c}} = \widetilde{S}/\zeta^n$. Similarly, $\widetilde{\zeta}$ defines a homeomorphism ζ_n on the quotient space $S_{\mathfrak{c}}$. Let

$$\phi_{\mathfrak{c}} = (\zeta_n)^{\overline{k}_n} \circ \eta_n.$$

Then $\widetilde{\phi}_{\mathfrak{c}}$ is a lift of $\phi_{\mathfrak{c}}$ by the covering map $\widetilde{S} \to S_{\mathfrak{c}}$.

The *quotient family* associated to the template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ is defined by

$$Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta}) = \{(S_{\mathfrak{c}}, \phi_{\mathfrak{c}}) \mid \mathfrak{c} \in I_{m_0}(\mathbb{Q})\}.$$

Example 1.1 (Penner Wheel as a sequence in a quotient family)**.** Let R_g^0 be the surface of genus g and two boundary components obtained by removing two disks on R_g surrounding the fixed points of ζ_n, and let ϕ_g^0 be the restriction of ϕ_g on R_g^0. Let $\widetilde{S}$ be as in Figure 3, a homeomorphism ζ

acting as vertical translation on a fundamental domain Σ_0 and its orbits. Let $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ be distinguished points on $\widetilde{S}$, and let $\widetilde{\eta} = \delta_{\widetilde{c}}\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}}$. Then (R_g^0, ϕ_g^0), $g \geq 2$ is a sequence in the quotient family associated to $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ parameterized by the sequence $\frac{1}{g} \in \mathcal{I}_1(\mathbb{Q})$.

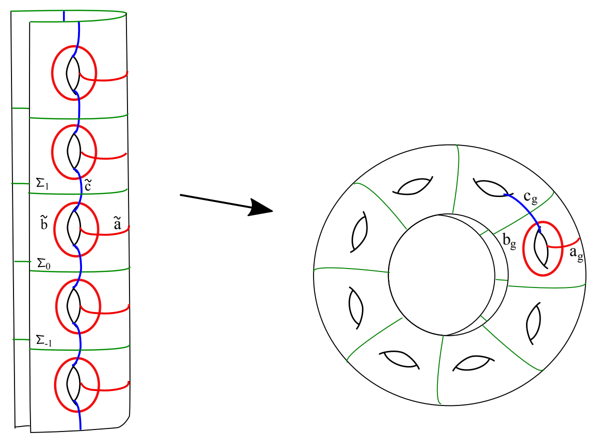

FIGURE 3. Cyclic covering $\widetilde{S}$ and quotient surface S_g for Penner's example.

1.4. Fibered faces and parameterizations of flow-equivalence classes.

Let $\mathrm{MCG}(S)$ be the group of mapping classes defined on a connected oriented surface of finite type S. Fibered face theory gives a way to partition the set of all mapping classes

$$\mathrm{MCG} = \bigcup_S \mathrm{MCG}(S)$$

into families with related dynamics. Each mapping class (S, ϕ) defines

(1) a 3-manifold M with mapping torus structure
$$M = [0,1] \times S/(x,1) \sim (\phi(x), 0);$$
(2) a distinguished fibration $\rho : M \to S^1$ defined by projection onto the second coordinate with *monodromy* (S, ϕ); and
(3) a one-dimensional oriented *suspension flow*, or *foliation with oriented leaves*, $\mathcal{L}$ on M whose leaves are the images of the leaves $\mathbb{R} \times \{x\}$ under the cyclic covering map $\mathbb{R} \times S \to M$ corresponding to the kernel of the map $\rho_* : \pi_1(M) \to \mathbb{Z}$.

Two mapping classes are said to be *flow-equivalent* if they define the same pair $(M, \mathcal{L})$. The induced homomorphism $\rho_* : H_1(M;\mathbb{Z}) \to \mathbb{Z}$ defines an element $\alpha \in H^1(M;\mathbb{R})$, called a *fibered element*.

In [11], William P. Thurston defines a semi-norm $|| \; ||$ on $H^1(M;\mathbb{R})$ with a convex polygonal unit norm ball, with compact closure if M is hyperbolic. Each cone V_F over an open top-dimensional face F is either *fibered* (all primitive integral elements are fibered) or contains no fibered elements. Since there is one primitive integral element on the ray through each rational points on the fibered face F, it follows that the union of rational points on fibered faces of oriented 3-manifolds has a natural surjection onto the set of all mapping classes on oriented surfaces of finite type. Explicitly, for α, a fibered element in a fibered cone V_F, let $\overline{\alpha}$ be its projection onto F along the rational ray, and let (S_α, ϕ_α) be its monodromy. For each fibered 3-manifold M and fibered face $F \subset H^1(M;\mathbb{R})$, define

$$\begin{array}{rcl} \mathfrak{f}_F : F(\mathbb{Q}) & \to & \mathrm{MCG} \\ \overline{\alpha} & \mapsto & (S_\alpha, \phi_\alpha) \end{array}$$

taking each $\overline{\alpha}$ to the monodromy (S_α, ϕ_α), where $\alpha \in V_F$ is the primitive integral element that is a positive multiple of $\overline{\alpha}$.

Let $\mathcal{C}$ be the set of all fibered faces, and let $\mathfrak{C}$ be the set of all flow-equivalence classes. Then we have a surjection

$$\mathfrak{f} : \mathcal{C} \to \mathfrak{C}$$

taking fibered faces to corresponding flow-equivalence classes.

Our first result shows that quotient families have natural parameterizations by linear segments on a fibered face.

Theorem A. *Each quotient family Q is contained in some flow equivalence class $\mathfrak{F}$. Let F be a fibered face with $\mathfrak{c}(F) = \mathfrak{F}$. Then there is an embedding*

$$\iota : I_{m_0} \hookrightarrow F,$$

such that

(1) *the image $\iota(I_{m_0})$ is a linear section of F in $H^1(M;\mathbb{R})$,*
(2) *ι restricts to a map $I_{m_0}(\mathbb{Q}) \to F(\mathbb{Q})$, and*
(3) *for all $\mathfrak{c} \in I_{m_0}(\mathbb{Q})$,*

$$(S_\mathfrak{c}, \phi_\mathfrak{c}) = \mathfrak{f}_F(\iota(c)).$$

Now consider the set of all pseudo-Anosov mapping classes on all oriented surfaces of finite type (possibly with boundary and/or punctures) $\mathcal{P} \subset \mathrm{MCG}$. By a result of Thurston [12] the (interior of the) mapping torus M of (S, ϕ) is hyperbolic if and only if (S, ϕ) is pseudo-Anosov.

Thus, Theorem A has the following immediate corollary.

Corollary 1.2. *A quotient family Q is contained in $\mathcal{P}$ if and only if $Q \cap \mathcal{P} \neq \emptyset$.*

1.5. Fibered faces and bounded normalized dilatations.

Let M be a hyperbolic 3-manifold with fibered face $F \subset H^1(M;\mathbb{R})$. David Fried [4] shows that the function

$$\alpha \mapsto \log \lambda(\phi_\alpha)$$

defined for α, a primitive integral element of the cone $V_F = F \cdot \mathbb{R}^+$, extends to a continuous convex function on V_F that is homogeneous of degree -1 and goes to infinity toward the boundary of V_F. The Thurston norm $|| \ ||$ on $H^1(M;\mathbb{R})$ has the property that $||\alpha|| = |\chi(S_\alpha)|$ for all integral elements α on a fibered cone [11]. Noting that normalized dilatation is the post-composition of Fried's function with the exponential function, we have the following.

Theorem 1.3 (Fried [4])**.** *Given a flow-equivalence class $\mathfrak{F} \subset \mathcal{P}$ and fibered face F with $\mathfrak{c}(F) = \mathfrak{F}$, the normalized dilatation function L extends uniquely to a continuous function*

$$L : V_F \to \mathbb{R}$$

that is constant on rays through the origin, and on F it defines a convex function that goes to infinity toward the boundary of F.

By this theorem, it suffices to think of L as a function on F thought of as the quotient of V_F by positive scalar multiplication.

Benson Farb, Christopher Leininger, and Dan Margalit's universal finiteness theorem states conversely that for any $C > 0$, there is a finite set of fibered 3-manifolds Ω_C such that for any pseudo-Anosov map (S,ϕ) with $L(S,\phi) < C$, there is some $M \in \Omega_C$ such that M is the mapping torus for (S^0,ϕ^0), where (S^0,ϕ^0) is the *fully-punctured* mapping class obtained from (S,ϕ) by removing the singularities of the ϕ ([2]).

1.6. Behavior of normalized dilatations and stability.

Our second result deals with the behavior of the normalized dilatations of a quotient family with pseudo-Anosov elements. We say $Q = Q(\widetilde{S},\widetilde{\zeta},\widetilde{\eta})$ is a *stable* family if there are integers $m_1 > 0$ such that for all for $x \in \Sigma_0$ and $m > m_1$, we have

$$(\widetilde{\zeta}\widetilde{\eta})^{m+1}(x) = \widetilde{\zeta}(\widetilde{\zeta}\widetilde{\eta})^m(x),$$

and Q is *bi-stable* if both $Q(\widetilde{S},\widetilde{\zeta},\widetilde{\eta})$ and $Q(\widetilde{S},\widetilde{\zeta}^{-1},\widetilde{\eta}^{-1})$ are stable.

Consider the function defined by

$$\begin{array}{rcl}\widetilde{\phi} : \widetilde{S} & \to & \widetilde{S} \\ x & \mapsto & \widetilde{\zeta}^{r-m_1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r}(x),\end{array}$$

where r is any integer such that $\widetilde{\zeta}^{-r}(x) \in \Sigma_0$. In Lemma 2.15, we show that if Q is stable, then $\widetilde{\phi}$ is a continuous open map, and if Q is bi-stable, then $\widetilde{\phi}$ is a homeomorphism.

Theorem B. *If Q is bi-stable, then we have the following:*

(1) *the map $\widetilde{\phi}$ defines a mapping class on $\widetilde{S}$ that commutes with the action of $\mathbb{Z}$, and hence defines a mapping class (S_0, ϕ_0), where $S_0 = \widetilde{S}/\widetilde{\zeta}$ and ϕ_0 is the mapping class on S induced by $\widetilde{\phi}$;*
(2) *the map $\iota : I_{m_0} \to F$ extends to the half open interval $[0, \frac{1}{m_0})$ and the monodromy at 0 equals (S_0, ϕ_0); and*
(3) *the value of the normalized stretch-factor $L(S_{\mathfrak{c}}, \phi_{\mathfrak{c}})$ converges to a value L_0 with*
$$1 < L_0 < \infty$$
as $\mathfrak{c}$ approaches 0.

If Q is not stable, then the map $\iota(t)$ converges to a point on the boundary of F as t approaches 0.

By Fried's theorem, we have the following immediate corollary.

Corollary 1.4. *If Q is bi-stable, then*
$$\lim_{c \to 0} L(S_{\mathfrak{c}}, \phi_{\mathfrak{c}}) = L(S_0, \phi_0),$$
and if Q is not stable, then $\lim_{c \to 0} \iota(c)$ lies on the boundary of F and
$$\lim_{c \to 0} L(S_{\mathfrak{c}}, \phi_{\mathfrak{c}}) = \infty.$$

Remark 1.5. It is not known to the author whether stability implies bi-stability for Q. However, bi-stability is a necessary condition for $\widetilde{\phi}$ to be invertible.

Remark 1.6. Up to now explicit examples and partial generalizations of Penner wheels have been studied without putting them in the context of fibered faces (see [1], [13], [14]). One benefit of seeing quotient families as elements of a single fibered face is the possibility of getting explicit defining equations for the geometric and homological stretch-factors via the Teichmüller and Alexander polynomials. We carry out some calculations in §3.

1.7. Idea of proofs.

In [8], Curtis T. McMullen makes the key observation that given a fibered hyperbolic 3-manifold M with monodromy $\phi : S \to S$ and pseudo-Anosov flow $\mathcal{L}$, one can study the transverse measures on $\mathcal{L}$ defined by points on the associated fibered face by lifting $\widetilde{\mathcal{L}}$ to the maximal abelian covering $\widetilde{M}^{\text{ab}}$ of M.

Topologically one can think of $\widetilde{M}^{\text{ab}}$ as a product

$$\widetilde{M}^{\text{ab}} = \mathbb{R} \times \widetilde{S}$$

for a surface $\widetilde{S}$ of infinite type, where the topologically trivial foliation $\widetilde{\mathcal{L}}$ with leaves

$$\{\mathbb{R} \times \{x\} \ : \ x \in \widetilde{S}\}$$

is a lift of $\widetilde{\mathcal{L}}$. The subtle geometric information associated to a point on the fibered cone, such as invariant transverse measure on $\mathcal{L}$ and expansion factor of the monodromy action, can be translated to information about the action of the covering automorphism group H^{ab} on the module of transversals of $\widetilde{\mathcal{L}}$.

In this paper, we make use of this idea, but in reverse. To build the desired 3-manifold M, a foliation $\mathcal{L}$, and a fibered face F from a template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$, we begin with the 3-manifold of infinite type: $\widetilde{M}' = \mathbb{R} \times \widetilde{S}$, a fixed point free, properly discontinuous action of a rank-2 free abelian group $H' = \langle T', Z' \rangle$ on $\widetilde{M}'$ that respects the product structure and a foliation $\widetilde{\mathcal{L}}'$ preserved by H'. We know that topologically $\widetilde{\mathcal{M}}'$ would have to be homeomorphic to an abelian covering of M, but the group action H' and the foliation $\widetilde{\mathcal{L}}'$ will typically need to be adjusted. To do this we use cutting and pasting on $\widetilde{M}'$ to create a new (though homeomorphic) $\widetilde{M}$ and an adjusted foliation $\widetilde{\mathcal{L}}$ and a covering automorphism group H. Then $M = \widetilde{M}/H$ and the quotient $\mathbb{R}^H$ of $H_1(M;\mathbb{R})$ defines a 2-dimensional dual subspace $W \subset H^1(M;\mathbb{R})$. To prove Theorem A, we define an inclusion

$$\iota : I_{m_0} \hookrightarrow F \cap W$$

where F is a fibered face of M, so that each mapping class in Q parameterized by a rational point of I_{m_0} is the monodromy parameterized by its image in F under ι.

By fibered face theory and Theorem A, there are two possible behaviors for the normalized dilatations of $(S_{\mathfrak{c}}, \phi_{\mathfrak{c}})$ as $\mathfrak{c} \to 0$ depending on whether $\iota(c)$ converges to the boundary of F (implying unboundedness) or to an interior point of f as $\mathfrak{c}$ approached 0 (implying convergence). In §2.3, we prove Theorem B by showing that Q is stable if and only if it is possible to find a quotient mapping class corresponding to the lower limit point of I_{m_0} such that ι extends. In §3, we illustrate how the results can be applied to give explicit computations of invariants for a given quotient family, namely two-variable specializations of the Alexander polynomial of M and the Teichmüller polynomial for F.

2. Construction of a One-Dimensional Flow-Equivalence Class

We begin this section by describing our construction in §2.1 starting with a template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ and using cutting and pasting on $\mathbb{R}\times\widetilde{S}$ to build an $H = \mathbb{Z}\times\mathbb{Z}$-covering $\widetilde{M}$ of our desired 3-manifold M and a one-dimensional foliation $\widetilde{\mathcal{L}}$ on $\widetilde{M}$. Using this we prove Theorem A in §2.2 and Theorem B in §2.3.

2.1. Building a manifold with a flow using coverings.

Our construction has three parts. We first build a topological model $\widetilde{M}'$ for the abelian covering of M and a projection $\mathfrak{h}' : \widetilde{M}' \to \mathbb{R}\times\mathbb{R}$ (§2.1.1). The map $\mathfrak{h}'$ allows us to specify a cutting locus whose connected components are preimages under $\mathfrak{h}'$ of connected line segments in $\mathbb{R}\times\mathbb{R}$ (§2.1.2). The cutting locus is re-pasted together using mappings defined by $\widetilde{\eta}$ (§2.1.3). This results in a new 3-manifold $\widetilde{M}$ with a one-dimensional foliation $\widetilde{\mathcal{L}}$ and an $H = \mathbb{Z}\times\mathbb{Z}$ action that preserves the leaves of $\widetilde{\mathcal{L}}$ and makes $\widetilde{M}$ a covering of $M = \widetilde{M}/H$. The map $\mathfrak{h}'$ defines a function $\mathfrak{h} : \widetilde{M} \to \mathbb{R}\times\mathbb{R}$ with jump discontinuities at the cutting loci. This will be useful for describing and visualizing the cross-sections of $\widetilde{\mathcal{L}}$, corresponding to rational points in I_{m_0}, all of which are homeomorphic to $\widetilde{S}$ but embedded differently in $\widetilde{M}$.

2.1.1. *Set up.* Let Σ_0 be a fundamental domain on $\widetilde{S}$ for the map $\widetilde{\zeta}$, with the following properties:

(i) Σ_0 is a connected closed submanifold of $\widetilde{S}$ (with boundary);
(ii) $\widetilde{\zeta}(\Sigma_0)\cap\Sigma_0$ is a finite disjoint union of closed arcs on the boundary $\partial\Sigma_0$ of Σ_0;
(iii) $\widetilde{\zeta}^2(\Sigma_0)\cap\Sigma_0 = \emptyset$; and
(iv) $\widetilde{S} = \bigcup_{i\in\mathbb{Z}} \zeta^i(\Sigma_0)$.

Let $\widetilde{M}' = \mathbb{R}\times\widetilde{S}$ and let $\widetilde{\mathcal{L}}'$ be the trivial 1-dimensional foliation defined on $\widetilde{M}'$ with oriented leaves

$$\{\mathbb{R}\times\{y\} \mid y\in\widetilde{S}\}.$$

The manifold $\widetilde{M}'$ comes with a natural $H' = \mathbb{Z}\times\mathbb{Z}$ action generated by the orientation preserving homeomorphisms T' and Z' defined by

$$\begin{array}{rcl} T' : \widetilde{M}' & \to & \widetilde{M}' \\ (t,x) & \mapsto & (t-1,x) \end{array}$$

and

$$\begin{array}{rcl} Z' : \widetilde{M}' & \to & \widetilde{M}' \\ (t,x) & \mapsto & (t, \widetilde{\zeta}(x)). \end{array}$$

Note that T' and Z' commute, act freely and properly discontinuously on $\widetilde{M}'$, and preserve the leaf structure of $\widetilde{\mathcal{L}}'$.

We now choose a continuous projection $\mathfrak{h}' : \widetilde{M}' \to \mathbb{R} \times \mathbb{R}$ so that powers of T' (and Z') correspond to integer shifts in the first (and second) coordinate. Let h be a continuous surjective function $h : \widetilde{S} \to \mathbb{R}$ (see Figure 4 for an illustration):

(1) each fiber of h is an immersed union of simple closed curves (rel. punctures) on $\widetilde{S}$ that split $\widetilde{S}$ into exactly two pieces;
(2) $\Sigma_0 \cap \widetilde{\zeta}^{-1}(\Sigma_0) = h^{-1}(0)$; and
(3) $h(\widetilde{\zeta}^k(x)) = h(x) + k$ for all $x \in \widetilde{S}$ and $k \in \mathbb{Z}$.

Let $\mathfrak{h}' : \widetilde{M}' \to \mathbb{R} \times \mathbb{R}$ be defined by $\mathfrak{h}' = \mathrm{id} \times h$.

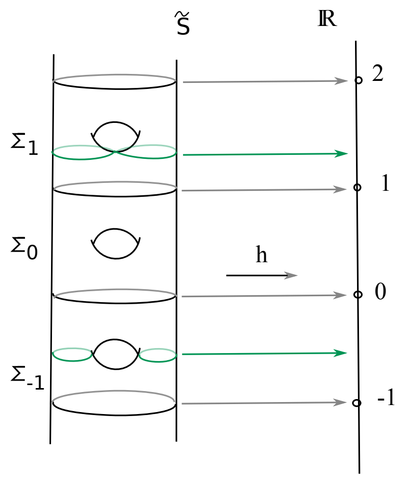

FIGURE 4. An example of a height function h on $\widetilde{S}$ that commutes with the action of $\widetilde{\zeta}$. The fundamental domain Σ_0 and all its images under powers of $\widetilde{\zeta}$ have genus one and two boundary components.

2.1.2. *Cutting.* We construct a cutting locus $\mathcal{G}'$ on $\widetilde{M}'$ by first defining a locus on $\mathbb{R} \times \mathbb{R}$, and taking the preimage by the map $\mathfrak{h}'$. Let $\Gamma_{0,0} \subset \mathbb{R} \times \mathbb{R}$ be the straight line segment connecting $(0,0)$ to $(\frac{1}{2}, m_0)$, and, for $(a,b) \in$

$\mathbb{R} \times \mathbb{R}$, let $\Gamma_{a,b} = (a,b) + \Gamma_{0,0}$ be the parallel translate of $\Gamma_{0,0}$ by (a,b) (see Figure 5). Let $X'_{a,b} = \mathfrak{h}'^{-1}(\Gamma_{a,b})$.

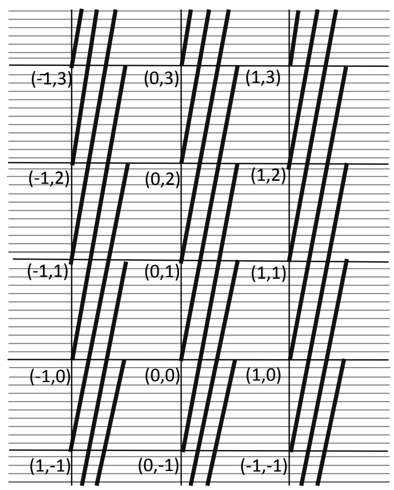

FIGURE 5. The image of the cutting loci $\Gamma_{a,b}$ in $\mathbb{R} \times \mathbb{R}$. Horizontal lines indicate the flow $\mathfrak{h}'(\widetilde{\mathcal{L}}')$.

Then we have the following:

(1) the projection $\sigma' : \widetilde{M}' \to \widetilde{S}$ to the second coordinate defines identifications $S\sigma'_{(a,b)} : X'_{a,b} \xrightarrow{\sim} \widetilde{\zeta}^b \Sigma_0 \cap \cdots \cap \widetilde{\zeta}^{m_0+b}\Sigma_0 S$ for each $(a,b) \in \mathbb{Z} \times \mathbb{Z}$;

(2) the maps T' and Z' satisfy

$$\mathfrak{h}'(T'(x,y)) = \mathfrak{h}'(x,y) + (-1,0) \quad \text{and} \quad \mathfrak{h}'(Z'(x,y)) = \mathfrak{h}'(x,y) + (0,1);$$

(3) $X'_{a,b} \cap X_{a',b'} = \emptyset$ for $(a,b) \neq (a',b')$; and

(4) $T'(X_{a,b}) = X'_{a-1,b}$ and $Z'(X'_{a,b}) = X'_{a,b+1}$.

Cut $\widetilde{M}'$ along

$$\mathcal{G}' = \bigcup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} X'_{a,b}\,;$$

i.e., remove the locus $\mathcal{G}'$ from $\widetilde{M}'$ replacing each $X'_{a,b}$ with two copies $X^{\text{left}}_{a,b}$ and $X^{\text{right}}_{a,b}$ of $X'_{a,b}$ intersecting only at the preimages of the endpoints of $\Gamma_{a,b}$ under $\mathfrak{h}'^{-1}$, and let $\widetilde{M}^{\text{cut}}$ be the result. Let

$$\mathcal{G}^{\text{cut}} = \bigcup_{(a,b)\in\mathbb{Z}\times\mathbb{Z}} X^{\text{left}}_{a,b} \cup X^{\text{right}}_{a,b}.$$

Let $q' : \widetilde{M}^{\mathrm{cut}} \to \widetilde{M}'$ be the quotient map identifying both $X^{\mathrm{left}}_{a,b}$ and $X^{\mathrm{right}}_{a,b}$ with $X'_{a,b}$.

Let $T^{\mathrm{cut}}, Z^{\mathrm{cut}} : \widetilde{M}^{\mathrm{cut}} \to \widetilde{M}^{\mathrm{cut}}$ be the lifts of the maps T' and Z' so that

$$T^{\mathrm{cut}}(X^{\mathrm{right}}_{a,b}) = X^{\mathrm{right}}_{a-1,b}, \quad T^{\mathrm{cut}}(X^{\mathrm{left}}_{a,b}) = X^{\mathrm{left}}_{a-1,b},$$

and

$$Z^{\mathrm{cut}}(X^{\mathrm{right}}_{a,b}) = X^{\mathrm{right}}_{a,b+1}, \quad Z^{\mathrm{cut}}(X^{\mathrm{left}}_{a,b}) = X^{\mathrm{left}}_{a,b+1},$$

Figure 6 gives a local illustration of the associated cuts along $\Gamma_{a,b}$ in $\mathbb{R} \times \mathbb{R}$, replacing each $\Gamma_{a,b}$ with a left and right copy attached along their boundaries.

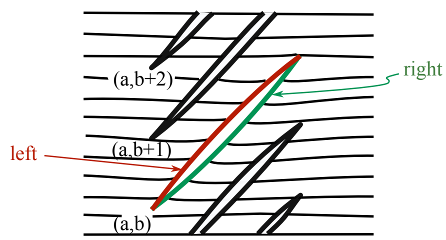

FIGURE 6. A local illustration of the associated cuts along $\Gamma_{a,b}$ in $\mathbb{R} \times \mathbb{R}$.

Let $\mathcal{L}^{\mathrm{cut}}$ be the lift of $\mathcal{L}'$ to $\widetilde{M}^{\mathrm{cut}}$. Then $\widetilde{\mathcal{L}}^{\mathrm{cut}}$ is a foliation of $\widetilde{M}^{\mathrm{cut}}$ by intervals whose boundaries lie on the left or right cut loci.

By the definitions, we have the following.

Lemma 2.1. *The maps T^{cut} and Z^{cut} are commuting self-homeomorphisms of $\widetilde{M}^{cut}$ and generate a $\mathbb{Z} \times \mathbb{Z}$ action on $\widetilde{M}^{cut}$ that is properly discontinuous and free and preserves the leaf structure of $\widetilde{\mathcal{L}}^{cut}$.*

2.1.3. *Pasting.* We are now ready to define the new manifold $\widetilde{M}$ and foliation $\widetilde{\mathcal{L}}$. For each $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, we define an identification $X^{\mathrm{left}}_{a,b}$ to $X^{\mathrm{right}}_{a,b}$ as follows. Let $\mathrm{id}_{\mathrm{r}} : X'_{a,b} \to X^{\mathrm{right}}_{a,b}$ be the identification of $X'_{a,b}$ with $X^{\mathrm{right}}_{a,b}$, and id_ℓ be the identification of $X'_{a,b}$ with $X^{\mathrm{left}}_{a,b}$. Let

$$p_{a,b} : X^{\mathrm{left}}_{a,b} \to X^{\mathrm{right}}_{a,b}$$

be the *pasting map* defined by the composition

$$X^{\mathrm{left}}_{a,b} \xleftrightarrow{\mathrm{id}_\ell} X'_{a,b} \xrightarrow{\sigma'} \sigma'(X_{a,b}) \xrightarrow{\widetilde{\zeta}^b \widetilde{\eta} \widetilde{\zeta}^{-b}} \sigma'(X_{a,b}) \xrightarrow{(\sigma')^{-1}} X'_{a,b} \xleftrightarrow{\mathrm{id}_{\mathrm{r}}} X^{\mathrm{right}}_{a,b}.$$

Lemma 2.2. *For each* $(a,b) \in \mathbb{Z}$, *we have*

$$T^{cut} \circ p_{a,b} = p_{a-1,b} \circ T^{cut} \qquad Z^{cut} \circ p_{a,b} = p_{a,b+1} \circ Z^{cut}.$$

Let $\widetilde{M}$ be obtained from $\widetilde{M}^{\text{cut}}$ by pasting each $X_{a,b}^{\text{left}}$ to $X_{a,b}^{\text{right}}$ by the map $p_{a,b}$; that is, $x \in X_{a,b}^{\text{left}}$ is identified with $y \in X_{a,b}^{\text{right}}$ if

$$y = p_{a,b}(x).$$

Let

$$q : \widetilde{M}^{\text{cut}} \to \widetilde{M}$$

be the quotient map,

$$X_{a,b} = q(X_{a,b}^{\text{left}})(= q(X_{a,b}^{\text{right}})),$$

and $\mathcal{G} = q(\mathcal{G}^{\text{cut}})$. Let

$$\tau : \widetilde{M} \to \widetilde{M}^{\text{cut}}$$

be the (non-continuous) lifting map that is the inverse map of q on $\widetilde{M} \setminus \mathcal{G}$ and the inverse of the restriction $q : X_{a,b}^{\text{left}} \to q(X_{a,b}^{\text{left}})$ on $X_{a,b}$.

By Lemma 2.2, T^{cut} and Z^{cut} define homeomorphisms T and Z on $\widetilde{M}$ with the property that

$$q \circ T^{\text{cut}} = T \circ q, \qquad q \circ Z^{\text{cut}} = Z \circ q.$$

Let $\widetilde{\mathcal{L}}$ be the foliation on $\widetilde{M}$ defined by $\widetilde{\mathcal{L}}^{\text{cut}}$ as follows. The inclusion

$$\widetilde{M}' \setminus \mathcal{G}' \subset \widetilde{M}^{\text{cut}}$$

defines an identification

$$\widetilde{M}' \setminus \mathcal{G}' = \widetilde{M} \setminus \mathcal{G}.$$

Thus, $\mathcal{L}'$ and $\widetilde{\mathcal{L}}^{\text{cut}}$ define foliations on $\widetilde{M}' \setminus \mathcal{G}'$, and we have left to consider only how the leaves of $\widetilde{\mathcal{L}}^{\text{cut}}$ meet near a point on $q(X_{a,b}^{\text{left}})$. If we think of the leaves of $\widetilde{\mathcal{L}}^{\text{cut}}$ as being oriented left to right, then they meet each $X_{a,b}^{\text{left}}$ only at endpoints and meet each $X_{a,b}^{\text{right}}$ only at initial points. Furthermore, these intersections are locally trivial products (as illustrated schematically in Figure 7). In $\widetilde{M}$, neighborhoods of these endpoints and initial points are identified under the map q according to the pasting map $p_{a,b}$. Since $p_{a,b}$ is a homeomorphism it follows that the resulting collection of leaves is a foliation. We have shown the following lemma.

Lemma 2.3. *The leaf structure defined by* $\widetilde{\mathcal{L}}^{cut}$ *on* $\widetilde{M}^{cut}$ *descends to a leaf structure for* $\widetilde{\mathcal{L}}$ *on* $\widetilde{M}$.

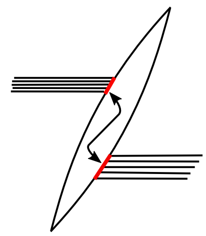

FIGURE 7. A schematic picture (in one-dimensional lower) of how leaves of the foliation $\widetilde{\mathcal{L}}^{\text{cut}}$ are pasted to form the leaves of $\widetilde{\mathcal{L}}$..

Since the $X_{a,b}$ are disjoint, we have the following.

Lemma 2.4. *The homeomorphisms T and Z commute and generate a free abelian group H of rank 2 acting freely and properly discontinuously on $\widetilde{M}$ and permuting the leaves of $\widetilde{\mathcal{L}}$.*

Remark 2.5. Following [8], we choose the notation T to correspond to translation in $\mathbb{R}\times\mathbb{R}$ by $(-1,0)$ in order to be compatible with the typical notation used for mapping tori:

$$\begin{aligned} M &= [0,1]\times S/_{(1,x)\,\simeq\,(0,\phi(x))} \\ &= \mathbb{R}\times S/_{(t,x)\,\simeq\,(t-1,\phi(x))}. \end{aligned}$$

Remark 2.6. Unlike the situation for $\widetilde{\mathcal{L}}'$ in $\widetilde{M}'$, the corresponding projection $\widetilde{M}\to\mathbb{R}$ need not be surjective on all leaves of $\widetilde{\mathcal{L}}$ in $\widetilde{M}$.

Let $\sigma:\widetilde{M}\to\widetilde{S}$ be the composition

$$\widetilde{M}\xrightarrow{\tau}\widetilde{M}^{\text{cut}}\xrightarrow{q'}\widetilde{M}'\xrightarrow{\sigma'}\widetilde{S}.$$

Let $\mathfrak{h}:\widetilde{M}\to\mathbb{R}\times\mathbb{R}$ be the composition

$$\widetilde{M}\xrightarrow{\tau}\widetilde{M}^{\text{cut}}\xrightarrow{q'}\widetilde{M}'\xrightarrow{\mathfrak{h}'}\mathbb{R}\times\mathbb{R}.$$

Then both σ and $\mathfrak{h}$ have jump discontinuities on $\mathcal{G}$, but their restrictions to $\mathcal{G}$ are continuous. In the next section, we define fibrations of $\widetilde{M}$ over $\mathbb{R}$ whose fibers are homeomorphic to $\widetilde{S}$ by the restriction of σ and so that each component of $\mathcal{G}$ is contained in a fiber.

2.2. PROOF OF THEOREM A.

Let $M=\widetilde{M}/H$ and let $\mathcal{L}$ be the foliation defined by taking the image of the leaves of $\widetilde{M}$ in M. We will show that M has a fibered face

F corresponding to cross-sections of $\mathcal{L}$, and there is a linear embedding

$$I_{m_0} \hookrightarrow F$$

so that $I_{m_0}(\mathbb{Q})$ maps to the rational points on the image of I_{m_0}, and for each $\mathfrak{c} \in I_{m_0}(\mathbb{Q})$, the image of $\mathfrak{c}$ in F has monodromy equal to $(S_\mathfrak{c}, \phi_\mathfrak{c})$.

Since $\widetilde{M} \to M$ is a regular abelian covering with automorphism group H, it is an intermediate covering of the maximal abelian covering of M, and hence there is a corresponding epimorphism $H_1(M;\mathbb{Z}) \to H$.

Identifying $H^1(M;\mathbb{R})$ with homomorphisms of $H_1(M;\mathbb{Z})$ to $\mathbb{R}$, we obtain an inclusion:

$$\mathrm{Hom}(H;\mathbb{R}) \to H^1(M;\mathbb{R}).$$

Let W be the image of this map, and let $z, u \in W$ be the basis elements dual to Z and T. Using the notation (r, s) for the linear combination $ru + sz$, so that

$$(r, s) \cdot (T^a Z^b) = ra + sb,$$

let $V_Q \subset W$ be the cone defined by

$$V_Q = \{(-r, -s) \in W \mid 0 < sm_0 < r\}.$$

To prove Theorem A, it suffices to prove the following proposition.

Proposition 2.7. *There is a fibered face $F \subset H^1(M;\mathbb{R})$ and a continuous map*

$$\iota : I_{m_0} \to F \cap W$$

with the property that for $\mathfrak{c} \in I_{m_0}(\mathbb{Q})$

$$(S_\mathfrak{c}, \phi_\mathfrak{c}) = \mathfrak{f}(\iota(c)).$$

Proof. The proof has three steps. In Step 1, we show that for each $\mathfrak{c} \in I_{m_0}(\mathbb{Q})$, there is a corresponding element $\alpha_\mathfrak{c} \in V_\mathbb{Q}$, and a fibration $\widetilde{\rho}_\mathfrak{c} : \widetilde{M} \to \mathbb{R}$ that lifts a fibration of $\rho_\mathfrak{c} : M \to S^1$ with monodromy equal to $(S_\mathfrak{c}, \phi_\mathfrak{c})$. We also define generators $T_\mathfrak{c}$ and $Z_\mathfrak{c}$ that generate H, such that $Z_\mathfrak{c}$ preserves the fibers. Finally, in Step 3, we show that the fibration descends to a circle bundle on M whose monodromy is $(S_\mathfrak{c}, \phi_\mathfrak{c})$.

Step 1. Building fibrations of $\widetilde{M}$ to $\mathbb{R}$ transverse to the foliation $\widetilde{\mathcal{L}}$. Take any $\mathfrak{c} = \frac{k}{n} \in I_{m_0}(\mathbb{Q})$, and let

$$\alpha_\mathfrak{c} = (-k, -n).$$

Then we have $\alpha_\mathfrak{c} \in V_\mathbb{Q}$, and, seeing $\alpha_\mathfrak{c}$ as a homomorphism, we have

$$\begin{array}{rcl} \alpha_\mathfrak{c} : H & \to & \mathbb{Z} \\ T^a Z^b & \mapsto & -an - bk. \end{array}$$

Let

$$Z_\mathfrak{c} = T^{-k} Z^n.$$

Then the kernel $K_\mathfrak{c}$ of $\alpha_\mathfrak{c}$ is freely generated by $Z_\mathfrak{c}$. Let w and $\overline{k}_n$ be the solutions to

$$wn + \overline{k}_n k = 1,$$

and let

$$T_\mathfrak{c} = T^w Z^{\overline{k}_n}.$$

Then $\alpha_\mathfrak{c}(T_\mathfrak{c}) = -1$.

We will define a fibration $\widetilde{\rho}_\mathfrak{c} : \widetilde{M} \to \mathbb{R}$ so that the fibers of $\widetilde{\rho}_\mathfrak{c}$ have the following properties:

(1) the restriction of σ to each fiber defines a homeomoorphism to $\widetilde{S}$;
(2) the elements of $K_\mathfrak{c}$ preserve each fiber; and
(3) the fibers are permuted by the action of Z and T.

Furthermore, each leaf of $\widetilde{\mathcal{L}}$ intersects each fiber of $\widetilde{\rho}_\mathfrak{c}$ exactly once.

We begin by first defining a suitable projection $p_\mathfrak{c} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and then pulling back by $\mathfrak{h}$ to $\widetilde{M}$.

Proposition 2.8. *For each $\mathfrak{c} = \frac{k}{n} \in I_{m_0}(\mathbb{Q})$, there is a continuous monotone increasing function*

$$g_\mathfrak{c} : \mathbb{R} \to \mathbb{R},$$

such that

(1) $g_\mathfrak{c}(r) = \frac{r}{\mathfrak{c}}$ *for all* $r \in \mathbb{Z}$*;*
(2) $g_\mathfrak{c}(x + r) = g_\mathfrak{c}(x) + \frac{r}{\mathfrak{c}}$ *for all* $r \in \mathbb{Z}$ *and* $x \in \mathbb{R}$*; and*
(3) *for each* $(a, b) \in \mathbb{Z}\times\mathbb{Z}$, $\Gamma_{a,b}$ *is contained in the graph* $y = g_\mathfrak{c}(x)+\frac{r}{k}$ *for some* $r \in \mathbb{Z}$*.*

Proof. Let $\Delta_{0,0}$ be the straight line segment on $\mathbb{R} \times \mathbb{R}$ connecting $p = (\frac{1}{2}, m_0)$ and $q_\mathfrak{c} = (1, \frac{1}{\mathfrak{c}})$. For $(a, b) \in \mathbb{R} \times \mathbb{R}$, let

$$\Delta_{a,b} = (a, b) + \Delta_{0,0}.$$

Let

$$\widetilde{R}_\mathfrak{c} = \bigcup_{r\in\mathbb{Z}} (\Gamma_{r,\frac{r}{\mathfrak{c}}} \cup \Delta_{r,\frac{r}{\mathfrak{c}}}).$$

Then $\widetilde{R}_\mathfrak{c}$ is the graph of a piecewise-linear, monotone increasing function $g_\mathfrak{c}$ with the desired properties. $\square$

Let

$$\widetilde{R}_{c,\xi} = \widetilde{R}_\mathfrak{c} + (0, -\frac{\xi}{k})$$

(see Figure 8). Then $\widetilde{R}_{c,\xi}$ intersects $\Gamma_{a,b}$ for some $(a, b) \in \mathbb{Z} \times \mathbb{Z}$ if and only if $\xi \in \mathbb{Z}$. Each $\widetilde{R}_{c,\xi}$ is the graph of a monotone increasing, piecewise linear, continuous function $g_{c,\xi}$ defined by $g_{\mathfrak{c},\xi}(\mathfrak{c}) = g_\mathfrak{c}(\mathfrak{c}) + \xi$. Thus, the

$\widetilde{R}_{c,\xi}$ partition $\mathbb{R} \times \mathbb{R}$ into a disjoint union, and we have a well-defined continuous function:

$$p_{\mathfrak{c}} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$$

defined by sending each $x \in \widetilde{R}_{c,\xi}$ to ξ.

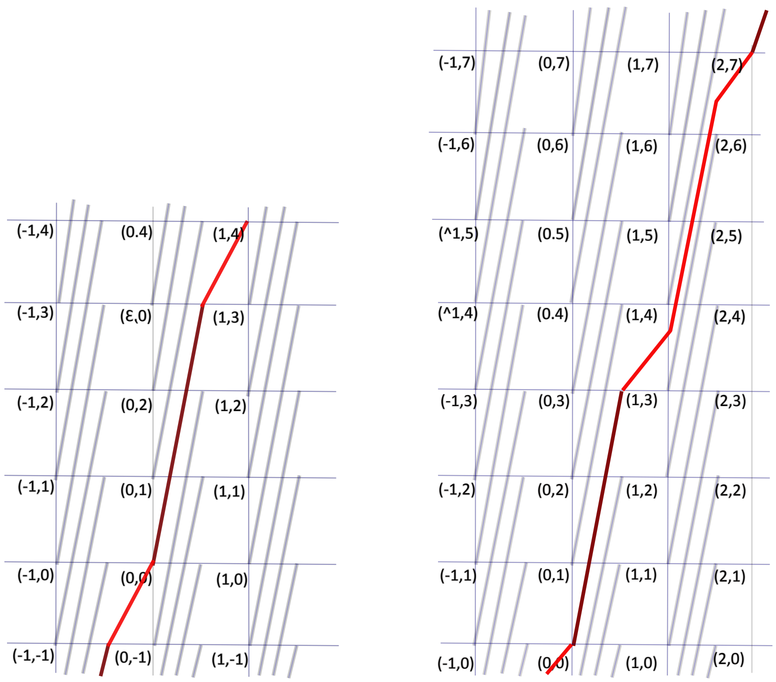

FIGURE 8. The loci $\widetilde{R}_{\mathfrak{c}} \subset \mathbb{R} \times \mathbb{R}$ for $\mathfrak{c} = \frac{1}{4}$ (left) and $\mathfrak{c} = \frac{2}{7}$ (right) when $m_0 = 3$.

Lemma 2.9. *For each $\xi \in \mathbb{R}$, the locus $\widetilde{R}_{c,\xi}$ has the following properties:*

(1) *translation by $(1, \frac{n}{k})$ generates an infinite cyclic action on $\widetilde{R}_{c,\xi}$ with fundamental domain given by $(0, -\frac{\xi}{k}) + \Gamma_{0,0} \cup \Delta_{0,0}$;*
(2) $(0,1) + \widetilde{R}_{c,\xi} = \widetilde{R}_{c,\xi-k}$*; and*
(3) $(-1,0) + \widetilde{R}_{c,\xi} = \widetilde{R}_{c,\xi-n}$.

Proof. To prove (1) we note that the statement holds for $\widetilde{R}_{\mathfrak{c}}$ by construction. Since translations by $(0,\xi)$ and by $(1, \frac{n}{k})$ commute on $\mathbb{R} \times \mathbb{R}$, the statement also holds for $\widetilde{R}_{c,\xi}$ for all ξ.

Properties (2) and (3) follow from verifying that $(0,1) \in \widetilde{R}_{c,-k}$ and $(0, \frac{n}{k}) \in \widetilde{R}_{c,-n}$. $\square$

We now apply Lemma 2.9 to $\widetilde{M}$. Let $\widetilde{\rho}_{\mathfrak{c}} = p_{\mathfrak{c}} \circ \mathfrak{h}$ and let $\widetilde{S}_{c,\xi} = \mathfrak{h}^{-1}(\widetilde{R}_{c,\xi}) \subset \widetilde{M}$.

Lemma 2.10. *The map $\widetilde{\rho}_{\mathfrak{c}}$ is a fibration with fibers $\widetilde{S}_{c,\xi}$, for all $\xi \in \mathbb{R}$ and for all $\xi \in \mathbb{R}$ we have the following:*

(1) *the fundamental domain of the action of $K_{\mathfrak{c}}$ on $\widetilde{S}_{c,\xi}$ is the set*

$$\mathfrak{h}^{-1}\left(\bigcup_{r=0}^{k-1} \Gamma_{r,\frac{rn}{k}} \cup \Delta_{r,\frac{rn}{k}}\right);$$

(2) *$K_{\mathfrak{c}} = \langle Z_{\mathfrak{c}} \rangle$ generates the set-wise stabilizer in H of $\widetilde{S}_{c,\xi}$;*
(3) *$Z(\widetilde{S}_{c,\xi}) = \widetilde{S}_{c,\xi-k}$ and $T(\widetilde{S}_{c,\xi}) = \widetilde{S}_{c,\xi-n}$;*
(4) *$Z_{\mathfrak{c}}(\widetilde{S}_{c,\xi}) = \widetilde{S}_{c,\xi}$, $T_{\mathfrak{c}}(\widetilde{S}_{c,\xi}) = \widetilde{S}_{c,\xi-1}$;*
(5) *$\widetilde{\rho}_{\mathfrak{c}}(Z_{\mathfrak{c}}(\widetilde{S}_{c,\xi})) = \widetilde{\rho}_{\mathfrak{c}}(\widetilde{S}_{c,\xi})$, $\widetilde{\rho}_{\mathfrak{c}}(T_{\mathfrak{c}}(\widetilde{S}_{c,\xi})) = \widetilde{\rho}_{\mathfrak{c}}(\widetilde{S}_{c,\xi}) - 1$; and*
(6) *$\widetilde{\rho}_{\mathfrak{c}}$ restricts to a homeomorphism to $\mathbb{R}$ on each leaf of $\widetilde{\mathcal{L}}$.*

Proof. The map $\widetilde{\rho}_{\mathfrak{c}}$ is continuous since each of the fibers $\widetilde{S}_{c,\xi}$ is either disjoint from $\mathcal{G}$ or contains components of $\mathcal{G}$. Since $\mathfrak{h} \circ Z_{\mathfrak{c}} \circ \mathfrak{h}^{-1}$ is a translation by $(k,n) = k(1,\frac{n}{k})$, Lemma 2.9 implies that $Z_{\mathfrak{c}}$ and all its powers preserve $\widetilde{S}_{\mathfrak{c},\xi}$. To show that $Z_{\mathfrak{c}}$ generates the stabilizer of $\widetilde{S}_{c,\xi}$, we note that any element of H is conjugate to a translation of $\mathbb{R} \times \mathbb{R}$. If $h \in H$ preserves $\widetilde{S}_{\mathfrak{c},\xi}$, it must be conjugate to a translation preserving $\widetilde{R}_{\mathfrak{c},\xi}$, i.e., translation by $m(1,\frac{n}{k})$ for some integer m. Since H furthermore preserves $\mathcal{G}$, and k and n are relatively prime, we must also have k divides m. Thus, h is a multiple of $Z_{\mathfrak{c}}$. This proves (1) and (2).

Items (3) and (4) follow directly from Lemma 2.9.

To prove (5), we note that the action of $T_{\mathfrak{c}}$ on $\widetilde{M}$ is conjugate by $\mathfrak{h}$ to translation in $\mathbb{R} \times \mathbb{R}$ by

$$T_{\mathfrak{c}} : (a,b) \mapsto (a,b) + (-w, \overline{k}_n).$$

Since translation by $(1,\frac{n}{k})$ stabilizes each $\widetilde{R}_{c,\xi}$, $(-w, \xi + \overline{k}_n)$ and $(0, (\xi + \overline{k}_n) + \frac{wn}{k})$ lie on the same fiber of $p_{\mathfrak{c}}$. Using the definition of w and $\overline{k}_n$, we have

$$(\xi + \overline{k}_n) + \frac{wn}{k} = \xi + \frac{\overline{k}_n k + wn}{k} = \xi + \frac{1}{k},$$

proving (5).

To prove (6), it suffices to show that $\widetilde{\rho}_{\mathfrak{c}}$ is 1-1 and onto on each leaf ℓ of $\widetilde{\mathcal{L}}$, or equivalently that ℓ intersects $\widetilde{S}_{c,\xi}$ exactly once for each $\xi \in \mathbb{R}$. This can be seen by verifying that each segment of $\widetilde{\mathcal{L}}'$ intersects each $(q' \circ \tau)(\widetilde{S}_{c,\xi})$ in exactly one point. $\square$

By the above, we have a setwise bijection

$$\begin{array}{rcl} \mathfrak{j}_{\mathfrak{c}} : \widetilde{M} & \to & \mathbb{R} \times \widetilde{S} \\ s & \mapsto & (\widetilde{\rho}_{\mathfrak{c}}(s), \sigma(s)) \end{array}$$

which is a homeomorphism outside the preimage of $\mathbb{Z} \times \widetilde{S}$. Outside this locus of discontinuity, the definition of the pasting map used to construct $\widetilde{M}$ gives

$$\mathfrak{j}_{\mathfrak{c}} \circ T_{\mathfrak{c}} \circ \mathfrak{j}_{\mathfrak{c}}^{-1}(s) = (\widetilde{\rho}_{\mathfrak{c}}(s) - 1, \widetilde{\phi}_{\mathfrak{c}}^{-1}(\sigma(s))).$$

Thus, we can also think of $\mathfrak{j}_{\mathfrak{c}}$ as a homeomorphism after cutting $\mathbb{R} \times \widetilde{S}$ along $\mathbb{Z} \times \widetilde{S}$ and pasting by the map $\widetilde{\phi}_{\mathfrak{c}}$.

Define a forward flow along $\widetilde{\mathcal{L}}$ by

$$\begin{array}{rcl} f_{\mathfrak{c}} : \mathbb{R} \times \widetilde{M} & \to & \widetilde{M} \\ (t, s) & \mapsto & \mathfrak{j}_{\mathfrak{c}}^{-1}(\widetilde{\rho}_{\mathfrak{c}}(s) + t, \phi_{\mathfrak{c}}^{r}(\sigma(s))), \end{array}$$

where

$$r = \begin{cases} \lfloor t \rfloor & \text{if } t \geq 0 \\ \lceil t \rceil & \text{if } t < 0. \end{cases}$$

Another way to think of $f_{\mathfrak{c}}$ is that for each $(t, s) \in \mathbb{R} \times \widetilde{M}$, $f_{\mathfrak{c}}(t, s)$ is the point on $\widetilde{S}_{\mathfrak{c},\xi+t}$ that lies on the leaf ℓ, where ℓ is the leaf containing the point s and $\xi = \widetilde{\rho}_{\mathfrak{c}}(s)$.

Step 2. Comparing $T_{\mathfrak{c}}$ with the forward flow on $\widetilde{M}$.

Lemma 2.11. *The flow map $f_{\mathfrak{c}}$ satisfies*

$$\sigma(f_{\mathfrak{c}}(t+1, s)) = \widetilde{\phi}_{\mathfrak{c}}(\sigma(f_{\mathfrak{c}}(t, s))).$$

Proof. This follows from comparing the definition of the pasting map in §2.1.3 with the definition of $\widetilde{\phi}_{\mathfrak{c}}$ given in §1.3. □

Corollary 2.12. *The map $T_{\mathfrak{c}}$ satisfies*

$$\sigma(s) = \widetilde{\phi}_{\mathfrak{c}}(\sigma(T_{\mathfrak{c}}(s)))$$

for $s \in \widetilde{M}$.

Proof. By Lemma 2.11, we have

$$\widetilde{\phi}_{\mathfrak{c}}(\sigma(T_{\mathfrak{c}}(s))) = \widetilde{\phi}_{\mathfrak{c}}(\sigma(f_{\mathfrak{c}}(-1, s))) = \sigma(f_{\mathfrak{c}}(0, s)) = \sigma(s).$$ □

Step 3. Descending to the quotient fibration.
Since $Z_{\mathfrak{c}}$ and $T_{\mathfrak{c}}$ commute, it follows that $T_{\mathfrak{c}}$ defines a covering automorphism $\overline{T}_{\mathfrak{c}}$ on the intermediate covering

$$M_{\mathfrak{c}} = \widetilde{M}/\langle Z_{\mathfrak{c}} \rangle \to M$$

over M. The projection $\widetilde{\rho}_\mathfrak{c}$ also descends to a fibration

$$\overline{\rho}_\mathfrak{c} : M_\mathfrak{c} \to \mathbb{R}$$

whose fibers are homeomorphic to $S_\mathfrak{c} = \widetilde{S}/\widetilde{\zeta}^n$. For $s \in \widetilde{M}$, let $\overline{s}$ be the image of s in $M_\mathfrak{c}$ and let $\overline{\sigma(s)}$ be the image of $\sigma(s)$ in $S_\mathfrak{c}$.

Since $Z_\mathfrak{c}$ is conjugate by $\mathfrak{j}_\mathfrak{c}$ to $\mathrm{id} \times \zeta^n$ on $\mathbb{R} \times \widetilde{S}$, the map $\mathfrak{j}_\mathfrak{c}$ descends to an identification

$$\begin{array}{rcl} \overline{j}_\mathfrak{c} : M_\mathfrak{c} & \to & \mathbb{R} \times S_\mathfrak{c} \\ \overline{s} & \mapsto & (\widetilde{\rho}_\mathfrak{c}(s), \sigma_\mathfrak{c}(s)). \end{array}$$

Then the element $\alpha_\mathfrak{c}$ defines a commutative diagram

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\mathfrak{j}_\mathfrak{c}"] \arrow[d, "/_{K_\mathfrak{c}}"'] & \mathbb{R} \times \widetilde{S} \arrow[d, "/_{\mathrm{id} \times \widetilde{\zeta}^n}"] \\ M_\mathfrak{c} \arrow[r, "\overline{\mathfrak{j}}_\mathfrak{c}"] \arrow[d, "/_{\langle \overline{T}_\mathfrak{c} \rangle}"'] & \mathbb{R} \times S_\mathfrak{c} \arrow[d, "/_{\mathbb{Z} \times S_\mathfrak{c}}"] \\ M \arrow[r] & S^1 \end{tikzcd}$$

where induced map $\overline{T}_\mathfrak{c} : M_\mathfrak{c} \to M_\mathfrak{c}$ is defined by

$$\begin{array}{rcl} \overline{j}_\mathfrak{c} \circ \overline{T}_\mathfrak{c} \circ j_\mathfrak{c}^{-1} : \mathbb{R} \times S_\mathfrak{c} & \to & \mathbb{R} \times S_\mathfrak{c} \\ (t, x) & \mapsto & (t - 1, \phi_\mathfrak{c}(x)). \end{array}$$

The bottom horizontal map in the diagram is the fibration $\rho_\mathfrak{c}$ associated to $\mathfrak{c}$, and the fibers are homeomorphic to $S_\mathfrak{c}$ and are cross-sections of the flow $\mathcal{L}$. The monodromy is defined by $\overline{T}_\mathfrak{c}$ and equals $(S_\mathfrak{c}, \phi_\mathfrak{c})$. This completes the proof of Proposition 2.7. □

2.3. Stable quotient families.

To prove Theorem B, we study the cohomology class in $H^1(M; \mathbb{Z})$ associated to $\alpha_0 = (-1, 0)$. This is the primitive element on the limit of the sequence of rays defined by $(-k, -n)$ as $\frac{k}{n}$ approaches 0. The kernel of α_0 restricted to H is generated by the map Z. For $r \in \mathbb{Z}$, let $\widetilde{S}_{0,r}$ be the surface defined by the image in $\widetilde{M}$ of $\{r\} \times \widetilde{S} \subset \widetilde{M}^{\mathrm{cut}}$. Unlike in the case of $\mathfrak{c} \in I_{m_0}(\mathbb{Q})$, there is no easily defined projection $\widetilde{\rho}_0$. (See Figure 9.)

We will show that if Q is a stable family, then for any $r \in \mathbb{Z}$, each leaf of $\widetilde{\mathcal{L}}$ passes through $\widetilde{S}_{0,r}$ exactly once for all $r \in \mathbb{Z}$, and if Q is not stable, then for each $r \in \mathbb{Z}$, there is at least one leaf of $\widetilde{\mathcal{L}}$ that does not pass through $\widetilde{S}_{0,r}$.

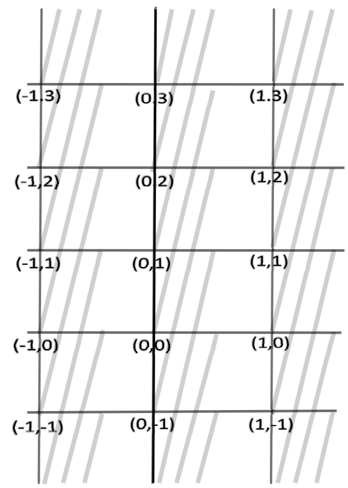

FIGURE 9. The surfaces $\widetilde{S}_{0,r}$ are sent by $\mathfrak{h}$ to the vertical lines $\{r\} \times \mathbb{R}$ where $r \in \mathbb{Z}$.

Lemma 2.13. *Let $x \in \widetilde{S}$, and let $r_1 < r_2$ be integers such that $\zeta^{-r_i}(x) \in \Sigma_0$ for $i = 1, 2$. Then we have*

$$\widetilde{\zeta}^{-m_1+r_1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r_1}(x) = \widetilde{\zeta}^{-m_1+r_2}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r_2}(x).$$

Proof. By assumption, $\Sigma_0 \cap \widetilde{\zeta}^2(\Sigma_0) = \emptyset$, and hence we have $r_2 = r_1 + 1$. This means that $\widetilde{\zeta}^{-r_2}(x) \in \Sigma_0 \cap \widetilde{\zeta}^{-1}(\Sigma_0)$, and hence

$$\widetilde{\eta}\widetilde{\zeta}^{-r_2}(x) = \zeta^{-r_2}(x).$$

By the definition of stability, we also have

$$\widetilde{\eta}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r_2}(x) = (\widetilde{\zeta}\widetilde{\eta})^{m_2}\widetilde{\zeta}^{-r_1}(x).$$

Putting this together gives the equalities

$$\begin{aligned} \widetilde{\zeta}^{-m_1+r_1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r_1}(x) &= \widetilde{\zeta}^{-m_1+r_2-1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r_2+1}(x) \\ &= \widetilde{\zeta}^{-m_1+r_2}\widetilde{\zeta}^{-1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}\widetilde{\zeta}^{-r_2}(x) \\ &= \widetilde{\zeta}^{-m_1+r_2}\widetilde{\eta}(\widetilde{\zeta}\widetilde{\eta})^{m_1-1}\widetilde{\zeta}\widetilde{\eta}\widetilde{\zeta}^{-r_2}(x) \\ &= \widetilde{\zeta}^{-m_1+r_2}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r_2}(x,), \end{aligned}$$

completing the proof. $\square$

Let $\widetilde{\phi}$ be defined by

$$\begin{aligned} \widetilde{\phi} : \widetilde{S} &\to \widetilde{S} \\ x &\mapsto \widetilde{\zeta}^{-m_1+r}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r}(x), \end{aligned}$$

where r is any integer such that $\widetilde{\zeta}^{-r}(x) \in \Sigma_0$. This is well defined by Lemma 2.13.

Lemma 2.14. *The map $\widetilde{\phi}$ commutes with $\widetilde{\zeta}$.*

Proof. Let $x \in \widetilde{\zeta}^r(\Sigma_0)$. Then we have

$$\begin{aligned} \widetilde{\phi}(\widetilde{\zeta}(x)) &= \widetilde{\zeta}^{-m_1+r+1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r-1}\widetilde{\zeta}(x) \\ &= \widetilde{\zeta}\widetilde{\zeta}^{-m_1+r}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r}(x) \\ &= \widetilde{\zeta}\widetilde{\phi}(x). \end{aligned}$$

□

Lemma 2.15. *If Q is stable, then the map $\widetilde{\phi}$ is a continuous, locally injective, open mapping, and if Q is bi-stable, then $\widetilde{\phi}$ is a homeomorphism.*

Proof. Since

$$\widetilde{\zeta}^{-m_1+r}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r}(x) \quad = \quad \widetilde{\zeta}^{r}(\widetilde{\zeta}^{-m_1}\widetilde{\eta}\widetilde{\zeta}^{m_1})\cdots(\widetilde{\zeta}^{-1}\widetilde{\eta}\zeta)\widetilde{\eta}\widetilde{\zeta}^{-r}(x),$$

we can think of $\widetilde{\phi}$ as the infinite composition of maps

$$\psi_i = \widetilde{\zeta}^{-i}\widetilde{\eta}\widetilde{\zeta}^{i},$$

where i goes from $-\infty$ to ∞; that is,

$$\widetilde{\phi} = \cdots \circ \psi_{i-1} \circ \psi_i \circ \psi_{i+1} \circ \cdots .$$

This is well defined and continuous since for each $x \in \widetilde{S}$, if we let r be such that $x \in \zeta^r(\Sigma_0)$, we have

$$\psi_i(x) = x$$

for all $i > -r$, and

$$\begin{aligned} \widetilde{\phi}(x) &= \cdots \psi_{-r-1}\psi_{-r}(x) \\ &= \psi_{-r-m_1}\cdots\psi_{-r}(x). \end{aligned}$$

Since ψ_i are all homeomorphisms, this implies that $\widetilde{\phi}$ is an open mapping.

If, in addition, Q is bi-stable, then a similar argument shows that the following map is well-defined and continuous. Take $x \in \widetilde{S}$, and let r be such that $x \in \widetilde{\zeta}^r(\Sigma_{m_0})$. Define

$$\widetilde{\phi}'(x) = \widetilde{\zeta}^{r}(\widetilde{\eta}^{-1}\widetilde{\zeta}^{-1})^{m_1}\widetilde{\zeta}^{-r}(x).$$

Then the stability of $Q(\widetilde{S}, \widetilde{\zeta}^{-1}, \widetilde{\eta}^{-1})$ implies that for $x \in \widetilde{S}$ and i large enough, $\psi_i^{-1}(x) = (x)$, and the composition

$$\cdots \circ \psi_{i+1}^{-1} \circ \psi_i^{-1} \circ \psi_{i-1}^{-1} \circ \cdots$$

is well defined and continuous. This map is the inverse of $\widetilde{\phi}$. □

Lemma 2.16. *The map $\widetilde{\phi}$ commutes with ζ.*

Proof. Take any $s \in \widetilde{S}$ and assume that $\widetilde{\zeta}^{-r}(s) \in \Sigma_0$. Then

$$\begin{aligned}\widetilde{\phi}(\widetilde{\zeta}(s)) &= \widetilde{\zeta}^{r+1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r-1}(\zeta(s)) \\ &= \widetilde{\zeta}^{r+1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r}(s) \\ &= \widetilde{\zeta}\widetilde{\zeta}^{r}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r}(s) \\ &= \widetilde{\zeta}(\widetilde{\phi}(s)).\end{aligned}$$

□

Lemma 2.17. *The quotient family Q is bi-stable if and only if for each $a \in \mathbb{Z}$ and any leaf ℓ of $\widetilde{\mathcal{L}}$, ℓ intersects $\widetilde{S}_{0,a}$ in a single point x_a. In this case, x_a satisfies*

$$\sigma(x_a) = \widetilde{\phi}(\sigma(x_{a+1}))$$

for all $a \in \mathbb{Z}$.

Proof. Let

$$\widetilde{\rho}_0 : \widetilde{M} \to \mathbb{R}$$

be the composition of $\mathfrak{h}$ with projection of $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ onto the first coordinate. Then $\widetilde{\rho}_0$ has jump discontinuities at the cut locus $\mathcal{G}$.

Assume Q is bi-stable. Let ℓ be any leaf of $\widetilde{\mathcal{L}}$. Then for each $a \in \mathbb{Z}$, ℓ intersects $\widetilde{S}_{0,a}$ in at most one point since the restriction of $\widetilde{\rho}_0$ to ℓ is monotone in small enough neighborhoods of

$$\bigcup_{a\in\mathbb{Z}} \widetilde{S}_{0,a}.$$

For each $a \in \mathbb{Z}$, let $\mathcal{G}_a = \bigcup_{b\in\mathbb{Z}} X_{a,b}$.

Suppose t_0 is a point on the intersection $X_{a,b} \cap \ell$ for some $a, b \in \mathbb{Z} \times \mathbb{Z}$. Then there is a well-defined orientation on ℓ such that the next time ℓ intersects $\mathcal{G}$ is at $X_{a,b-1}$. This orientation defines an ordering on the sequence $t_0, t_1, t_2, \ldots$ of intersections of ℓ with $\mathcal{G}$ that occur on ℓ after t_0. Then $t_i \in X_{a,b-i}$ for each $i = 1, 2, \ldots,$ and

$$\begin{aligned}\sigma(t_i) &= \widetilde{\zeta}^{b-i+1}\widetilde{\eta}\widetilde{\zeta}^{-b+i-1}(\sigma(t_{i-1})) \\ &= \widetilde{\zeta}^{b-i+1}\widetilde{\eta}\widetilde{\zeta}^{-b+i-1}\widetilde{\zeta}^{b-i+2}\widetilde{\eta}\widetilde{\zeta}^{-b+i-2}(\sigma(t_{i-2})) \\ &= \widetilde{\zeta}^{b-i}(\widetilde{\zeta}\widetilde{\eta})^{2}\widetilde{\zeta}^{-b+i-2}(\sigma(t_{i-2})) \\ &= \widetilde{\zeta}^{b}(\widetilde{\zeta}\widetilde{\eta})^{b}\widetilde{\zeta}^{-b}(t_0).\end{aligned}$$

If Q is stable, then this sequence must terminate after at most m_1 steps. Since this is true for any starting point $t_0 \in \ell \cap \mathcal{G}_a$, it follows that $\ell \cap \mathcal{G}_a$ is finite and has at most m_1 elements. Furthermore, if t_0 is the first time ℓ meets $\mathcal{G}_a$ and t_m is the last, then

$$\begin{aligned}\sigma(x_{a+1}) &= \widetilde{\zeta}^{m+1}\widetilde{\eta}\widetilde{\zeta}^{1-m}\sigma(t_m) = \sigma(t_m) \\ &= \widetilde{\zeta}^{r}(\widetilde{\zeta}\widetilde{\eta})^{r}\widetilde{\zeta}^{-4}(\sigma(t_0)) = \widetilde{\phi}(\sigma(t_0)) = \widetilde{\phi}(\sigma(x_a),\end{aligned}$$

where x_a is the intersection of ℓ with $\widetilde{S}_{0,a}$ and x_{a+1} is the intersection of ℓ with $\widetilde{S}_{0,a+1}$.

Since all leaves of $\widetilde{\mathcal{L}}$ must intersect $\mathcal{G}$ in at least one point (since $m_0 \geq 1$), we have shown that for each $\ell \in \widetilde{\mathcal{L}}$, ℓ intersects $\widetilde{S}_{0,a}$ and $\widetilde{S}_{0,a+1}$ for some a. Our argument also shows by induction that this ℓ intersects $\widetilde{S}_{0,a'}$ for all $a' \geq a$.

Using the same argument on the backward flow on ℓ starting from the point x_a, the stability of $Q(\widetilde{S}, \widetilde{\zeta}^{-1}, \widetilde{\eta}^{-1})$ implies that ℓ intersects every $\widetilde{S}_{0,a'}$ for $a' < a$.

Conversely, suppose that for each ℓ and each $a \in \mathbb{Z}$, we have ℓ intersecting $\widetilde{S}_a$ in a single point x_a. Again, we consider the ordered intersections of ℓ with $\mathcal{G}_a$. This must be a finite sequence, since if not, then ℓ would never reach $\widetilde{S}_{a+1}$. If $t_0, \ldots, t_k$ are intersections, starting with the one on ℓ immediately following x_a, then as before, we have

$$\widetilde{\phi}(\sigma(x_a)) = \widetilde{\phi}(\sigma(t_0)) = \sigma(t_k) = \sigma(x_{a+1}). \qquad \square$$

Remark 2.18. We have shown in the proof of Lemma 2.17 that Q is stable if and only if for all $a \in \mathbb{Z}$ and leaf $\ell \in \widetilde{\mathcal{L}}$ passing through a point $x_a \in \widetilde{S}_a$, the leaf ℓ passes through $\widetilde{S}_{a'}$ for all $a' \geq a$. A priori, there may be other leaves of ℓ that stay in between S_a and S_{a+1}.

By Lemma 2.17, if Q is bi-stable, then ι extends continuously to 0, and $\iota(0)$ has monodromy (S, ϕ), where $S = \widetilde{S}/\widetilde{\zeta}$ and ϕ is the map induced by $\widetilde{\phi}$. Thus, α_0 must lie in the interior of the fibered face and, by continuity of L, we have

$$\lim_{c \to 0} L(\alpha_{\mathfrak{c}}) = L(\alpha_0) = L(S, \phi).$$

Conversely, if Q is not stable, then α_0 must lie on the boundary of the fibered cone, and the sequence $L(\alpha_{\mathfrak{c}})$ diverges as $\mathfrak{c}$ approaches 0. This completes the proof of Theorem B.

3. Penner Example

In this section, we illustrate Theorem A and Theorem B using Penner's sequence (R_g, ψ_g) (recall Figure 1).

Let $Q = Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$, where $\widetilde{S}$ is the infinite surface drawn in Figure 3 as a stack of copies Σ_i, $i \in \mathbb{Z}$, of a fundamental domain Σ, and $\widetilde{\zeta}$ sends each Σ_i homeomorphically to Σ_{i+1}, $\Sigma_i \cap \Sigma_{i+1} \subset \partial\Sigma_i$, $\widetilde{\zeta}^2(\Sigma_i) \cap \Sigma_i = \emptyset$, and $\widetilde{\eta} = \delta_{\widetilde{c}} \delta_{\widetilde{b}}^{-1} \delta_{\widetilde{a}}$.

For a mapping class (S, ϕ), where S has punctures or boundary components, let $\overline{S}$ be the *closure* of S, that is, the closed surface obtained by filling in the punctures and boundary components, and let $(\overline{S}, \overline{\phi})$ be the induced mapping class. Then we observe the following.

Proposition 3.1. *For $g \geq 3$,*

$$(R_g, \psi_g) = (\overline{S}_{\frac{1}{g}}, \overline{\phi}_{\frac{1}{g}}),$$

where $(S_{\frac{1}{g}}, \phi_{\frac{1}{g}}) = \iota(\frac{1}{g}) \in Q$.

Proof. It suffices to observe that $\phi_{\frac{1}{g}} = \zeta_g \delta_c \delta_b^{-1} \delta_a$. □

Corollary 3.2. *For all $\mathfrak{c} \in (0, \frac{1}{2})$, $(S_{\mathfrak{c}}, \phi_{\mathfrak{c}})$ is pseudo-Anosov.*

We now use Theorem B to show that $L(S_{\frac{1}{g}}, \phi_{\frac{1}{g}})$ is bounded for $g \geq 2$. By Theorem B, it suffices to show the following.

Proposition 3.3. *The family Q is a bi-stable quotient family.*

Proof. The support of $\widetilde{\eta}$ is contained in the union of annular neighborhood of $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, which are all contained in

$$\Sigma_0 \cup \Sigma_1,$$

and hence $m_0 = 1$. Thus, we have

$$\widetilde{\eta}(\Sigma_0) \subset \Sigma_0 \cup \Sigma_1,$$

and

$$\widetilde{\zeta}\widetilde{\eta}(\Sigma_0 \cup \Sigma_1) \subset \Sigma_1 \cup \Sigma_2.$$

On Σ_1, $\delta_{\widetilde{a}}$ and $\delta_{\widetilde{b}}$ act trivially, while the map $\delta_{\widetilde{c}}$ sends $\Sigma_1 \cup \Sigma_2$ to the union of $\Sigma_1 \cup \Sigma_2$ and a small annular neighborhood of $\widetilde{c}$. Thus,

$$\widetilde{\zeta}\widetilde{\eta}(\Sigma_1 \cup \Sigma_2) \subset \Sigma_2 \cup \Sigma_3 \cup \widetilde{\zeta}(A_{\widetilde{c}}),$$

where $A_{\widetilde{c}} \subset \Sigma_0 \cup \Sigma_1$ is a small annular neighborhood of $\widetilde{c}$. The map $\widetilde{\eta}$ acts trivially on $\Sigma_2 \cup \Sigma_3 \cup \widetilde{\zeta}(A_{\widetilde{c}})$. Thus, we have, for all $x \in \Sigma_0$,

$$(\widetilde{\zeta}\widetilde{\eta})^3(x) = \widetilde{\zeta}(\widetilde{\zeta}\widetilde{\eta})^2(x).$$

Now consider $Q(\widetilde{S}, \widetilde{\zeta}^{-1}, \widetilde{\eta}^{-1})$. The support of

$$\eta^{-1} = \delta_{\widetilde{a}} \delta_{\widetilde{b}}^{-1} \delta_{\widetilde{c}}$$

is $\Sigma_0 \cup \Sigma_1$ and its image is the union of annuli around $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$. Thus,

$$\widetilde{\zeta}^{-1}\widetilde{\eta}^{-1}(\Sigma_0 \cup \Sigma_1) \subset \Sigma_{-1} \cup \widetilde{\zeta}^{-1}(A_{\widetilde{c}}).$$

The action of $\delta_{\widetilde{c}}$ acts trivially on this set, and hence

$$(\widetilde{\zeta}^{-1}\widetilde{\eta}^{-1})^2(\Sigma_0 \cup \Sigma_1) \subset \Sigma_{-2} \cup \Sigma_{-1}.$$

The map $\widetilde{\eta}^{-1}$ acts trivially on this set, and hence Q is bi-stable. □

By Theorem B, it follows that

$$\lim_{g\to\infty} L(S_{\frac{1}{g}}, \phi_{\frac{1}{g}}) = L(S_0, \phi_0),$$

where $S_0 = \widetilde{S}/\widetilde{\zeta}$ and ϕ_0 is defined by the ζ-equivariant map $\widetilde{\phi}$. This gives an alternative to Penner's proof in [10] that for some constant $C > 0$,

$$\log(\lambda(\phi_{\frac{1}{g}})) \leq \frac{C}{g}$$

for $g \geq 2$.

3.1. Limiting mapping class.

We describe the limiting mapping class (S, ϕ) for Q at 0 explicitly. Figure 10 gives a picture of $S = \widetilde{S}/\zeta$ with images a, b, and c of the curves $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$, and the image d of $\widetilde{d} = \Sigma_0 \cap \zeta(\Sigma_0)$. Then, since the map $\widetilde{\zeta}$ descends to the identity map, ϕ is the mapping class on the torus with two boundary components given by the composition $\phi = \delta_{\mathfrak{c}} \circ \delta_b^{-1} \circ \delta_a$.

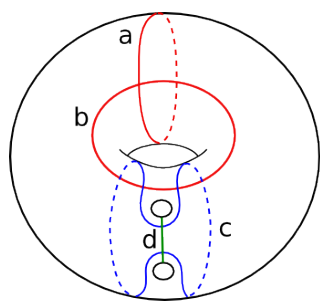

FIGURE 10. The limiting mapping class for Penner's sequence.

3.2. Alexander and Teichmüller polynomial.

Let M be the mapping torus of the quotient family Q.

Proposition 3.4. *The first Betti number of M equals 2.*

Proof. The first cohomology group of $H^1(S;\mathbb{Z})$ is generated by duals to $[a]$, $[b]$, and $[d]$, the relative homology classes defined by a, b, and d in $H_1(S, \partial S; \mathbb{Z})$. With respect to this basis, the action of ϕ on the first cohomology group $H^1(S, \mathbb{Z})$ is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The invariant cohomology is 1-dimensional, and hence $b_1(M) = 2$. $\square$

The cohomology class generating the invariant cohomology is dual to the path d between the two punctures on S, and the corresponding cyclic covering $\widetilde{S} \to S_0$ is the one drawn in Figure 3, with fundamental domain $\Sigma = S \setminus [d]$ the surface S slit at d. Let $\widetilde{\zeta}$ generate the group of covering automorphisms of $\mathbb{R} \times \widetilde{S} \to M$. Then $Z = \widetilde{\zeta} \times \{\mathrm{id}\}$ and $T = T_{\widetilde{\phi}}$ define covering automorphisms, and hence generators for $H_1(M;\mathbb{Z})$. Let $u, t \in H^1(M;\mathbb{Z})$ be duals to Z and T, respectively.

Let τ be the train track for ϕ_0 given by smoothings near intersections of the union of a, b, and c as in Figure 11. Endow a, b, and c with orientations.

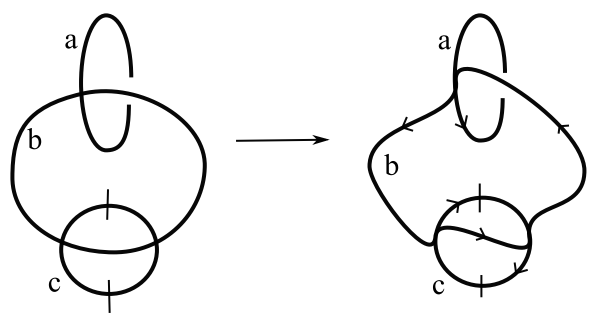

FIGURE 11. Traintrack τ for the quotient mapping class (S_0, ϕ_0).

Then τ lifts to a train track with oriented edges $\widetilde{\tau}$ on $\widetilde{S}$. To understand the action of $\widetilde{\phi}$, it suffices to know where $\widetilde{\phi}$ sends each closed curve the lifts $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ of a, b, and c on $\widetilde{S}$. For example, $\delta_{\widetilde{c}}$ acts by the identity on $\widetilde{a}$ and $\widetilde{c}$, and $\delta_{\widetilde{c}}(\widetilde{b})$ is shown in Figure 12.

When considering the algebraic action of $\widetilde{\phi}_0$, we count $\widetilde{c}^-$ as $-\widetilde{c}$. Thus, the algebraic action of $\widetilde{\phi}$ on $(\mathbb{Z}[t,t^{-1}])^{\widetilde{a},\widetilde{b},\widetilde{c}}$ as the matrix

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1-t \\ 1-t^{-1} & 2(1-t^{-1}) & 1+(1-t)(1-t^{-1}) \end{bmatrix}.$$

The Alexander polynomial Δ is the characteristic polynomial of the action of $\widetilde{\phi}$ on $H_1(\widetilde{S};\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$-module. More precisely, there is a free presentation of $H_1(\widetilde{M};\mathbb{Z})$ as a $\mathbb{Z}[t,t^{-1}]$-module given by

$$(\widetilde{\mathbb{Z}}[t,t^{-1}])^{\widetilde{a},\widetilde{b},\widetilde{c}} \xrightarrow{\mathcal{A}-uI} (\widetilde{\mathbb{Z}}[t,t^{-1}])^{\widetilde{a},\widetilde{b},\widetilde{c}} \longrightarrow H_1(\widetilde{M};\mathbb{Z}),$$

and the determinant of $\mathcal{A} - uI$ is the generator of the first fitting ideal for this presentation, that is, the *Alexander polynomial* of M. In our example,

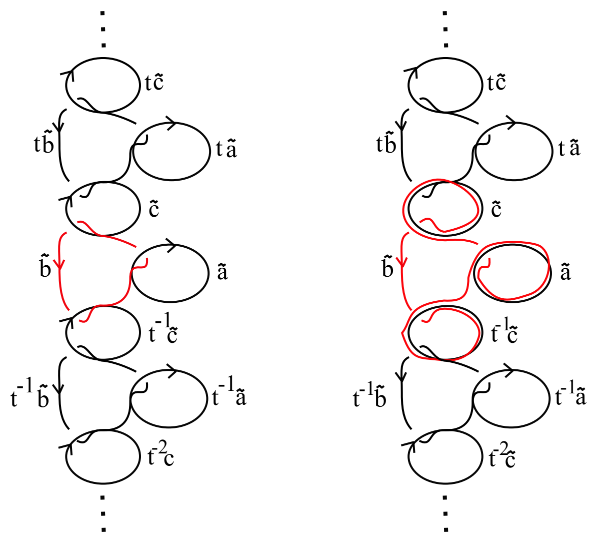

FIGURE 12. Lift of τ to $\widetilde{S}$ and action of $\delta_{\widetilde{c}}$ on $\widetilde{b}$.

this is given by.

$$\Delta(u,t) = \Theta(u,-t) = u^2 - u(5 - t - t^{-1}) + 1.$$

Considering $\widetilde{a}$, $\widetilde{b}$, and $\widetilde{c}$ as simple closed curves, we forget the sign of $\widetilde{c}^-$, and the action of $\widetilde{\phi}$ on $(\mathbb{Z}[t,t^{-1}])^{\widetilde{a},\widetilde{b},\widetilde{c}}$ becomes

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1+t \\ 1+t^{-1} & 2(1+t^{-1}) & 1+(1+t)(1+t^{-1}). \end{bmatrix}.$$

Thus, the Teichmüller polynomial is the characteristic polynomial of this matrix

$$\Theta(u,t) = u^2 - u(5 + t + t^{-1}) + 1.$$

Remark 3.5. In general, our group H is a quotient group of G isomorphic to $\mathbb{Z} \times \mathbb{Z}$, and Δ is the image of Δ_M in $\mathbb{Z}[H]$. Since, in this example, $b_1(M) = 2$, $\Delta(u,t)$ is the Alexander polynomial Δ_M of M and not a specialization.

3.3. Fibered face.

The fibered face of a 3-manifold M associated to a flow equivalence class can be found explicitly from the Tecihmüller polynomial of the face and the Alexander polynomial of M by a result of McMullen [9], which we recall here.

Let H be a finitely generated free abelian group. Write $f \in \mathbb{Z}H$ as

$$f = \sum_{h \in H_0} a_h h$$

where $H_0 \subset H$ is a finite subset and $a_h \neq 0$ for all $h \in H_0$. This representation for f is unique, and we call H_0 the *support* of f. If the convex hull of the points in H_0 is non-degenerate in $H \otimes \mathbb{R}$, then there is a corresponding norm on $\mathrm{Hom}(H;\mathbb{R})$ given by

$$||\alpha||_f = \max\{|\alpha(h_1) - \alpha(h_2)| \; : \; h_1, h_2 \in H_0\},$$

and the norm ball for $|| \; ||_f$ is convex polyhedral.

McMullen shows in [9] that if F is a fibered face of a hyperbolic 3-manifold, Δ and Θ_F are the Alexander and Teichmüller polynomials, respectively, and $b_1(M) \geq 2$, then the Thurston norm $|| \; ||$ restricted to the cone $V = F \cdot \mathbb{R}^+$ has the property that

$$||\alpha|| = ||\alpha||_\Delta \leq ||\alpha||_{\Theta_F}$$

for all $\alpha \in V_F$.

Lemma 3.6. *The fibered cone C in $H^1(M;\mathbb{R})$ associated to Penner wheels is given by elements $(a,b) \in H^1(M;\mathbb{R})$, satisfying*

$$a > |b|,$$

and the Thurston norm is given by

$$||(a,b)||_T = \max\{2|a|, 2|b|\}.$$

3.4. Dilatations and normalized dilatations.

The dilatation $\lambda(\phi_\alpha)$ corresponding to primitive integral points $\alpha = (a,b)$ in V is the largest solution of the polynomial equation

$$\Theta(x^a, x^b) = 0.$$

In particular, Penner's examples (R_g, ψ_g) correspond to the points $(g,1) \in V$, and we have the following.

Proposition 3.7. *The dilatation of ψ_g is given by the largest root of the polynomial*

$$\Theta(x^g, x) = x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1.$$

The limiting mapping class (S_0, ϕ_0) corresponds to the specialization

$$\theta(x) = \Theta(x,1) = x^2 - 7x + 1,$$

so $\lambda(\phi_0) = \frac{1}{2}(7 + 3\sqrt{5}) \approx 6.8541$.

The symmetry of Θ with respect to $x \mapsto -x$ and convexity of L on fibered faces implies that the minimum normalized dilatation realized on the fibered face must occur at $(a,b) = (1,0)$, i.e., at the monodromy (S_0, ϕ_0). Thus, we have the following.

Proposition 3.8. *For all monodromies (S', ϕ') of primitive integral elements in the cone V,*

$$L(S', \phi') \geq L(S_0, \phi_0) = \lambda(\phi_0)^2 \approx 46.9787.$$

3.5. Orientability.

A pseudo-Anosov mapping class is *orientable* if it has orientable invariant foliations, or equivalently the geometric and homological dilatations are the same, and the spectral radius of the homological action is realized by a real (possibly negative) eigenvalue (see, for example, [7, p. 5]). Given a polynomial f, the largest complex norm amongst its roots is called the *house of* f, denoted $|f|$. Thus, ψ_g is orientable if and only if

$$|\Delta(x^g, x))| = |\Theta(x^g, x)|. \tag{3.1}$$

Proposition 3.9. *The mapping classes (R_g, ψ_g) are orientable if and only if g is even.*

Proof. The homological dilatation of ψ_g is the largest complex norm amongst roots of

$$\Delta(x^g, x) = x^{2g} + x^{g+1} - 5x^g + x^{g-1} + 1.$$

Let λ be the real root of $\Delta(x^g, x)$ with largest absolute value. Plugging λ into $\Theta(x^g, x)$ gives

$$\Theta(\lambda^g, \lambda) = -2\lambda^{g+1} - 2\lambda^{g-1} \neq 0,$$

while for $-\lambda$, we have

$$\Theta(-\lambda^g, -\lambda) = (-\lambda)^{g+1} - (\lambda)^{g+1} + (-\lambda)^{g-1} - (\lambda^{g-1})$$

which equals 0 if and only if g is even. $\square$

3.6. Boundary behavior.

By Lemma 3.6, we can extend the parameterization

$$\mathfrak{f} : I_2 = (0, \frac{1}{2}) \to F$$

to

$$\begin{aligned} \mathfrak{f} : (-1, 1) &\to F \\ \mathfrak{c} &\mapsto \frac{1}{|\chi(\mathfrak{c})|}(1, \mathfrak{c}), \end{aligned}$$

Lemma 3.10. *The sequence of mapping classes associated to $\mathfrak{f}(\frac{n-1}{n})$ is conjugate to*

$$(\widetilde{S}/\widetilde{\zeta}^n, \zeta_n \delta_{\zeta_n^{-1}(c)} \delta_b^{-1} \delta_a).$$

Proof. Let $R : \widetilde{S} \to \widetilde{S}$ be a rotation around an axis that passes through $\widetilde{a} \cup \widetilde{b}$ in three points, preserves each of $\widetilde{a}$ and $\widetilde{b}$, and $R\widetilde{c} = \zeta^{-1}(\widetilde{c})$. Then we have

$$\begin{aligned} R^{-1}\widetilde{\zeta}R &= \widetilde{\zeta}^{-1} \\ R^{-1}\delta_{\widetilde{a}}R &= \delta_{\widetilde{a}} \\ R^{-1}\delta_{\widetilde{b}}R &= \delta_{\widetilde{b}} \\ R^{-1}\delta_{\widetilde{c}}R &= \delta^{-1}_{\widetilde{\zeta}^{-1}(\widetilde{c})}. \end{aligned}$$

We have

$$\widetilde{\zeta}^{-1}\widetilde{\eta} = \widetilde{\zeta}^{-1}\delta_{\widetilde{c}}\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}} = \widetilde{\zeta}^{-1}R\delta_{\widetilde{\zeta}^{-1}(c)}R\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}}.$$

Conjugating by R, we have

$$\begin{aligned} R\widetilde{\zeta}^{-1}\widetilde{\eta}R &= R\widetilde{\zeta}^{-1}R\delta_{\widetilde{\zeta}^{-1}(c)}R\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}}R \\ &= \widetilde{\zeta}\delta_{\widetilde{\zeta}^{-1}(c)}\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}} \end{aligned}$$

□

By Lemma 3.10, the mapping classes $(S_{\frac{n-1}{n}}, \phi_{\frac{n-1}{n}})$, also known as the *reverse Penner sequence*, are the same as the mapping classes for the sequence $(\frac{1}{n})$ in the family $Q' = Q(\widetilde{S}, \widetilde{\zeta}, \delta_{\zeta^{-1}(\widetilde{c})}\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}})$. In fact, Q and Q' are equal, but parameterized so that $\iota'(c) = \iota(1-c)$. By Lemma 3.6, Q' is not stable, and it follows that $\lim_{n\to\infty} L(S_{\frac{n-1}{n}}, \phi_{\frac{n-1}{n}}) = \infty$.

4. How Common Are Quotient Families?

A set of pseudo-Anosov mapping classes $\mathcal{F} \subset \mathcal{P}$ has *small dilatation* if there is some L so that for each $(S, \phi) \in \mathcal{F}$, $L(S, \phi) < L$, or in other words,

$$\log(\lambda(\phi)) < \frac{\log(L)}{|\chi(S)|}.$$

Given a pseudo-Anosov mapping class (S, ϕ), let (S^0, ϕ^0) be the mapping class obtained by removing any singularities of the invariant stable foliation. Then (S^0, ϕ^0) is called the *fully-punctured representative* of (S, ϕ). Given a collection $\mathcal{F}$ of pseudo-Anosov mapping classes, let $\mathcal{F}^0$ be the collection of fully-punctured representatives of elements of $\mathcal{F}$. One may ask if the following analog of the Farb–Leininger–Margalit universal finiteness theorem is true.

Question 4.1 (Penner-wheel question)**.** If $\mathcal{F}$ is a small dilatation collection of pseudo-Anosov mapping classes, is $\mathcal{F}^0$ contained in a finite union of (fully-punctured) quotient families?

One can also weaken the question as follows.

Definition 4.2. For $\kappa > 0$, a pseudo-Anosov mapping class (S, ϕ) is *κ-quasi-periodic* if there is a subsurface $Y \subset S$ and a mapping class $\zeta : Y \to Y$ such that

(1) for some m, ζ^m is a product of Dehn twists along boundary parallel curves on Y (i.e., ζ is *periodic rel boundary on* Y); and
(2) the support of $\zeta\phi$ has topological Euler characteristic χ bounded by
$$-\kappa < \chi < 0.$$

In other words, (x, ϕ) is κ-quasi-periodic if $\phi = \zeta\eta$ for some η with κ-small support.

A mapping class (S, ϕ) is *strongly κ-quasi-periodic* if $Y = S$. A family of mapping classes $\mathcal{F} \subset \mathcal{P}$ is *(strongly) quasi-periodic* if for some $\kappa > 0$ all its members are *(strongly) κ-quasi-periodic.*

Question 4.3 (Ferris-wheel question)**.** If $\mathcal{F}$ is a small dilatation collection of pseudo-Anosov mapping classes, is $\mathcal{F}^0$ contained in a quasi-periodic family?

Penner-type sequences and quotient families of mapping classes are strongly quasi-periodic where we can take $\kappa = m_1|\chi(\Sigma)|$. Small dilatations families, such as those found in [6] and [5], are quasi-periodic in the weaker sense and have not been shown to be strongly quasi-periodic.

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References

[1] Max Bauer, *Examples of pseudo-Anosov homeomorphisms*, Trans. Amer. Math. Soc. **330** (1992), no. 1, 333–359.

[2] Benson Farb, Christopher J. Leininger, and Dan Margalit, *Small dilatation pseudo-Anosov homeomorphisms and 3-manifolds*, Adv. Math. **228** (2011), no. 3, 1466–1502.

[3] Benson Farb and Dan Margalit, *A Primer on Mapping Class Groups*. Princeton Mathematical Series, 49. Princeton, NJ: Princeton University Press, 2012

[4] David Fried, *Flow equivalence, hyperbolic systems and a new zeta function for flows*, Comment. Math. Helv. **57** (1982), no. 2, 237–259.

[5] Eriko Hironaka, *Small dilatation mapping classes coming from the simplest hyperbolic braid*, Algebr. Geom. Topol. **10** (2010), no. 4, 2041-060.

[6] Eriko Hironaka and Eiko Kin, *A family of pseudo-Anosov braids with small dilatation*, Algebr. Geom. Topol. **6** (2006), 699–738.

[7] Erwan Lanneau and Jean-Luc Thiffeault, *On the minimum dilatation of pseudo-Anosov homeromorphisms on surfaces of small genus*, Ann. Inst. Fourier (Grenoble) **61** (2011), no. 1, 105–144.

[8] Curtis T. McMullen, *Polynomial invariants for fibered 3-manifolds and Teichmüller geodesics for foliations*, Ann. Sci. École Norm. Sup. (4) **33** (2000), no. 4, 519–560.

[9] Curtis T. McMullen, *The Alexander polynomial of a 3-manifold and the Thurston norm on cohomology*, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 2, 153–171.

[10] R. C. Penner, *Bounds on least dilatations*, Proc. Amer. Math. Soc. **113** (1991), no. 2, 443–450.

[11] William P. Thurston, *A norm for the homology of 3-manifolds*, Mem. Amer. Math. Soc. **59** (1986), no. 339, i–vi and 99–130.

[12] William P. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. (N.S.) **19** (1988), no. 2, 417–431.

[13] Chia-Yen Tsai, *The asymptotic behavior of least pseudo-Anosov dilatations*, Geom. Topol. **13** (2009), no. 4, 2253–2278.

[14] Aaron D. Valdivia, *Sequences of pseudo-Anosov mapping classes and their asymptotic behavior*, New York J. Math. **18** (2012), 609–620.

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