# QUOTIENT FAMILIES OF MAPPING CLASSES

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ABSTRACT. We define *quotient families* of mapping classes parameterized by rational points on an interval generalizing an example of Penner. This gives an explicit construction of families of mapping classes in a single flow-equivalence class of monodromies of a fibered 3-manifold $M$. The special structure of quotient families helps to compute useful invariants such as the Alexander polynomial of the mapping torus, and (in the case when $M$ is hyperbolic) the Teichmüller poliynomial of the associated fibered face. These in turn give useful information about the homological and geometric dilatations of the mapping classes in the quotient family.

## 1. INTRODUCTION

In [Pen2] Penner explicitly constructed a sequence of pseudo-Anosov mapping classes, sometimes called *Penner wheels*, with asymptotically small dilatations. In this paper we define a generalization of Penner wheels called *quotient families*, and put them in the framework of the Thurston-Fried-McMullen fibered face theory [Thu1] [Fri] [McM1]. Specifically, we show that each quotient family corresponds naturally to a linear section of a fibered face of a 3-manifold. Putting quotient families in the fibered face context helps to determine their Nielsen-Thurston classification, and in the pseudo-Anosov case makes it possible to compute dilatations via the Teichmüller polynomial.

1.1. **Pseudo-Anosov mapping classes, dilatations, and fibered faces.** Let $S$ be a connected oriented surface of finite type with negative Euler characteristic $\chi(S)$. A *mapping class* $\phi : S \to S$ is an orientation preserving homeomorphism modulo isotopy. The Nielsen-Thurston classification states that mapping classes are either periodic, reducible, or pseudo-Anosov, where $(S, \phi)$ is *pseudo-Anosov* if $\phi$ preserves a pair of transverse measured singular *stable* and *unstable* foliations $(\mathcal{F}^{\pm}, \nu^{\pm})$ and $\phi^*(\nu^{\pm}) = \lambda \mu^{\pm}$ for some $\lambda > 0$ [Thu2]. The constant $\lambda$ is uniquely determined by $(S, \phi)$ and is called its *dilatation*. The singularities of $\mathcal{F}^{\pm}$ are called the the *singularities of* $\phi$ (see also [FM]).

In [Pen2], Penner constructed a sequence of pseudo-Ansoov mapping classes $(R_g, \psi_g)$, for $g \geq 3$, where $R_g$ is a closed surface of genus $g \geq 2$ and $\lambda(\psi_g)^g \leq 11$. Using this he showed that the minimum expansion factor $l_g$ for pseudo-Anosov mapping classes of genus $g$ behaves like $\log(l_g) \asymp \frac{1}{g}$.

Since then, fibered face theory (recalled below) has been applied to show that pseudo-Anosov mapping classes with bounded *normalized dilatation* $L(S, \phi) := \lambda(\phi)^{|\chi(S)|}$ are commonplace: such families can be found in the set of monodromies of any hyperbolic fibered 3-manifold that has first Betti number greater than or equal to two [McM1]. Furthermore, the Universal Finiteness Theorem of Farb-Leininger-Margalit implies that any family of pseudo-Anosov mapping classes with bounded normalized dilatations is contained in the set of monodromies of a finite set of fibered 3-manifolds up to fiber-wise Dehn fillings [FLM].

1.2. **Penner wheels.** Penner wheels are defined as follows. Consider the genus $g$ surface $R_g$ as a surface with rotational symmetry of order $g$ fixing two points, as drawn in Figure 1, and let $\zeta_g$ be the counterclockwise rotation by the angle $\frac{2\pi}{g}$. For a simple closed curve $\gamma$ on a surface, let $\delta_\gamma$ be the right Dehn twist centered at $\gamma$. Let $\eta_g = \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g}$ be the product of Dehn twists centered along the labeled curves $a_g, b_g, c_g$ drawn in Figure 1. Then Penner's sequence consists of the pairs $(R_g, \psi_g)$, where $\psi_g = \zeta_g \eta_g$.
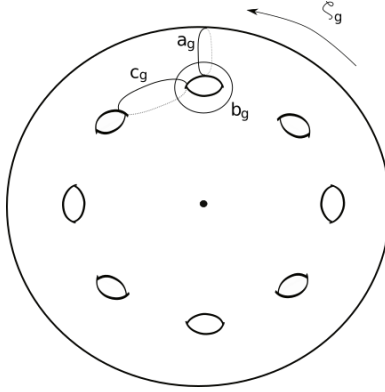


FIGURE 1. Penner wheel on a surface of genus $g$ with rotational symmetry fixing two points, one of which is marked in the figure as a central dot.

To define quotient families, consider a triple $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ satisfying the following:

(1) $\widetilde{S}$ is an oriented surface of infinite type with a properly discontinuous, orientation-preserving, fixed-point free infinite cyclic action generated by
$$\widetilde{\zeta} : \widetilde{S} \to \widetilde{S};$$

(2) $\widetilde{S}/\widetilde{\zeta}$ is a surface of finite genus and number of punctures; and

(3) the action of $\widetilde{\zeta}$ has a fundamental domain $\Sigma_0$, a compact, connected, oriented surface of finite type, so that the support of $\widetilde{\eta}$ is strictly contained in
$$\Sigma_0 \cup \widetilde{\zeta}\Sigma_0 \cup \cdots \cup \widetilde{\zeta}^{m_0}\Sigma_0.$$

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We say that the triple $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ forms a *template* of *width* $m_0$.

Let $I_{m_0}(\mathbb{Q})$ be the rational points on the open interval $I_{m_0} = (0, \frac{1}{m_0})$. From a template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ with width $m_0$ we define an associated *quotient family* $Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ parameterized by $I_{m_0}(\mathbb{Q})$ as follows. For $c \in I_{m_0}(\mathbb{Q})$, where $c = \frac{k}{n}$ is in reduced form, define a mapping class $(S_c, \phi_c)$ as follows. Let $\widetilde{\eta}_n$ be the composition

$$\widetilde{\eta}_n := \underset{r \in \mathbb{Z}}{\circ} \widetilde{\zeta}^{rn} \widetilde{\eta} \widetilde{\zeta}^{-rn},$$

which is well-defined on $\widetilde{S}$ since $n > m_0$, and hence the supports of $\widetilde{\zeta}^{rn} \widetilde{\eta} \widetilde{\zeta}^{-rn}$ are disjoint for distinct $r$. The map $\widetilde{\eta}_n$ is invariant under conjugation by $\widetilde{\zeta}^n$, and thus defines a well-defined map $\eta_n$ on the quotient space $S_c = \widetilde{S}/\zeta^n$. Similarly, $\widetilde{\zeta}$ defines a map $\zeta_n$ on the quotient space $S_c$. Let

$$\phi_c := (\zeta_n)^{\overline{k}_n} \circ \eta_n,$$

where $\overline{k}_n k = 1 (\mathrm{mod}\ n)$. The *quotient family* associated to the template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ is defined by

$$Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta}) := \{(S_c, \phi_c) \mid c \in I_{m_0}(\mathbb{Q})\}.$$

For example, the surfaces $R_g^0$ obtained from Penner's sequence by removing the two centers of rotation is a sequence in a quotient family where $(\widetilde{S}, \widetilde{\zeta})$ has fundamental domain $\Sigma_0$ homeomorphic to a torus with one boundary component and the copies $\zeta^r \Sigma_0$ are attached along disjoint arcs on the boundary of each copy of $\Sigma_0$. The map $\widetilde{\eta}$ is defined by $\widetilde{\eta} = \delta_{\widetilde{c}} \delta_{\widetilde{b}}^{-1} \delta_{\widetilde{a}}$, where $\widetilde{a}, \widetilde{b}$, and $\widetilde{c}$ are simple closed curves so that $\widetilde{a} \cup \widetilde{b} \cup \widetilde{c}$ is a lift of $a_g \cup b_g \cup c_g$ drawn in Figure 1 for each $g$.

### 1.3. Fibered faces and parameterizations of flow-equivalence classes.
Let $\mathrm{MCG}(S)$ be the group of mapping classes defined on $S$. Thurston's fibered face theory [Thu1] gives a way to partition the set of all mapping classes

$$\mathrm{MCG} = \{(S, \phi)\ \phi \in \mathrm{MCG}(S),\ S \text{ a connected orientable surface of finite type}\}$$

into families with related dynamics. Each mapping class $(S, \phi)$ defines (up to isotopic equivalence)

(1) a mapping torus $M = [0, 1] \times S/(x, 1) \sim (\phi(x), 0)$;
(2) a distinguished fibration $\rho : M \to S^1$ induced by projection onto the second coordinate with *monodromy* $(S, \phi)$; and
(3) a one-dimensional oriented *suspension flow*, or *foliation*, $\mathcal{L}$ on $M$ whose leaves are the images of the leaves $\mathbb{R} \times \{x\}$ under the cyclic covering map $\mathbb{R} \times S \to M$ defined by $\rho_* : \pi_1(M) \to \mathbb{Z}$.

Two mapping classes are said to be *flow-equivalent* if they determine the same isotopy class of pairs $(M, \mathcal{L})$. The induced homomorphism $\rho_* : H_1(M; \mathbb{Z}) \to \mathbb{Z}$ determines an element $\alpha \in H^1(M; \mathbb{R})$, called a *fibered element*.

Thurston defined a semi-norm $|| \ ||$ on $H^1(M; \mathbb{R})$ with a convex polygonal unit norm ball. Each cone $V_F$ over an open top-dimensional face $F$ is either *fibered* if all primitive integral elements are fibered, or contains no fibered elements. The primitive integral elements of

any fibered cone are in one-to-one correspondence with rational points on the fibered face $F$.

It follows from this discussion that the set of all mapping classes on arbitrary connected orientable surfaces of finite type is parameterized by the union of rational points on fibered faces of 3-manifolds. Explicitly, for $\alpha$ a fibered element in a fibered cone $V_F$, let $\overline{\alpha}$ be its projection onto $F$ along the rational ray, and let $(S_\alpha, \phi_\alpha)$ be its monodromy. For each fibered 3-manifold $M$ and fibered face $F \subset H^1(M; \mathbb{R})$ define

$$\mathfrak{f}_F : F(\mathbb{Q}) \quad \to \quad \text{MCG}$$
$$\overline{\alpha} \quad \mapsto \quad (S_\alpha, \phi_\alpha)$$

taking each $\overline{\alpha}$ to the monodromy $(S_\alpha, \phi_\alpha)$, where $\alpha \in V_F$ is the primitive integral element $\alpha \in V_F$ that is a positive multiple of $\overline{\alpha}$. Let $\mathfrak{C}$ be the set of all flow-equivalence classes, and let $\mathcal{C}$ be the set of all fibered faces. Then we have a bijection

$$\mathfrak{c} : \mathcal{C} \to \mathfrak{C}$$

taking fibered faces to corresponding flow-equivalence classes, and parameterizations of the elements of each flow equivalence class $\mathfrak{F} \in \mathfrak{C}$ by the set of rational points on $F = \mathfrak{c}(\mathfrak{F})$.

Our first result characterizes quotient families in terms of the above parameterization of MCG.

**Theorem A.** *Each quotient family $Q$ is contained in some flow equivalence class $\mathfrak{F}$. Let $F = \mathfrak{c}^{-1}(\mathfrak{F})$. Then there is an embedding*

$$\iota : I_{m_0} \hookrightarrow F,$$

*such that*

   *(1) the image $\iota(I_{m_0})$ is a linear section of $F$ in $H^1(M; \mathbb{R})$,*
   *(2) $\iota$ restricts to a map $I_{m_0}(\mathbb{Q}) \to F(\mathbb{Q})$, and*
   *(3) for all $c \in I_{m_0}(\mathbb{Q})$,*

$$(S_c, \phi_c) = \mathfrak{f}_F(\iota(c)).$$

Now consider the set of pseudo-Anosov mapping classes $\mathcal{P} \subset \text{MCG}$. By a theorem of Thurston [Thu2] the mapping torus $M$ of $(S, \phi)$ is hyperbolic if and only if $(S, \phi)$ is pseudo-Anosov.

Thus, Theorem A has the following immediate corollary:

**Corollary 1.1.** *Let $Q$ be a quotient family. Then $Q$ is contained in $\mathcal{P}$ if and only if $Q \cap \mathcal{P} \neq \emptyset$.*

1.4. **Fibered faces and bounded normalized dilatations.** Let $M$ be a hyperbolic 3-manifold with fibered face $F \subset H^1(M; \mathbb{R})$. Fried [Fri] showed that the function

$$\alpha \mapsto \log \lambda(\phi_\alpha)$$

defined for $\alpha$ a primitive integral element of the cone $V_F = F \cdot \mathbb{R}^+$ extends to a continuous convex function on $V_F$ that is homogeneous of degree $-1$ and goes to infinity toward the

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boundary of $V_F$. The Thurston norm $||\ ||$ on $H^1(M;\mathbb{R})$ has the property that $||\alpha|| = |\chi(S_\alpha)|$ for all integral elements $\alpha$ on a fibered cone [Thu1], and hence

$$L(S_\alpha, \phi_\alpha) = \lambda(\phi_\alpha)^{||S_\alpha||}.$$

Noting that $L$ is the composition of Fried's function with the exponential function, we have the following.

**Proposition 1.2** (Fried)**.** *Given a flow-equivalence class $\mathfrak{F} \subset \mathcal{P}$ with $F = \mathfrak{c}(\mathfrak{F})$, the function*

$$
\begin{aligned}
L : F(\mathbb{Q}) &\to \mathbb{R} \\
\overline{\alpha} &\mapsto L(S_\alpha, \phi_\alpha)
\end{aligned}
$$

*extends uniquely to a continuous convex function $L : F \to \mathbb{R}$ that goes to infinity toward the boundary of $F$ and satisfies*

$$1 < c_K \leq L(\overline{\alpha}) < C_K$$

*for any compact subset $K \subset F$, where $c_K$ is the minimum of $L$ on $F$.*

Farb-Leininger-Margalit's [FLM] *Universal Finiteness Theorem* states conversely that for any $C > 0$, there is a finite set of fibered 3-manifolds $\Omega$ such that for any pseudo-Anosov map $(S, \phi)$ with $L(S, \phi) < C$, there is some $M \in \Omega$ such that $M$ is the mapping torus for $(S^0, \phi^0)$, where $(S^0, \phi^0)$ is the mapping class obtained from $(S, \phi)$ by removing the singularities of the $\phi$.

1.5. **Behavior of normalized dilatations and stability.** Our second result deals with the behavior of the normalized dilatations of a quotient family with pseudo-Anosov elements. We say $Q = Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ is a *stable* family if, for some $m_1$,

$$(\widetilde{\zeta}\widetilde{\eta})^{m+1}(x) = \widetilde{\zeta}(\widetilde{\zeta}\widetilde{\eta})^m(x),$$

for each $x \in \Sigma_0 \cup \cdots \widetilde{\zeta}^{m_0}(\Sigma_0)$, and all $m \geq m_1$. Consider the function defined by

$$
\begin{aligned}
\widetilde{\phi} : \widetilde{S} &\to \widetilde{S} \\
x &\mapsto \widetilde{\zeta}^{r-m_1}(\widetilde{\zeta}\widetilde{\eta})^{m_1}\widetilde{\zeta}^{-r},
\end{aligned}
$$

where $r$ is the greatest integer such that $\widetilde{\zeta}^{-r}(x) \in \Sigma_0$. If $Q$ is stable, then $\widetilde{\phi}$ is a well-defined homeomorphism (see Lemma 2.11).

**Theorem B.** *The following are equivalent for a quotient family $Q$:*

*(1) $Q$ is stable;*

*(2) the map $\widetilde{\phi}$ defines a mapping class on $\widetilde{S}$ that commutes with the action of $\mathbb{Z}$, and hence defines a mapping class $(S, \phi)$, where $S = \widetilde{S}/\zeta$ and $\phi$ is the mapping class on $S$ induced by $\widetilde{\phi}$;*

*(3) the map $\iota$ extends to $0$ so that $\iota(0)$ lies in the interior of $F$; and*

*(4) the value of the normalized stretch-factor $L(S_c, \phi_c)$ is bounded as $c$ approaches $0$.*

By Fried's theorem, it follows that if $Q$ is stable then,

$$\lim_{c\to 0} L(S_c, \phi_c) = L(S, \phi);$$

and if $Q$ is not stable, then $\lim_{c\to 0} \iota(c)$ lies on the boundary of $F$ and

$$\lim_{c\to 0} L(S_c, \phi_c) = \infty.$$

**Remark 1.3.** Up to now explicit examples and partial generalizations of Penner wheels have been studied without putting them in the context of fibered faces (see [Bau] [Tsa] [Val]). One benefit of seeing quotient families as elements of a single fibered face is the possibility of getting explicit defining equations for the geometric and homological stretch-factors via the Teichmüller and Alexander polynomials. We carry out some explicit calculations in Section 3.

1.6. **Outline of proofs and organization.** To prove Theorem A, we start with $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ and construct a 3-manifold $\widetilde{M}$ of infinite type with one dimensional foliation $\widetilde{\mathcal{L}}$ preserved by a properly discontinuous, fixed-point free action of the rank 2 free abelian group $H = \mathbb{Z} \times \mathbb{Z}$ on $\widetilde{M}$. Setting $M = \widetilde{M}/H$, defines a 3-manifold, with first Betti number satisfying $b_1(M) \geq 2$, equipped with a foliation $\mathcal{L}$. We show that the quotient family $Q = Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ is contained in the flow equivalence class defined by $\mathcal{L}$, and is associated to rational points on a linear segment contained in the associated fibered face $F$. This defines the map $\iota : I_{m_0} \to F$. We complete the proof of Theorem A in Section 2.3.

By fibered face theory, there are two possible behaviors for $(S_c, \phi_c)$ as $c \to 0$, depending on whether $\iota(c)$ converges to the boundary of $F$ or to an interior point of $f$ as $c$ approached 0. In Section 2.4, we show that the bounded case occurs if and only if $Q$ is stable Theorem B. In Section 3, we illustrate how the results can be applied to explicit examples.

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## 2. Construction of a one-dimensional flow-equivalence class

In this section, we prove Theorem A by starting with a template $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$, and building a 3-manifold $\widetilde{M}$ of infinite type, a one-dimensional foliation $\widetilde{\mathcal{L}}$, and a properly discontinuous, fixed-point free action of a group $H = \mathbb{Z} \times \mathbb{Z}$ on $\widetilde{M}$ with the following properties:

(1) $H$ preserves the leaf structure of $\widetilde{\mathcal{L}}$;
(2) there is an open cone $V$ in $\mathrm{Hom}(H; \mathbb{R})$ so that for all primitive integral $\alpha \in V$
   (a) the kernel $K_\alpha$ preserves the fibers of a fibration $\pi_\alpha : \widetilde{M} \to \mathbb{R}$; and
   (b) the leaves of $\widetilde{\mathcal{L}}$ are sections of $\pi_\alpha$.

The above properties imply that $\pi_\alpha$ and $\widetilde{\mathcal{L}}$ define a trivialization $\widetilde{M} = \widetilde{S}_\alpha \times \mathbb{R}$. Setting $M = \widetilde{M}/H$ and $\mathcal{L} = \widetilde{\mathcal{L}}/H$, the fibrations $\pi_\alpha$ descend to fibrations of $M$ whose fibers, homeomorphic to $S_\alpha := \widetilde{S}_\alpha/Z_\alpha$, are cross-sections of $\mathcal{L}$. The first return map $\phi_\alpha$ defines a mapping class on $S_\alpha$, and the pairs $(S_\alpha, \phi_\alpha)$ constructed in this way all lie in the flow-equivalence class defined by the pair $(M, \mathcal{L})$.
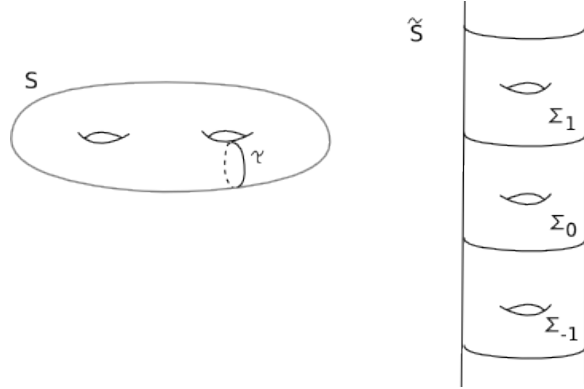


FIGURE 2. The surface $\widetilde{S}$ and the quotient surface $S = \widetilde{S}/\widetilde{\zeta}$.

2.1. **Constructing $\widetilde{M}$, $\widetilde{\mathcal{L}}$ and $H$.** Starting with $(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$, let $S := \widetilde{S}/\widetilde{\zeta}$, and let $\Sigma_0$ be a fundamental domain on $\widetilde{S}$ for the map $\widetilde{\zeta}$, with the following properties:

(i) $\Sigma_0$ is a connected closed subset of $\widetilde{S}$;
(ii) $\widetilde{\zeta}(\Sigma_0) \cap \Sigma_0$ is a finite disjoint union of closed arcs on the boundary $\partial \Sigma_0$ of $\Sigma_0$; and
(iii) $\widetilde{S} = \bigcup_{i \in \mathbb{Z}} \zeta^i(\Sigma_0)$.

Let $\widetilde{M'} = \mathbb{R} \times \widetilde{S}$, and let $\widetilde{\mathcal{L}}'$ be the flow defined on $\widetilde{M'}$ with oriented leaves $\mathbb{R} \times \{x\}$. We construct $\widetilde{M}$ and $\widetilde{\mathcal{L}}$ using cutting and pasting on $\widetilde{M'}$.

Choose any continuous surjective function $h : \widetilde{S} \to \mathbb{R}$ with the properties:

(1) each fiber of $h$ is an immersed union of simple closed curves (rel. punctures) on $\widetilde{S}$ that split $\widetilde{S}$ into exactly two pieces;
(2) $\Sigma_0 \cap \widetilde{\zeta}^{-1}(\Sigma_0) = h^{-1}(0)$; and
(3) $h(\widetilde{\zeta}^k(x)) = h(x) + k$ for all $x \in \widetilde{S}$ and $k \in \mathbb{Z}$.

For example, $h$ could be the height function on $\widetilde{S}$ illustrated in Figure 2.

Let $\mathfrak{h}' : \widetilde{M'} \to \mathbb{R} \times \mathbb{R}$ be defined by $\mathfrak{h}' = \text{id} \times h$. Let

$$
\begin{aligned}
T' : \widetilde{M'} &\to \widetilde{M'} \\
(x, y) &\mapsto (x - 1, y)
\end{aligned}
$$

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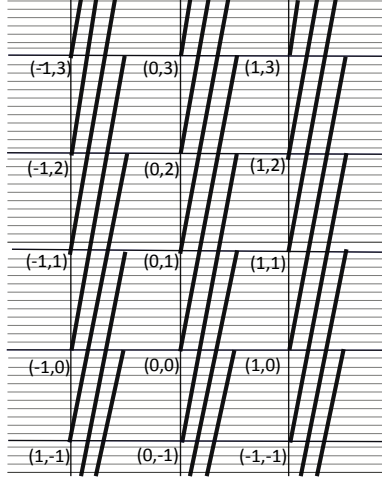
FIGURE 3. The images of the cutting loci $\Gamma_{a,b}$ under $h \times \mathrm{id} : \widetilde{M}' \to \mathbb{R} \times \mathbb{R}$. Horizontal lines indicate the flow $\mathcal{L}'$.

and

$$
\begin{aligned}
Z' : \widetilde{M}' &\to \widetilde{M}' \\
(x, y) &\mapsto (x, \widetilde{\zeta}(y)).
\end{aligned}
$$

Then $T'$ and $Z'$ generate a properly discontinuous, fixed-point free $\mathbb{Z} \times \mathbb{Z}$ action on $\widetilde{M}'$, and

$$
M' = \widetilde{M}'/\langle T', Z' \rangle = S^1 \times S.
$$

2.2. **Cutting and pasting.** We construct a cutting locus $\mathcal{G}$ on $\widetilde{M}'$ by defining a locus on $\mathbb{R} \times \mathbb{R}$, and taking the preimage by the map $\mathfrak{h}'$. Let $\Gamma_{0,0} \subset \mathbb{R} \times \mathbb{R}$ be the straight line segment connecting $(0,0)$ to $(\frac{1}{2}, m_0)$, and, for $(a, b) \in \mathbb{R} \times \mathbb{R}$, let $\Gamma_{a,b} = (a, b) + \Gamma_{0,0}$ be the parallel translate of $\Gamma_{0,0}$ by $(a, b)$. Then we have the following:

(1) the projection $\sigma' : \widetilde{M}' \to \widetilde{S}$ to the second coordinate gives an identification

$$
\sigma'(\mathfrak{h}'^{-1}(\Gamma_{a,b})) = \widetilde{\zeta}^b \Sigma_0 \cap \cdots \cap \widetilde{\zeta}^{m_0+b} \Sigma_0;
$$

(2) the maps $T'$ and $Z'$ satisfy

$$
\mathfrak{h}'(T'(x,y)) = \mathfrak{h}'(x,y) + (-1, 0) \quad \text{and} \quad \mathfrak{h}'(Z'(x,y)) = \mathfrak{h}'(x,y) + (0, 1);
$$

(3) $T'(\Gamma_{a,b}) = \Gamma_{a-1,b}$ and $Z'(\Gamma_{a,b}) = (\Gamma_{a,b+1})$; and
(4) $\Gamma_{a,b}$ are pairwise disjoint.

Cut $\widetilde{M}'$ along

$$
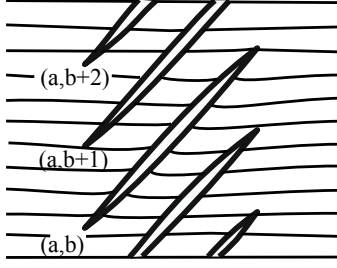\mathcal{G} = \bigcup_{(a,b) \in \mathbb{Z} \times \mathbb{Z}} \mathfrak{h}'^{-1}(\Gamma_{a,b}),
$$

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FIGURE 4. A figurative local picture $(\widetilde{M}', \mathcal{L}')$ of cut along the slits $\mathfrak{h}'^{-1}(\Gamma_{a,b})$, for $(a,b) \in \mathbb{Z} \times \mathbb{Z}$. The top and bottom sides of the slits are pasted together using the map $\widetilde{\eta}$ to form $(\widetilde{M}, \mathcal{L})$.

i.e., remove the locus $\mathcal{G}$ from $\widetilde{M}'$ and replace it with two copies of $\mathcal{G}$ that only intersect at $\bigcup_{(a,b) \in \mathbb{Z} \times \mathbb{Z}} \mathfrak{h}'^{-1}(\partial \Gamma_{a,b})$. creating a slit with a well defined *top* and *bottom*. Figure 4 gives a local illustration of the cut $\widetilde{M}'$.

Let $\widetilde{M}$ be obtained from the cut $\widetilde{M}'$ by pasting, by the map $\widetilde{\zeta}^b \widetilde{\eta} \widetilde{\zeta}^{-b}$, the top to the bottom of the slit associated to $\mathfrak{h}'^{-1}(\Gamma_{a,b})$ after identification with $\widetilde{\zeta}^b \Sigma_0 \cap \cdots \cap \widetilde{\zeta}^{m_0+b} \Sigma_0$. Identifying each component of $\mathcal{G}$ with the top copy of the corresponding slit in $\widetilde{M}'$ defines a bijective map

$$\tau : \widetilde{M} \to \widetilde{M}',$$

where $\tau \circ Z = Z' \circ \tau$, and $\tau \circ T = T' \circ \tau$. The locus of discontinuity of $\tau$ is contained in $\mathcal{G}$.

Let

$$\mathfrak{h} : \widetilde{M} \to \mathbb{R} \times \mathbb{R}$$

be the composition $\mathfrak{h} := \mathfrak{h}' \circ \tau$. Let $\sigma : \widetilde{M} \to \widetilde{S}$ be the composition $\sigma := \sigma' \circ \tau$, and let $\pi : \widetilde{M} \to \mathbb{R}$ be the composition of $\tau$ (or equivalently $\mathfrak{h}$) with projection onto the first coordinate.

The foliation $\widetilde{\mathcal{L}}'$ defines a foliation $\widetilde{\mathcal{L}}$ on $\widetilde{M}$, by cutting $\widetilde{\mathcal{L}}'$ along $\mathcal{G}$, which intersects each leaf transversally, and pasting as above.

**Lemma 2.1.** *The automorphisms $T'$ and $Z'$ define well-defined actions $T$ and $Z$ on $\widetilde{M}$ satisfying*

(1) $\sigma \circ T = \sigma$;
(2) $\sigma \circ Z = \zeta \circ \sigma$; *and*
(3) $T$ *and* $Z$ *preserve the leaf structure on* $\widetilde{\mathcal{L}}$.

*Proof.* Properties (1) and (2) follow from the corresponding properties of $T'$, $Z'$, and $\sigma'$, and the fact that the cutting and pasting locus $\mathcal{G}$ is (set-wise) invariant under $T'$ and $Z'$. Since $\sigma \circ Z = \zeta \circ \sigma$, the gluing maps commute with $Z$ and $T$ proving (3). □

Let $H = \langle T, Z \rangle$, and let $M = \widetilde{M}/H$. In the next section, we prove that the quotient family $Q$ consists of monodromies of fibrations of $M$.

**Remark 2.2.** We choose the notation $T$ to correspond to translation in $\mathbb{R} \times \mathbb{R}$ by $(-1, 0)$ in order to be compatible to the notation used for mapping tori:

$$M = [0,1] \times S/(1, x) \simeq (0, \phi(x)) = \mathbb{R} \times S/(t, x) \simeq (t-1, \phi(x))$$

(cf. [McM1]).

**Remark 2.3.** Although projection of $\widetilde{M}'$ to the first coordinate is surjective on leaves of $\widetilde{\mathcal{L}}'$, the corresponding projection $\pi_0 : \widetilde{M} \to \mathbb{R}$ is not necessarily surjective on individual leaves of $\widetilde{\mathcal{L}}$ in $\widetilde{M}$. We show that surjection of $\pi_0$ on all leaves of $\widetilde{\mathcal{L}}$ is equivalent to stability in Section 2.4.

2.3. **Proof of Theorem A.** To find a suitable fibered face $F$ in $H^1(M; \mathbb{R})$, we first fix coordinates. Since $\widetilde{M} \to M$ is a regular abelian covering with automorphism group $H$, it is an intermediate covering of the maximal abelian covering of $M$, and hence there is a corresponding quotient map $H_1(M; \mathbb{Z}) \to H$.

Identifying $H^1(M; \mathbb{R})$ with homomorphisms of $H_1(M; \mathbb{Z})$ to $\mathbb{R}$, we obtain an inclusion:

$$\mathrm{Hom}(H; \mathbb{R}) \to H^1(M; \mathbb{R}).$$

Thus, the dual elements of $Z$ and $T$ in $\mathrm{Hom}(H; \mathbb{R})$ define element $z, u \in H^1(M; \mathbb{R})$ that form a basis for a two dimensional subspace $W \subset H^1(M; \mathbb{R})$. Using the notation $(a, b) = au + bz$, let $V \subset W$ be the cone defined by

$$V = \{(-a, -b) \in W \mid 0 < bm_0 < a\}.$$

We will show that $V$ is contained in a fibered cone for $M$. Let $S = \widetilde{S}/\widetilde{\zeta}$. Then $\Sigma_0$ is obtained from $S$ by removing a finite collection of simple closed curves, or arcs connecting punctures on $S$.

**Lemma 2.4.** *For $c = \frac{k}{n}$,*

$$|\chi(S_c)| = n|\chi(S)|.$$

*Proof.* The surface $S_c = \widetilde{S}/\zeta^n$ is an $n$-fold covering of $S = \widetilde{S}/\zeta$. $\qquad\square$

By Lemma 2.4, can write $V = F_Q \cdot \mathbb{R}^+$, where $F_Q$ is defined by

$$F_Q := \left\{ \left( -\frac{1}{|\chi|(S)|}, -\frac{k}{n|\chi|(S)|} \right) \in V \;\middle|\; 0 < \frac{k}{n} < \frac{1}{m_0} \right\}.$$

We prove Theorem A by showing the following.

**Proposition 2.5.** *The Thurston norm ball in $H^1(M; \mathbb{R})$ has a fibered face $F$ satisfying*

$$F_Q \subset F \cap W;$$

*and the map*

$$\iota : I_{m_0} \rightarrow F$$
$$c \mapsto \frac{1}{|\chi(S)|}(-1, -c)$$

*has the property that for $c \in I_{m_0}(\mathbb{Q})$*

$$(S_c, \phi_c) = \mathfrak{f}(\iota(c)).$$

We prove Proposition 2.5 in three steps.

**Step 1. Fibrations of $\widetilde{M}$ transverse to the foliation $\widetilde{\mathcal{L}}$.**
For $c = \frac{k}{n} \in I_{m_0}(\mathbb{Q})$, let $\alpha_c$ be the primitive integral element in $V$ on the ray defined by $\iota(c)$. Seen as a homomorphism, we have

$$\alpha_c : H \rightarrow \mathbb{Z}$$
$$T^a Z^b \mapsto -an - bk,$$

and the kernel $K_c$ of $\alpha_c$ is freely generated by $Z_c := T^{-k} Z^n$. Let $w, \bar{k}_n$ be the solutions to $wn + \bar{k}_n k = 1$, and let $T_c := T^w Z^{\bar{k}}$. Then $\alpha_c(T_c) = -1$.

We will define a fibration $\pi_c : \widetilde{M} \rightarrow \mathbb{R}$ so that the fibers of $\pi_c$ have the following properties:

(1) each fiber of $\pi_c$ is preserved by the action of the kernel of $\pi_c$;
(2) the fibers of $\pi_c$ are permuted by the actions of $Z$ and $T$ are invariant under the action of the kernel of $\alpha_c$; and
(3) each fiber of $\pi_c$ intersects each leaf of $\mathcal{L}$ exactly once.

We begin by first defining a suitable projection $p_c : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, and then pulling back by $\mathfrak{h}$ to $\widetilde{M}$.

**Proposition 2.6.** *There is a continuous monotone increasing function $f_c : \mathbb{R} \rightarrow \mathbb{R}$ such that*

*(1) for $r \in \mathbb{Z}$, $f_c(r) = \frac{rn}{k}$;*
*(2) $f_c(x) - f_c(x + r) = \xi_r$, for some constant $\xi_r \in \mathbb{R}$ independent of $x$;*
*(3) for $r \in \mathbb{Z}$, $f_c(x + r) = f_c(x) + \frac{r}{c}$; and*
*(4) each $\Gamma_{a,b}$ is contained in the locus $y = f_c(x) + \frac{r}{k}$ for some $r \in \mathbb{Z}$.*

*Proof.* We constructive an explicit example. Let $\Delta_{0,0}$ be the straight line segment on $\mathbb{R} \times \mathbb{R}$ connecting $p := (\frac{1}{2}, m_0)$ and $q_c := (1, \frac{n}{k})$. Let

$$\Delta_{a,b} := (a, b) + \Delta_{0,0},$$

for $(a, b) \in \mathbb{R} \times \mathbb{R}$. Let

$$\widetilde{R}_c := \bigcup_{r \in \mathbb{Z}} (\Gamma_{r, \frac{rn}{k}} \cup \Delta_{r, \frac{rn}{k}}),$$

and

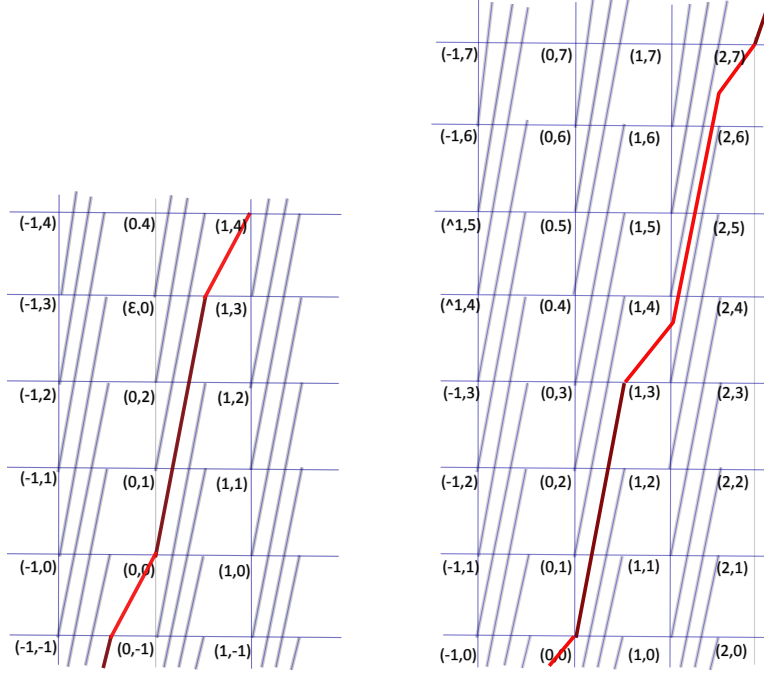$$\widetilde{R}_{c,\xi} := \widetilde{R}_c + (0, \xi).$$

FIGURE 5. The loci $\mathcal{R}_c \subset \mathbb{R} \times \mathbb{R}$ for $c = \frac{1}{4}$ (left) and $c = \frac{2}{7}$ (right) when $m_0 = 3$.

(See Figure 5). Then $\widetilde{R}_c$ is the graph of a piecewise-linear, monotone increasing function, $f_c$ with the desired properties. □

**Lemma 2.7.** *The loci $\mathcal{R}_{c,\xi}$ have the following properties.*

*(a) the $\widetilde{R}_{c,\xi}$ are fibers of a fibration $p_c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ that sends $\widetilde{R}_{c,\xi}$ to $\xi$;*

*(b) translation by $(r, \frac{rn}{k})$, for $r \in \mathbb{Z}$ defines an infinite cyclic action on $R_{c,\xi}$ with fundamental domain given by $(0,\xi) + \Gamma_{0,0} \cup \Delta_{0,0}$; and*

*(c) $(0,1) + \widetilde{R}_{c,\xi} = \widetilde{R}_{c,\xi+1}$ and $(-1,0) + \widetilde{R}_{c,\xi} = \widetilde{R}_{c,\xi+\frac{n}{k}}$.*

*Proof.* To verify propery (1) it suffices to check that the $R_{c,\xi}$ are connected and partition $\mathbb{R} \times \mathbb{R}$. This follows from the fact that each $\widetilde{R}_{c,\xi}$ is the graph of a monotone increasing, piecewise linear, continuous function on $\mathbb{R}$: $\Gamma_{r,\frac{rn}{k}}$ is a straight line segment connecting $(r, \frac{rn}{k})$ to $(r + \frac{1}{2}, \frac{rn}{k} + m_0)$, and $\Delta_{r,\frac{rn}{k}}$ is a straight line segment connecting $(r + \frac{1}{2}, \frac{rn}{k} + m_0)$ to $(r + 1, \frac{(r+1)n}{k})$. In other words, $\widetilde{R}_{c,\xi} = \{(x,y) \mid y = f(x) - \xi\}$ for a monotone increasing continuous function $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(r) = (0, \frac{rn}{k})$, for each $r \in \mathbb{Z}$. To prove (2) we note that $(1, \frac{n}{k}) \in R_{c,0}$, and to prove (3) we note that since $(1, \frac{n}{k}) \in R_{c,0}$, $(0, \frac{n}{k}) \in (-1,0) + R_{c,0}$, □

We now apply Lemma 2.7 to $\widetilde{M}$. Let $\pi_c = p_c \circ h$, and let $\widetilde{S}_{c,\xi} := \mathfrak{h}^{-1}(\widetilde{R}_{c,\xi}) \subset \widetilde{M}$.

**Lemma 2.8.** *The map $\pi_c$ has the following properties:*

*(1)* $\pi_c$ *is a fibration with fibers* $S_{c,\xi}$*;*

*(2)* $K_c = \langle Z_c \rangle$ *generates the set-wise stabilizer in* $H$ *of* $\widetilde{S}_{c,\xi}$*;*

*(3)* *the fundamental domain of the action of* $K_c$ *on* $\widetilde{S}_c$ *is*

$$\Sigma_c := \mathfrak{h}^{-1}\left( \bigcup_{r=0}^{k} \Gamma_{r,\frac{rn}{k}} \cup \Delta_{r,\frac{rn}{k}} \right).$$

*(4)* $Z(\widetilde{S}_{c,\xi}) = \widetilde{S}_{c,\xi+1}$ *and* $T(\widetilde{S}_{c,\xi}) = \widetilde{S}_{c,\xi+\frac{n}{k}}$*;*

*(5)* $T_c(\widetilde{S}_{c,\xi}) = \widetilde{S}_{c,\xi+\frac{1}{k}}$*; and*

*(6)* $\pi_c$ *restricts to a homeomorphism on each leaf of* $\widetilde{\mathcal{L}}$*.*

*Proof.* The map $\pi_c$ is a continuous since $p_c$ identifies the images of points of discontinuity of $\mathfrak{h}$, and the fibers are $\widetilde{S}_{c,\xi}$. This proves (1). The map $\mathfrak{h} \circ Z_c \circ \mathfrak{h}^{-1} : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ is translation by $(k,n) = k(1, \frac{n}{k})$ and hence preserves $R_{c,\xi}$. It follows that $Z_c$ stabilizes $\widetilde{S}_{c,\xi}$ proving (2). Item (3) follows from pulling back the fundamental domain of translation by $(k,n)$ on $R_c$. Item (4) follows from the definitions of $T$ and $Z$ and properties of $T'$ and $Z'$. To prove (5) we note that the action of $T_c$ on $\widetilde{M}$ is conjugate by $\mathfrak{h}$ to translation in $\mathbb{R} \times \mathbb{R}$ by

$$(-w, \overline{k}) + (0, \xi) = (-w, \xi + \overline{k}).$$

Since translation by $(1, \frac{n}{k})$ stabilizes each $R_{c,\xi}$, $(-w, \xi + \overline{k})$ and $(0, (\xi + \overline{k}) + \frac{wn}{k})$ lie on the same fiber of $\pi_c$. Using the definition of $w$ and $\overline{k}$, we have

$$(\xi + \overline{k}) + \frac{wn}{k} = \frac{\xi k + \overline{k}k + wn}{k} = \xi + \frac{1}{k}$$

proving (5). Let $\ell$ be a leaf of $\widetilde{\mathcal{L}}$. Then $\pi_c$ restricted to $\ell$ is equal to $p_c \circ \mathfrak{h}$ restricted to $\ell$, which is 1-1 and continuous. Since $\pi_c$ is locally a trivial fibration, $\pi_c$ is locally invertible on $\ell$, proving (6). $\qquad\square$

**Step 2. Comparing forward flow on $\widetilde{M}$ and $T_c$ on fibers of $\pi_c$.**

Let

$$f_{\widetilde{\mathcal{L}},c}(\xi, s) : \mathbb{R} \times \widetilde{S}_{c,0} \to \widetilde{M}$$

where $f_{\widetilde{\mathcal{L}},c}(\xi, s)$ is the unique point on $\widetilde{S}_{c,\xi}$ that lies on the same leaf of $\mathcal{L}$ as $s$. The map $\sigma$ allows us to find the difference between flow along $f_{\widetilde{\mathcal{L}},c}$ and $T$.

**Lemma 2.9.** *For each $\xi \in \mathbb{R}$, and $s \in \widetilde{S}_{c,0}$, we have*

$$\sigma(f_{\widetilde{\mathcal{L}},c}(\xi - \frac{1}{k}, s)) = \widetilde{\eta}_n(\sigma(f_{\widetilde{\mathcal{L}},c}(\xi, s))),$$

*Proof.* Recall that $\mathcal{L}$ (and $\mathcal{L}'$) are trivial outside of

$$\bigcup_{m \in \mathbb{Z}} \widetilde{S}_{c,\frac{m}{k}}.$$

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Let $\ell_s$ be the segment of the leaf of $\mathcal{L}$ through $s$ that lies between $\widetilde{S}_{c,\xi}$ and $\widetilde{S}_{c,\xi-\frac{1}{k}}$. Then $\sigma$ restricted to $\mathcal{L}$ has a single jump discontinuity at a point on $\mathfrak{h}^{-1}(\Gamma)$. The claim follows by the definition of the pasting maps. $\qquad\square$

**Lemma 2.10.** *The map $T_c$ satisfies*

$$
\begin{aligned}
T_c : \widetilde{M} &\to \widetilde{M} \\
f_{\widetilde{\mathcal{L}},c}(\xi, s) &\mapsto f_{\widetilde{\mathcal{L}},c}(\xi - \frac{1}{k}, \Phi_c(s)).
\end{aligned}
$$

*where for all $s \in \widetilde{S}_{c,0}$,*

$$
\sigma \circ \Phi_c(s) = \widetilde{\zeta^k}\widetilde{\eta}_n(\sigma(s)).
$$

*Proof.* Fix $s \in \widetilde{S}_{c,0}$, and $\xi \in \mathbb{R}$. Using Lemma 2.9 we have

$$
\begin{aligned}
\sigma(T_c(f_{\widetilde{\mathcal{L}},c}(\xi, s))) &= \sigma(T^w Z^{\overline{k}}(f_{\mathcal{L}}(\xi - \frac{1}{k}, s)) \\
&= \zeta^{\overline{k}}(\widetilde{\eta}_n(\sigma(s))).
\end{aligned}
$$

$\qquad\square$

**Step 3.  Descending to the quotient surface $S_c$.** Let $\overline{\sigma}_c$ be the map making the following diagram commute:

$$
\begin{array}{ccc}
\widetilde{S}_c & \xrightarrow{\ \sigma\ } & \widetilde{S} \\
{\scriptstyle /Z_c}\big\downarrow & & \big\downarrow{\scriptstyle /\langle\widetilde{\zeta}^n\rangle} \\
\widetilde{S}_c/K_c & \xrightarrow[\ \overline{\sigma}_c\ ]{} & S_c
\end{array}
$$

Then we have a commutative diagram

$$
\begin{array}{ccc}
\widetilde{M} \simeq \mathbb{R} \times \widetilde{S}_c & & \\
{\scriptstyle /\langle Z_c\rangle}\big\downarrow & \searrow{\scriptstyle \pi_c} & \\
\mathbb{R} \times S_c & \xrightarrow[\ \overline{\pi}_c\ ]{} & \mathbb{R} \\
{\scriptstyle /\langle \overline{T}_c\rangle}\big\downarrow & & \big\downarrow{\scriptstyle /\mathbb{Z}} \\
M & \xrightarrow{\qquad} & S^1
\end{array}
$$

where $\overline{T}_c$ is the automorphism of $S_c \times \mathbb{R}$ induced by $T_c$ and $\overline{\pi}_c$ is the projection to the first factor. It follows that $\alpha_c$ is a fibered element with fiber $S_c$.

To show that $(S_c, \phi_c)$ is the monodromy of $\alpha_c$, we consider the lift of the monodromy to the fibration $\pi_c : \widetilde{M} \to \mathbb{R}$. By Lemma 2.10, we have

$$
\sigma(T_c(f_{\widetilde{\mathcal{L}},c}(\xi, s)))) = \widetilde{\zeta^s} \circ \widetilde{\eta}_n(\sigma(f_{\widetilde{\mathcal{L}},c}(\xi, s))).
$$

It follows that $\phi_c = \zeta_n^{\overline{k}}\eta_n$, finishing the proof of Theorem A.

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2.4. **Stable quotient families.** To prove Theorem B, we define the projection

$$\pi_0 : \widetilde{M} \to \mathbb{R}$$

by composing $h : \widetilde{M} \to \mathbb{R} \times \mathbb{R}$ with projection onto the first coordinate, and show that $\pi_0$ descends to a fibration of $M$ whose fibers intersect each leaf of $\mathcal{L}$ transversally in a single point. We also describe the mapping class given by the first return map.

Recall that $Q = Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ is stable if for some $m_1$,

$$(\widetilde{\zeta\eta})^{m_1}(s) = \zeta^{-m+m_1}(\widetilde{\zeta\eta})^m(s),$$

for all $m \geq m_1$ and $s \in \Sigma_0 \cup \cdots \cup \widetilde{\zeta}^{m_0}\Sigma_0$. Let $\phi_0'$ be defined by

$$\begin{aligned}
\widetilde{\phi}_0' : \Sigma_0 \cup \cdots \cup \widetilde{\zeta}^{m_0}\Sigma_0 &\to \widetilde{S} \\
s &\mapsto \zeta^{-m_1}(\widetilde{\zeta\eta})^{m_1}(s).
\end{aligned}$$

Since $\widetilde{\phi}_0'$ is the identity on $\Sigma_0 \cap \zeta^{-1}(\Sigma_0)$ and on $\zeta^{m_0}(\Sigma_0) \cap \zeta^{m_0+1}(\Sigma_0)$, it extends to a function

$$\widetilde{\phi}_0 : \widetilde{S} \to \widetilde{S}$$

by identity outside the domain of $\widetilde{\phi}_0'$.

Consider the function defined by

$$\begin{aligned}
\widetilde{\phi} : \widetilde{S} &\to \widetilde{S} \\
s &\mapsto \widetilde{\zeta}^r \widetilde{\phi}_0 \widetilde{\zeta}^{-r}(s),
\end{aligned}$$

where $r$ is any integer such that $\widetilde{\zeta}^{-r}(s) \in \Sigma_0$.

**Lemma 2.11.** *The map $\widetilde{\phi}$ is well-defined and continuous, with continuous inverse.*

*Proof.* By definition $\widetilde{\eta}$ is the identity when restricted to $\Sigma_0 \cap \widetilde{\zeta}^{-1}(\Sigma_0)$. Suppose $\widetilde{\zeta}^{-r}(s)$ and $\widetilde{\zeta}^{-r+1}(s)$ lie in $\Sigma_0$. Then $\widetilde{\zeta}^{-r}(s) \in \Sigma_0 \cap \zeta^{-1}\Sigma_0$, and we have

$$\begin{aligned}
\widetilde{\phi}(s) &= \widetilde{\zeta}^r \widetilde{\phi}_0 \widetilde{\zeta}^{-r}(s) \\
&= \widetilde{\zeta}^r \widetilde{\zeta}^{-m_1}(\widetilde{\zeta\eta})^{m_1}\widetilde{\zeta}^{-r}(s) \\
&= \widetilde{\zeta}^{r-1}\widetilde{\zeta}^{-m_1}(\widetilde{\zeta\eta})^{m_1+1}\widetilde{\zeta}^{-r}(s) \\
&= \widetilde{\zeta}^{r-1}\widetilde{\zeta}^{-m_1}(\widetilde{\zeta\eta})^{m_1}\widetilde{\zeta\eta}(\widetilde{\zeta}^{-r}(s)) \\
&= \widetilde{\zeta}^{r-1}\zeta^{-m_1}(\widetilde{\zeta\eta})^{m_1}\widetilde{\zeta}^{-r+1}(s).
\end{aligned}$$

This shows that $\widetilde{\phi}$ is well-defined and continuous. The inverse of $\widetilde{\phi}$ is defined by

$$\widetilde{\phi}^{-1}(s) = \widetilde{\zeta}^r \widetilde{\phi}_0^{-1} \widetilde{\zeta}^{-r}(s).$$

where $r$ is any integer such that $\zeta^{-r}(s) \in \zeta^{m_0}(\Sigma_0)$. Since $\widetilde{\eta}$ is the identity on $\zeta^{m_0}(\Sigma_0) \cap \zeta^{m_0+1}(\Sigma_0)$, the proof that $\widetilde{\phi}^{-1}$ is well-defined and continuous is the same as for $\widetilde{\phi}$. $\square$

**Lemma 2.12.** *The map $\widetilde{\phi}$ commutes with $\zeta$.*

*Proof.* Take any $s \in \widetilde{S}$, and assume that $\widetilde{\zeta}^{-r}(s) \in \Sigma_0$. Then

$$
\begin{aligned}
\widetilde{\phi}(\widetilde{\zeta}(s)) &= \widetilde{\zeta}^{r+1}(\widetilde{\zeta\eta})^{m_1}\widetilde{\zeta}^{-r-1}(\zeta(s)) \\
&= \widetilde{\zeta}^{r+1}(\widetilde{\zeta\eta})^{m_1}\widetilde{\zeta}^{-r}(s) \\
&= \widetilde{\zeta}(\widetilde{\zeta}^{r}(\widetilde{\zeta\eta})^{m_1}\widetilde{\zeta}^{-r}(s)) \\
&= \widetilde{\zeta}(\widetilde{\phi}(s)).
\end{aligned}
$$

$\square$

**Lemma 2.13.** *If the quotient family $Q$ is stable, then for any leaf $\widetilde{\ell}$ of $\widetilde{\mathcal{L}}$, then setting $x_t$ to be the point of intersection of $\widetilde{\ell}$ with $\pi_0^{-1}(t)$ we have*

$$
\sigma(x_1) = \widetilde{\phi}(\sigma(x_0)).
$$

*Proof.* The map $\sigma$ restricts to a homeomorphism on $\widetilde{S}_t = \pi_0^{-1}(t)$ for all $t \in \mathbb{Z}$, since the cut loci only intersect these fibers at their boundary.

Let $\sigma_0 : \widetilde{S}_0 \to \widetilde{S}$ be this homeomorphism. Let $\widetilde{\ell}$ be the leaf of $\widetilde{\mathcal{L}}$ that intersects $\widetilde{S}_0$ at $x_0$. This is necessarily unique since $\mathfrak{h}(\widetilde{S}_0)$ intersects the cut loci only at endpoints, where the gluing map is trivial.

Consider the sequence $t_1, t_2, \ldots$ of intersections of $\widetilde{\ell}$ with the cut loci of $\pi_0^{-1}([0,1])$ (see Figure 4). Let $t_0 = x_0$. A priori, the sequence $t_i$ may be finite or infinite.

Then we have the following. If $\zeta^{-r}(\sigma(t_0)) \in \Sigma_0$, then

$$
\begin{aligned}
\sigma(t_1) &= \zeta^r \eta \zeta^{-r}(\sigma(t_0)) \\
\sigma(t_2) &= \zeta^{r-1}\eta\zeta^{1-r}\zeta^r\eta\zeta^{-r}(\sigma(t_0)) = \zeta^{r-2}(\zeta\eta)^2\zeta^{-r}(\sigma(x_0)) \\
&\cdots \\
\sigma(t_m) &= \zeta^{r-(m-1))}\eta\zeta^{m-1-r}\sigma(t_{m-1}) = \zeta^{r-(m-1)}(\zeta\eta)^{m-1}\zeta^{-r}(\sigma(t_0)) = \widetilde{\phi}(\sigma(t_0)).
\end{aligned}
$$

The condition for stability implies that for $m > m_1$,

$$
\sigma(t_{m+1}) = \zeta^r\zeta^{-m_1}(\zeta\eta)^{m_1}\zeta^{-r}(\sigma(x_0)) = \widetilde{\phi}(\sigma(t_0)) = \sigma(t_m).
$$

Thus we can break $\widetilde{\ell}$ into a finite number of segments $\widetilde{\ell}_0, \widetilde{\ell}_1, \ldots, \widetilde{\ell}_{m_1+1}$ where $\widetilde{\ell}_0 = \widetilde{\ell} \cap \pi_0^{-1}((-\infty, 0])$, $\widetilde{\ell}_i$ is the segment of $\widetilde{\ell}$ between $t_{i-1}$ and $t_i$, for $i = 1, \ldots, m_1$, and

$$
\widetilde{\ell}_{m_1+1} = \widetilde{\ell} \setminus (\widetilde{\ell}_1 \cup \cdots \cup \widetilde{\ell}_{m_1}).
$$

Since $\alpha$ is constant on $I = \widetilde{\ell}_{m_1+1} \cap \pi_0^{-1}([0,1])$ the leaf is identical to a leaf in the trivial foliation $\widetilde{\ell}'$. Thus, $\widetilde{\ell}_{m_1+1}$ and hence $\widetilde{\ell}$ intersects $\widetilde{S}_1$ at some point $x_1$. Furthermore, since $\alpha$ is constant on $I$, we have

$$
\sigma(x_1) = \sigma(t_{m_1+1}) = \widetilde{\phi}(\sigma(x_0)).
$$

$\square$

**Corollary 2.14.** *The quotient family $Q$ is stable if and only if each leaf $\widetilde{\ell}$ of $\widetilde{\mathcal{L}}$ intersects every fiber of $\pi_0$ exactly once.*

By Lemma 2.13, if $Q$ is stable then $\iota$ extends continuously to 0, and $\iota(0)$ has monodromy $(S, \phi)$, where $S = \widetilde{S}/\widetilde{\zeta}$ and $\phi$ is the map induced by $\widetilde{\phi}$. By continuity of $L$, we have

$$\lim_{c \to 0} L(\alpha_c) = L(\alpha_0) = L(S, \phi).$$

Conversely, if the limit does not exist, then $\alpha_0$ must lie on the boundary of the fibered cone, and hence $Q$ could not have been stable. This completes the proof of Theorem B.

## 3. PENNER EXAMPLE

In this section we illustrate Theorem A and Theorem B using Penner's sequence $(R_g, \psi_g)$ (see Figure 1).

Consider the quotient family $Q = Q(\widetilde{S}, \widetilde{\zeta}, \widetilde{\eta})$ where $\widetilde{S}$ is the infinite surface drawn in Figure 6 as a stack of copies $\Sigma_i$, $i \in \mathbb{Z}$, of a fundamental domain $\Sigma$, $\widetilde{\zeta}$ sends each $\Sigma_i$ to $\Sigma_{i+1}$, and $\widetilde{\eta} = \delta_{\widetilde{c}} \delta_{\widetilde{b}}^{-1} \delta_{\widetilde{a}}$.
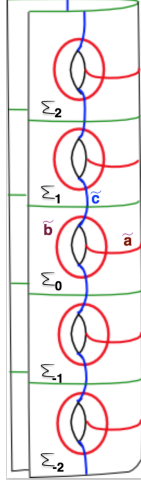


FIGURE 6. The surface $\widetilde{S}$ with fundamental domain homeomorphic to a once punctured torus.

For a mapping class $(S, \phi)$, where $S$ has punctures or boundary components, let $\overline{S}$ be the *closure* of $S$, that is, the closed surface obtained by filling in the punctures and boundary components, and let $(\overline{S}, \overline{\phi})$ be the induced mapping class. Then we have the following.

**Proposition 3.1.** *For $g \geq 3$,*

$$(R_g, \psi_g) = (\overline{S}_{\frac{1}{g}}, \overline{\phi}_{\frac{1}{g}}),$$

*where $(S_{\frac{1}{g}}, \phi_{\frac{1}{g}}) = \iota(\frac{1}{g}) \in Q$.*

We now use Theorem B to show that $L(S_{\frac{1}{g}}, \phi_{\frac{1}{g}})$ is bounded for $g \geq 3$ by showing the following.

**Proposition 3.2.** *The family Q is a stable quotient family.*

*Proof.* Let $\mathfrak{S}_r = \bigcup_{i=r}^{\infty} \Sigma_i$, and for any simple closed curve $\gamma$ embedded on an oriented surface surface, let $\mathcal{A}_\gamma$ denote a small annular neighborhood of $\gamma$. To see that $Q$ is of stable type, observe that we have:

(i) $\widetilde{\zeta}(\widetilde{\eta}(\mathfrak{S}_0)) \subset \mathcal{A}_{\widetilde{c}} \cup \mathfrak{S}_1$;
(ii) $\widetilde{\eta}(\mathfrak{S}_1) = \mathcal{A}_{\widetilde{c}} \cup \mathfrak{S}_1$;
(iii) $\widetilde{\eta}(\mathcal{A}_{\widetilde{c}}) \subset \mathcal{A}_{\widetilde{b}} \cup \mathcal{A}_{\widetilde{c}}$; and
(iv) $\widetilde{\zeta}(\mathcal{A}_{\widetilde{b}} \cup \mathcal{A}_{\widetilde{c}} \cup \mathfrak{S}_1) \subset \mathfrak{S}_1 \setminus \mathcal{A}_{\widetilde{c}}$.

It follows that for all $x \in \Sigma_0$, $\widetilde{\zeta}(\widetilde{\zeta}\widetilde{\eta})^2(x) = \widetilde{\zeta}(x)$, and thus $Q$ is stable. $\qquad\square$

By Theorem B it follows that

$$\lim_{g \to \infty} L(S_{\frac{1}{g}}, \phi_{\frac{1}{g}}) = L(S, \phi)$$

where $S = \widetilde{S}/\widetilde{\zeta}$ and $\phi$ is defined by the $\zeta$-equivariant map $\widetilde{\phi}$. This gives an alternative to Penner's proof in [Pen2] that for some constant $C > 0$

$$\log(\lambda(\phi_{\frac{1}{g}})) \leq \frac{C}{g}.$$

3.1. **Train tracks for the quotient family.** Train track theory, introduced by W. Thurston, gives a way to compute the dilatation of any individual pseudo-Anosov mapping class (see [FLP] [CB]). We recall the theory briefly.

A *train track* is an embedded graph with trivalent edges $\tau$ and smoothings of the edges locally near each vertex as in Figure 7. A simple closed curve $\gamma$ is carried on $\tau$ if there is
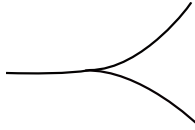


FIGURE 7. Smoothing near a vertex of a train track

an isotopy of $\gamma$ to a curve on $\tau$ that is locally embedded at each vertex. A train track $\tau$ on $S$ is *compatible* with a pseudo-Anosov map $\phi : S \to S$ if for any simple closed curve $\Gamma$ on $S$ there is an $n > 0$ such that for all $k \geq n$ $\phi^k(\gamma)$ is carried by $\tau$. In particular, any train track $\tau$ fills $S$. That is, all its complementary components are either disks or punctured disks.

If $(S, \phi)$ is pseudo-Anosov and $\tau$ is a compatible train track, then we can represent $\phi$ up to isotopy as a graph map on $\tau$, that is, a map that takes vertices of $\tau$ to vertices, and edges of $\tau$ to edge paths. This defines a linear map on $\mathbb{R}^\mathcal{E}$ where $\mathcal{E}$ is the set of edges of $\tau$. Let $\mathcal{V}$ be the set of vertices of $\tau$. The *space of allowable weights on $\tau$* is defined to be the quotient space $\mathcal{W}$ in the exact sequence

$$0 \to \mathcal{W} \to \mathbb{R}^\mathcal{E} \to \mathbb{R}^\mathcal{V},$$

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where the second map takes $v \in \mathcal{V}$ to $e_1 - e_2 - e_3$, for $e_1$ and $e_2$ the two edges that meet in a cusp at $v$, and $e_3$ the third edge or *branch* of $\tau$ at $v$. Then the action of $\phi$ defines a linear map $T$ on $\mathcal{W}_\tau$ called the transition matrix.

Recall that a non-negative integer matrix $T$ is *Perron-Frobenius* (or PF) if its powers are eventually positive. This implies that $T$ has a positive eigenvector with eigenvalue equal to the spectral radius of $T$ (unique up to positive scalar multiplication). This is called the *PF eigenvalue*, which we write as $\lambda(T)$.

Any non-negative integer matrix $T$ defines a directed graph $D$ with vertices associated to the rows (or columns) of $T$, so that the entry in the $i$th row and $j$th column equal to the number of directed edges on $D$ from the $i$th vertex to the $j$th vertex. Then $T$ is the directed adjacency matrix of $D$. A directed graph is PF if and only if there is a directed path from any vertex to any other, and the set of lengths of directed cycles is relatively prime. The following is well-known:

**Proposition 3.3.** *A non-negative integer matrix $T$ is PF if and only if its corresponding directed graph $D$ is PF.*

The transition matrix $T$ defined on the weight space of a compatible train track under the action of a pseudo-Anosov mapping class is a PF map, and $\lambda(\phi) = \lambda(T)$. If $D$ is the directed graph associated to $T$, we also write $\lambda(D) = \lambda(T)$.

**Lemma 3.4.** *For all $c = \frac{k}{n} \in (0, \frac{1}{2})$, $(S_c, \phi_c)$ is pseudo-Anosov and there is a compatible train track $\tau_n \subset S_c$ depending only on $n$.*

The proof of Lemma 3.4 uses Penner's semi-group criterion, as is done in [Pen2]. Suppose there are multi-curves

$$\gamma_1 = a_1 \cup \cdots \cup a_n, \quad \text{and} \quad \gamma_2 = b_1 \cup \cdots \cup b_m,$$

finite disjoint unions of simple closed curves, so that each pair of components $a$ of $\gamma_1$ and $b$ of $\gamma_2$ intersect minimally, and $\phi_c$ is defined by a multiple of positive Dehn twists on components of $\gamma_1$ and negative Dehn twists on components of $\gamma_2$.

Consider the train track obtained from $\gamma_1$ and $\gamma_2$ by changing every crossing as in the diagram below.
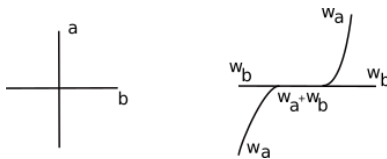


FIGURE 8. Smoothings of a multi-curve $\gamma_1 = a_1 \cup \cdots \cup a_n$, $\gamma_2 = b_1 \cup \cdots \cup b_n$.

Penner's semi-group criterion states that this train track is compatible with any mapping class that can be written as a product of positive multiples of $\delta_{a_i}$ and negative multiples of $\delta_{b_j}$ [Pen1]. One can also see from Figure 7 that the weight space $\mathcal{W}$ is isomorphic to $\mathbb{R}^{A \cup B}$, where $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_m\}$.
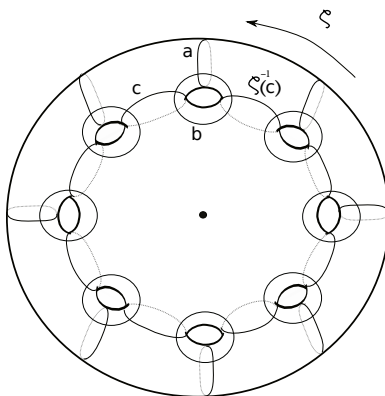
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FIGURE 9. Multicurves defining the mapping classes in $Q$.

*Proof of Lemma 3.4.* Let $c = \frac{k}{g}$. Let $\gamma_1$ be the union of the orbits of $a$ and $c$ under $\zeta$, and let $\gamma_2$ be the union of the orbits of $b$. Then $\delta_c \delta_b^{-1} \delta_a$ is carried by the train track $\tau$ obtained by smoothing $\gamma_1 \cup \gamma_2$ as above, and since $\gamma_1 \cup \gamma_2$ is invariant under application of $\zeta$, $\tau$ carried $\phi_c$. $\qquad\square$

The following lemma (see [Pen2]) also makes computations easier.

**Lemma 3.5.** *The weight space $\mathcal{W}$ is isomorphic to the space of weights on the components of $\gamma_1$ and $\gamma_2$.*

*Proof.* One can see that any choice of weights on the components of $\gamma_1$ and $\gamma_2$ uniquely determines an element of $\mathcal{W}$ as shown in Figure 8. $\qquad\square$

As a corollary, we see that the dilatation of any element $(S_c, \phi_c) \in Q$, $c = \frac{k}{n}$ is the PF eigenvalue of a PF matrix of dimension $3n \times 3n$. This matrix is encapsulated in the directed graph shown in Figure 10.
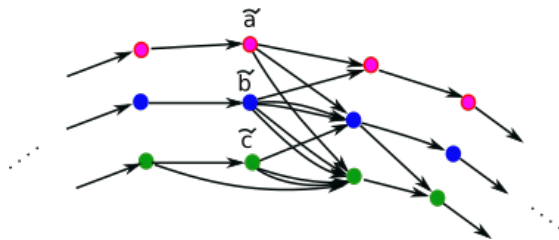


FIGURE 10. Directed graph associated to the PF transition matrix for $(S_{\frac{1}{n}}, \phi_{\frac{1}{n}})$.

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3.2. **Limiting mapping class.** We describe the limiting mapping class $(S, \phi)$ for $Q$ at 0 explicitly. Figure 11 gives a picture of $S = \widetilde{S}/\zeta$ with images $a, b$, $c$ of the curves $\widetilde{a}, \widetilde{b}$ and $\widetilde{c}$, and the image $d$ of $\widetilde{d} = \Sigma_0 \cap \zeta(\Sigma_0)$. Then $\phi$ is the mapping class on the torus with two boundary components given by the composition $\phi = \rho \circ \delta_c \circ \delta_b^{-1} \circ \delta_a$, where $\rho$ is the map induced by $\widetilde{\zeta}$.
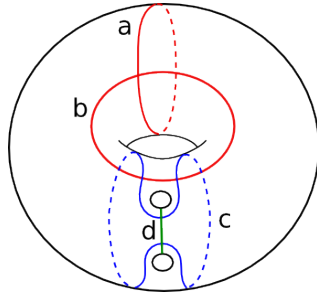


FIGURE 11. The limiting mapping class for Penner's sequence.

Then we have the following.

**Proposition 3.6.** *The map $\iota$ extends to 0, and $\iota(0) = (S, \phi)$.*

3.3. **Alexander and Teichmüller polynomial.** Let $M$ be the mapping torus of the quotient family $Q$.

**Proposition 3.7.** *The first Betti number of $M$ equals 2.*

*Proof.* The first cohomology group of $H^1(S; \mathbb{Z})$ is generated by duals to $[a]$, $[b]$ and $[d]$, the relative homology classes defined by $a, b$ and $d$ in $H_1(S, \partial S; \mathbb{Z})$. With respect to this basis, the action of $\phi$ on the first cohomology group $H^1(S, \mathbb{Z})$ is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The invariant cohomology is 1-dimensional, and hence $b_1(M) = 2$. $\square$

The cohomology class generating the invariant cohomology is dual to the path $d$ between the two punctures on $S$, and the corresponding cyclic covering $\widetilde{S} \to S$ is the one drawn in Figure 6, with fundamental domain $\Sigma = S[d]$ where $S[d]$ is the surface $S$ slit at $d$. Let $\widetilde{\zeta}$ generate the group of covering automorphisms, and $Z = \widetilde{\zeta} \times \{\text{id}\}$ and $T = T_{\widetilde{\phi}}$ define generators for $H_1(M; \mathbb{Z})$. Let $u, t \in H_1(M; \mathbb{Z})$ be duals to $Z$ and $T$ respectively. We consider these as multiplicative elements in $H = H_1(M; \mathbb{Z})$, and write elements of $\mathbb{Z}H$ as polynomials in $u, t$.

Let $\tau$ be the train track for $\phi$ given by smoothing the union of $a, b$ and $c$ at the intersections (see [Pen2]). The space $\widetilde{W}_\tau$ of allowable weights on the lift $\widetilde{\tau}$ of $\tau$ in $\widetilde{S}$ is a module

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over $\mathbb{Z}H$, and the Teichmüller polynomial is the characteristic polynomial of the transition matrix for the action of $\widetilde{\phi}$ on $\widetilde{W}_\tau$.

There is an isomorphism

$$\mathbb{R}^{\{a,b,c\}} \to \widetilde{W}_\tau,$$

given by sending any choice of weights on the edges $a, b, c$ to the unique extension to a set of allowable weights on edges of $\tau$. Thus, the elements $a, b, c$ define a basis for $\widetilde{W}_\tau$. The action of $\widetilde{\phi}$ on $\widetilde{W}_\tau$ written as a matrix with respect to this basis is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1+t \\ 1+t^{-1} & 2(1+t^{-1}) & 1+(1+t)(1+t^{-1}). \end{bmatrix}.$$

The Teichmüller polynomial is the characteristic polynomial of this matrix:

$$\Theta(u,t) = u^2 - u(5 + t + t^{-1}) + 1.$$

The Alexander polynomial $\Delta$ is the characteristic polynomial of the action of the lift $\widetilde{\phi}$ of $\phi$ on the first homology of $\widetilde{S}$. The lifts of $a, b$ and $c$ generate $H_1(\widetilde{S}; \mathbb{Z})$ as a $\mathbb{Z}[t, t^{-1}]$ module, and the action of $\widetilde{\phi}$ on these generators is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1-t \\ 1-t^{-1} & 2(1-t^{-1}) & 1+(1-t)(1-t^{-1}) \end{bmatrix}.$$

We thus have

$$\Delta(u,t) = \Theta(u,-t) = u^2 - u(5 - t - t^{-1}) + 1.$$

3.4. **Fibered face.** The fibered face of a 3-manifold $M$ associated to a flow equivalence class can be found explicitly from the Tecihmüller polynomial of the face, and the Alexander polynomial of $M$ by a result of McMullen [McM2], which we recall here.

Let $H$ be a finitely generated free abelian group. Write $f \in \mathbb{Z}H$ as

$$f = \sum_{h \in H_0} a_h h$$

where $H_0 \subset H$ is a finite subset, and $a_h \neq 0$ for all $h \in H_0$. This representation for $f$ is unique, and we call $H_0$ the *support* of $f$. If $H_0$ is in general position in $H \otimes \mathbb{R}$ considered as a Euclidean space, then there is a corresponding norm on $\mathrm{Hom}(H; \mathbb{R})$ given by

$$\|\alpha\|_f = \max\{|\alpha(h_1) - \alpha(h_2)| \; : \; h_1, h_2 \in H_0\},$$

and the norm ball for $\|\ \|_f$ is convex polyhedral.

McMullen showed in [McM2] that if $F$ is a fibered face of a hyperbolic 3-manifold, $\Delta$ and $\Theta_F$ are the Alexander and Teichmüller polynomials, and $b_1(M) \geq 2$, then the Thurston norm $\|\ \|$ restricted to the cone $C = F \cdot \mathbb{R}^+$ has the property that

$$\|\alpha\| = \|\alpha\|_\Delta \leq \|\alpha\|_{\Theta_F}$$

for all $\alpha \in V_F$.

**Lemma 3.8.** *The fibered cone $C$ in $H^1(M;\mathbb{R})$ associated to Penner wheels is given by elements $(a,b) \in H^1(M;\mathbb{R})$, satisfying*

$$a > |b|,$$

*and the Thurston norm is given by*

$$\|(a,b)\|_T = \max\{2|a|, 2|b|\}.$$

3.5. **Orientability.** A pseudo-Anosov mapping class is *orientable* if it has orientable invariant foliations, or equivalently the geometric and homological dilatations are the same, and the spectral radius of the homological action is realized by a real (possibly negative) eigenvalue (see, for example, [LT] p. 5). Given a polynomial $f$, the largest complex norm amongst its roots is called the *house of $f$*, denoted $|f|$. Thus, $\psi_g$ is orientable if and only if

$$|\Delta(x^g, x))| = |\Theta(x^g, x)|. \tag{1}$$

**Proposition 3.9.** *The mapping classes $(R_g, \psi_g)$ are orientable if and only if $g$ is even.*

*Proof.* By Equation (1), the homological dilatation of $\psi_g$ is the largest complex norm amongst roots of

$$\Delta(x^g, x) = x^{2g} + x^{g+1} - 5x^g + x^{g-1} + 1.$$

Let $\lambda$ be the real root of $\Delta(x^g, x)$ with largest absolute value. Plugging $\lambda$ into $\Theta(x^g, x)$ gives

$$\Theta(\lambda^g, \lambda) = -2\lambda^{g+1} - 2\lambda^{g-1} \neq 0.$$

while for $-\lambda$ we have

$$\Theta(-\lambda^g, -\lambda) = (-\lambda)^{g+1} - (\lambda)^{g+1} + (-\lambda)^{g-1} - (\lambda^{g-1}).$$

It follows that $|\Delta(x^g, x)| = \lambda = |\Theta(x^g, x)|$ if and only if $g$ is even. $\qquad\square$

3.6. **Dilatations and normalized dilatations.** The dilatation $\lambda(\phi_\alpha)$ corresponding to primitive integral points $\alpha = (a, b)$ in $C$ is the largest solution of the polynomial equation

$$\Theta(x^a, x^b) = 0.$$

In particular, Penner's examples $(R_g, \psi_g)$ correspond to the points $(g, 1) \in C$, and we have the following.

**Proposition 3.10.** *The dilatation of $\psi_g$ is given by the largest root of the polynomial*

$$\Theta(x^g, x) = x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1.$$

Specializing $\Theta$ at $t = 1$ and $u = x$ gives

$$\theta(x) = x^2 - 7x + 1,$$

so $\lambda(\phi) = \frac{1}{2}(7 + 3\sqrt{5})$ and

$$\lim_{g \to \infty} L(R_g, \psi_g) = L(S, \phi) \approx 46.9787.$$

The symmetry of $\Theta$ with respect to $x \mapsto -x$ and convexity of $L$ on fibered faces implies the minimum normalized dilatation realized on the fibered face must occur at $(a, b) = (1, 0)$. Thus, we have the following.

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**Proposition 3.11.** *The minimum normalized dilatation for the monodromies in $C$ is given by $L(S, \phi) \approx 46.9787$.*

3.7. **Boundary Behavior.** By Lemma 3.8 we can extend the parameterization $\mathfrak{f} : I_2 = (0, \frac{1}{2}) \to F$ to

$$\mathfrak{f} : (-1, 1) \to F$$
$$c \mapsto \frac{1}{|\chi(c)|}(1, c),$$

**Lemma 3.12.** *The sequence of mapping classes associated to $\mathfrak{f}(\frac{n-1}{n})$ is conjugate to*

$$(\widetilde{S}/\widetilde{\zeta}^n, \zeta_n \delta_{\zeta_n^{-1}(c)} \delta_b^{-1} \delta_a).$$

*Proof.* Let $R : \widetilde{S} \to \widetilde{S}$ be a rotation around an axis that passes through $\widetilde{a} \cup \widetilde{b}$ in 3 points, preserves each of $\widetilde{a}$ and $\widetilde{b}$, and $R\widetilde{c} = \zeta^{-1}(\widetilde{c})$. Then we have

$$
\begin{aligned}
R^{-1}\widetilde{\zeta}R &= \widetilde{\zeta}^{-1} \\
R^{-1}\delta_{\widetilde{a}}R &= \delta_{\widetilde{a}} \\
R^{-1}\delta_{\widetilde{b}}R &= \delta_{\widetilde{b}} \\
R^{-1}\delta_{\widetilde{c}}R &= \delta_{\widetilde{\zeta}^{-1}(\widetilde{c})}^{-1}.
\end{aligned}
$$

We have

$$\widetilde{\zeta}^{-1}\widetilde{\eta} = \widetilde{\zeta}^{-1}\delta_{\widetilde{c}}\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}} = \widetilde{\zeta}^{-1}R\delta_{\widetilde{\zeta}^{-1}(c)}R\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}}$$

Conjugating by $R$ we have

$$
\begin{aligned}
R\widetilde{\zeta}^{-1}\widetilde{\eta}R &= R\widetilde{\zeta}^{-1}R\delta_{\widetilde{\zeta}^{-1}(c)}R\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}}R \\
&= \widetilde{\zeta}\delta_{\widetilde{\zeta}^{-1}(c)}\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}}
\end{aligned}
$$

$\square$

By Lemma 3.12, the mapping classes $(S_{\frac{n-1}{n}}, \phi_{\frac{n-1}{n}})$, also known as the *reverse Penner sequence*, are the same as the mapping classes for the sequence $(\frac{1}{n})$ in the family $Q' = Q(\widetilde{S}, \widetilde{\zeta}, \delta_{\zeta^{-1}(\widetilde{c})}\delta_{\widetilde{b}}^{-1}\delta_{\widetilde{a}})$. In fact, $Q$ and $Q'$ are equal, but parameterized so that $\iota'(c) = \iota(1-c)$. By Lemma 3.8, $Q'$ is not stable, and it follows that $\lim_{n \to \infty} L(S_{\frac{n-1}{n}}, \phi_{\frac{n-1}{n}}) = \infty$.

## 4. How common are quotient families?

Quotient families have special structure, and any quotient family can be reconstructed from any single element of the family.

**Proposition 4.1.** *Let $(S, \phi)$ be a mapping class. Then $(S, \phi)$ belongs to a quotient family if and only if $\phi = r \circ \widetilde{\eta}$, where*

*(i) $r$ is periodic of order $m \geq 2$ with fundamental domain $\Sigma$ with bounded by $b_-$ and $b_+ = \zeta b_-$,*

*(ii) $\eta$ has support*

$$Y \subset \Sigma \cup \zeta\Sigma \cup \cdots \cup \zeta^{m-1}\Sigma.$$

*Proof.* Let $\widetilde{S}$ and $\zeta$ by taking the cyclic covering of $S$ corresponding to the map $H_1(S; \mathbb{Z}) \to \mathbb{Z}$ given by intersection number with $b_-$. Let $\Sigma'$ be a lift of $\Sigma$. Then $\eta$ determines a map $\widehat{\phi}$ with support contained in

$$\Sigma' \cup \zeta\Sigma' \cup \cdots \cup \zeta^{m-1}\Sigma'.$$

Thus $(S, \phi)$ lies in the quotient family defined by $(\widetilde{S}, \zeta, \widehat{\phi})$. □

**Question 4.2.** *Are there one-dimensional linear sections of fibered faces that do not contain any quotient family?*

**Definition 4.3.** For $\kappa > 0$, a mapping $(S, \phi)$ $\kappa$-*quasi-periodic* if there is a subsurface $Y \subset S$ and a mapping class $r : Y \to Y$ such that

(1) the boundary of $Y$ is a finite union of simple closed curves on $S$;
(2) $r^k$ is a product of Dehn twists along boundary parallel curves on $Y$ ($r$ is *periodic rel boundary on $Y$*); and
(3) the support $A$ of $\phi \circ r$ has topological Euler characteristic bounded by

$$-\kappa < \chi(A) < 0.$$

A mapping class $(S, \phi)$ is *strongly $\kappa$-quasi-periodic* if $Y = S$, and $(S, \phi)$ is *(strongly) quasi-periodic* it is (strongly) $\kappa$-quasi-periodic for some $\kappa > 0$.

A family of mapping classes $\mathcal{F} \subset \mathcal{P}$ is a *(strongly) quasi-periodic* if for some $\kappa > 0$ all its members are *(strongly) $\kappa$-quasi-periodic*.

Penner-type sequences and quotient families of mapping classes are strongly $\kappa$-quasi-periodic, where we can take $\kappa = m_1|\chi(\Sigma)|$.

Given $\ell > 1$, let $\mathcal{P}_\ell \subset \mathcal{P}$ be the elements with normalized dilatation less than $\ell$. It is known, for example, that if $\ell > \left(\frac{3+\sqrt{5}}{2}\right)^2$, then $\mathcal{P}_\ell$ is infinite [Hir1] [AD] [KT].

**Question 4.4** (Quasi-periodicity question)**.** *For each $\ell$, is $\mathcal{P}_\ell$ a quasi-periodic family? Is it strongly quasi-periodic?*

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