QUOTIENT FAMILIES OF MAPPING CLASSES

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Abstract. We define quotient families of mapping classes parameterized by rational points on an interval. We show that the elements of a quotient family are the monodromies of a single fibered 3-manifold and correspond to rational points on a linear subsegment of a fibered face of the Thurston norm ball. We study the behavior of the mapping classes and their invariants near the endpoints of the interval of parameterization, and give a criterion for when an endpoint lies in the interior or on the boundary of a fibered face.

1. Introduction

While the dynamics and geometry of the monodromies of a fibered 3-manifold have been well-studied using Thurston's fibered face theory (see, for example, [Thu1], [Fri], [McM1]), there are few concrete descriptions of the mapping classes themselves. In this paper, we investigate a construction of families of mapping classes generalizing *Penner-type sequences* defined in [Pen] (cf. [Val]). We show that these quotient families correspond to rational points on a linear section of a fibered face. Thus, we can think of them as describing a particular kind of linear deformation space of mapping classes.

In the case when the mapping classes are pseudo-Anosov, quotient families can be used to analyze the behavior of dilatations and for showing that certain types of pseudo-Anosov mapping classes realize small dilatations relative to the Euler characteristic of the surface (see [Pen] [Tsa] [Val] [Hir3]).

1.1. Quotient families.

Throughout this paper, a surface is a connected, complete, oriented, 2-dimensional manifold possibly with boundary.

Definition 1.1. A surface is of *finite type* if it is obtained from a closed surface with finite genus by removing a finite number (possibly zero) of embedded open disks.

Definition 1.2. A $\mathbb{Z}$-*surface* is a pair $(\widetilde{S}, \zeta)$, where $\widetilde{S}$ is a surface with a self-homeomorphism $\zeta : \widetilde{S} \to \widetilde{S}$ that acts properly discontinuously with infinite order and finite type quotient. Then, for some compact surface Σ of finite type, we have

(1)
$$\widetilde{S} = \bigcup_{i \in \mathbb{Z}} \Sigma_i,$$

where each Σ_i is homeomorphic to Σ,

This work was partially supported by a grant from the Simons Foundation (#209171 to Eriko Hironaka).

(2) $b_- = \Sigma_0 \cap \Sigma_{-1}$ is a finite union of disjoint arcs or closed curves on the boundary of Σ_0, and

(3) ζ is a homeomorphism of $\widetilde{S}$ to itself satisfying $\zeta(\Sigma_i) = \Sigma_{i+1}$ for all $i \in \mathbb{Z}$.

The surface Σ, which we identify with Σ_0, is a *fundamental domain* for $(\widetilde{S}, \zeta)$.

Let $b^+ = \zeta b^-$, and $b_i^\pm = \zeta^i(b^\pm)$. We can think of $\widetilde{S}$ as being obtained by gluing together i copies of Σ, namely $\Sigma_i = \zeta^i \Sigma$, by identifying b_i^+ with b_{i+1}^-, where $b_i^\pm = \zeta^i b^\pm$ (see Figure 1).

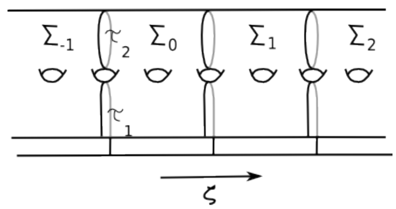

FIGURE 1. Example of a $\mathbb{Z}$-surface with fundamental domain Σ_0 and $b^- = \tau_1 \cup \tau_2$.

Definition 1.3. A *mapping class* on a surface S of finite type is a self-homeomorphism up to isotopies that fix the boundary point-wise. The group of mapping classes on a surface S of finite type is denoted by $\mathrm{Mod}(S)$.

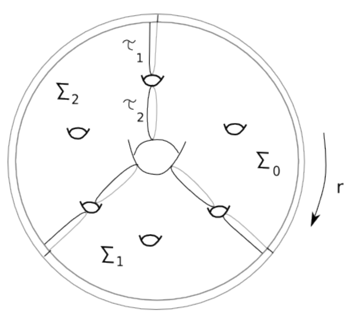

FIGURE 2. Quotient surface by the action of $\zeta \mapsto \zeta^3$.

Definition 1.4. For a $\mathbb{Z}$-surface $(\widetilde{S}, \zeta)$ with fundamental domain Σ, a *mapping class* is a self-homeomorphism up to bounded isotopies, that is, there is a k so that the isotopy moves each point in Σ_i to a point in Σ_{i+r}, for some $|r| < k$.

Definition 1.5. A mapping class (X, f) is *supported* on a subsurface $Y \subset X$ if f has a representative that is the identity outside of Y. Such a mapping class restricts to mapping classes on the connected components of Y.

Given a $\mathbb{Z}$-surface $(\widetilde{S}, \zeta)$, let $\widehat{\phi} : \widetilde{S} \to \widetilde{S}$ be a mapping class supported on a subsurface $Y \subset \widetilde{S}$, where, for some $m_0 \geq 1$,

$$Y \subset \Sigma_0 \cup \Sigma_1 \cup \cdots \cup \Sigma_{m_0}.$$

Let $J_{m_0} = (0, \frac{1}{m_0}) \cap \mathbb{Q}$ be the rational points in the open interval from 0 to $\frac{1}{m_0}$. Elements $c \in J_{m_0}$ will also be represented by their unique reduced fraction form:

$$c = \frac{k}{m},$$

where k, m are positive, relatively prime integers.

Given $c = \frac{k}{m}$, let (X_c, f_c) be defined by

$$S_c = \widetilde{S}/\zeta^m,$$

and

$$\phi_c = r^a \eta,$$

where

(1) $r : S_c \to S_c$ is the order m automorphism induced by ζ,
(2) $\eta : S_c \to S_c$ is defined by identifying Y with its image in S_c by the quotient map, and letting η be the restriction $\widehat{\phi}$ on Y, and the identify map on the complement of Y in S_c, and
(3) a is an integer satisfying $ak = 1 (\text{mod } m)$.

Definition 1.6. The collection

$$\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi}) = \{(S_c, \phi_c) \ : c \in J_{m_0}\}$$

is called the *(cyclic) quotient family associated to* $(\widetilde{S}, \zeta, \widehat{\phi})$, and J_{m_0} the *parameter set.*

Definition 1.7. A triple $(\widetilde{S}, \zeta, \widehat{\phi})$ is of *stable type* and $(\zeta\widehat{\phi})^m$ *stabilizes* if there is a mapping class $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ that commutes with ζ and satisfies

$$\widetilde{\phi}(s) = \zeta^{-m}(\zeta\widehat{\phi})^m(s)$$

for all $m \geq m_0$ and $s \in \Sigma$. Otherwise, we say $(\widetilde{S}, \zeta, \widehat{\phi})$ is of *unstable type.* If $(\widetilde{S}, \zeta, \widehat{\phi})$ is of stable type, then $\widetilde{\phi}$ determines a mapping class ϕ on the quotient surface $S = \widetilde{S}/\zeta$, which is the same as Σ_0 with b_- and the identification $x = \zeta(x)$ for $x \in b_-$.

Theorem A. *If (S_c, ϕ_c) is pseudo-Anosov for one $c \in J_{m_0}$, then it is pseudo-Anosov for all $c \in J_{m_0}$. If $(\widetilde{S}, \zeta, \widetilde{\phi})$ is of stable type, then*

$$\lim_{c \to 0} L(S_c, \phi_c) = L(S, \phi).$$

If it is of unstable type, then

$$\lim_{c \to 0} L(S_c, \phi_c) = \infty.$$

1.2. **Fibered faces.** Thurston's theory of fibered faces [Thu1] gives a way to associate fibrations of a 3-manifold to rational points on open affine polyhedra F.

Let M be a fibered 3-manifold with first Betti number $b = b_1(M)$. An element $\alpha \in H^1(M;\mathbb{Z})$ is said to be *fibered* if it is dual to the fiber S_α of a fibration $\psi_\alpha : M \to S^1$. The monodromy of (S, ϕ_α) is a mapping class $\phi_\alpha : S_\alpha \to S_\alpha$.

In [Thu1], Thurston defines a semi-norm $||\ ||$ on $\mathbb{R}^b$ so that if $\alpha \in H^1(M;\mathbb{Z})$ is fibered with monodromy (S_α, ϕ_α) then

$$||\psi_\alpha|| = |\chi(S_\alpha)|,$$

where χ is the topological Euler characteristic.

Theorem 1.8 (Thurston [Thu1])**.** *Let M be a fibered 3-manifold. Then there is a top-dimensional face F of the norm ball*

$$\{\alpha \in \mathbb{R}^b \mid ||\alpha|| = 1\}$$

such that for all rational rays through the face F, the cohomology class corresponding to the primitive integral element on the ray is induced by a fibration of M to S^1.

Definition 1.9. Let $\Psi(M, F)$ be the set of fibrations of M to the circle corresponding to rational points on the fibered face F. Let $\mathcal{P}(M, F)$ be the collection of monodromies (S_α, ϕ_α) associated to elements of $\mathcal{P}(M, F)$. We think of $\mathcal{P}(M, F)$ as an *open neighborhood* in the set of all mapping classes

$$\mathrm{Mod} = \bigcup_S \mathrm{Mod}(S),$$

where the union is over all surfaces S of finite type.

Remark 1.10. When a fibered 3-manifold M has $b_1(M) = 1$, then each fibered face F is a single point, and hence the corresponding monodromy is *isolated* from the point of view of deformations mentioned above. To study the dynamical behavior of a pseudo-Anosov mapping class (S, ϕ), however, little is changed when we remove the neighborhood of the orbit flow of a periodic point. For pseudo-Anosov mapping classes, the number of periodic points of a given period grows exponentially. By removing a small neighborhood of one such orbit from S, we obtain a new surface S^0 and restriction map $\phi^0 = \phi|_{S^0}$. The mapping torus M^0 for (S^0, ϕ^0) can be obtained from M by removing a solid torus. When $b_1(M^0) > b_1(M)$, it is possible to study positive dimensional deformations of the dynamics of an isolated (S, ϕ) by considering the larger deformation spaces of mapping classes whose dynamical properties are essentially identical.

Theorem 1.11 (Thurston [Thu2])**.** *A fibered 3-manifold M is hyperbolic if and only if it has a monodromy that is pseudo-Anosov. Conversely, if M is hyperbolic, then any fibration of M over the circle has a pseudo-Anosov monodromy.*

If M is hyperbolic, then its Thurston semi-norm is a norm, and the norm ball is a convex polyhedron (see [Thu1]). The top dimensional faces F of the Thurston norm ball have the property that either all the integral points in the cone $V_F = F \cdot \mathbb{R}^+$ are fibered or none are. In the former case we say F is a *fibered face.*

Let $\overline{\alpha}$ be a rational point on a fibered face F. The integral element α with relatively prime coefficients on the ray emanating from the origin and passing through $\overline{\alpha}$ is called the *primitive element associated to* $\overline{\alpha}$, and is the unique point on the ray through α corresponding to a fibered element with connected fiber. By these definitions, the denominator of α equals the absolute value $|\chi(S_\alpha)|$ of the topological Euler characteristic of S_α.

Definition 1.12. Let $\mathcal{P} \subset \mathrm{Mod}$ be the subcollection consisting of pseudo-Anosov elements. Then

$$\mathcal{P} = \bigcup_{(M,F)} \mathcal{P}(M,F),$$

where M is a hyperbolic 3-manifold, and F is a fibered face. Thus, the $\mathcal{P}(M,F)$ can be thought of as neighborhoods or deformation spaces of pseudo-Anosov mapping classes.

Assume M is hyperbolic. The restriction of the normalized dilatation L to $\mathcal{P}(M,F)$ defines a real-valued function on the rational points $F_{\mathbb{Q}}$ of F, by

$$\begin{array}{rcl} L_F : F_{\mathbb{Q}} & \to & \mathbb{R} \\ \overline{\alpha} & \mapsto & L(S_\alpha, \phi_\alpha). \end{array}$$

Variations of the following theorem can also be found in [Mat] and [McM1].

Theorem 1.13 (Fried [Fri])**.** *The function L_F extends to a continuous convex function on F going to infinity toward the boundary of F.*

As a consequence, if F has dimension greater than or equal to 1 (or, equivalently, $b_1(M) \geq 2$) then compact subsets of F give rise to families of mapping classes with bounded normalized dilatation and unbounded topological Euler characteristic.

1.3. **Quotient families and fibered faces.** The second main result of this paper describes how quotient families sit on fibered faces.

Theorem B. *Let $\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$ be a quotient family. Then there is a 3-manifold M, and an embedding $\overline{\alpha} : (0, \frac{1}{m_0}) \to V$, where V is a one-dimensional linear section of a fibered face F so that*

(i) the image of the rational points of $(0, \frac{1}{m_0})$ equals $V \cap F_{\mathbb{Q}}$, and
(ii) for all $c \in J_{m_0}$, (S_c, ϕ_c) is conjugate to $(S_{\alpha(c)}, \phi_{\alpha(c)})$, where $\alpha(c)$ is the primitive integral element of the ray emanating from the origin and passing through $\overline{\alpha}(c)$.

Finally, we show how the behavior of the monodromies as c tends to 0 is determined by whether the triple $(\widetilde{S}, \zeta, \widehat{\phi})$ is of stable or unstable type.

Theorem C. *Let α' be the continuous extension of α from $[0, \frac{1}{m_0}]$ to the closure of F. If $(\widetilde{S}, \zeta, \widehat{\phi})$ is of stable type, then $\alpha'(0)$ lies in the interior of F and has monodromy (S, ϕ). Otherwise, $\alpha'(0)$ lies on the boundary of F.*

Theorem 1.11, Theorem 1.13, Theorem B and Theorem C imply Theorem A.

1.4. **Polynomial invariants.** One of the advantages of realizing a family of mapping classes as monodromies on a single fibered face F of a 3-manifold M is that one can compute invariants such as homological and geometric dilatations for all the mapping classes using the multivariable Alexander and Teichmüller polynomials associated to M and F. Furthermore, these polynomials can be computed by studying the monodromy of any single rational element of F (cf. [McM2] [McM1]). In particular, by studying the dynamics of a single member of a quotient family, one can determine whether or not the family is of stable type. We give some examples in Section 5.

1.5. **Penner-type sequences.** Penner shows in [Pen] that if δ_g is the minimum dilatation for pseudo-Anosov mapping classes on closed surfaces of genus g, then

$$\log(\delta_g) \asymp \frac{1}{g}. \tag{1}$$

As part of his proof, Penner constructs an explicit sequence of pseudo-Anosov mapping classes $\phi_g : S_g \to S_g$ and proves that

$$\lambda(\phi_g)^g \leq C, \qquad \text{for some constant } C. \tag{2}$$

Penner's sequence is defined as follows. Consider a sequence (S_g, ϕ_g) of mapping classes on compact surfaces shown in Figure 3 (for $g = 12$). For each genus g, let S_g is a surface of genus g with 2 boundary components, and $\phi_g = r_g c_g b_g a_g$. By Penner's semi-group criterion, (S_g, ϕ_g) is a pseudo-Anosov map for each $g \geq 2$.

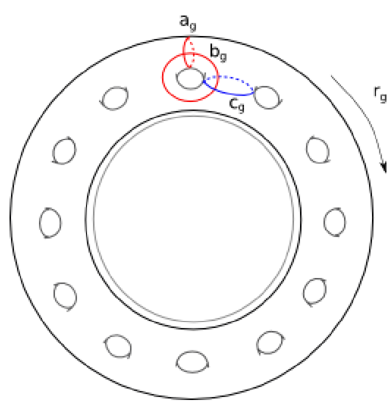

FIGURE 3. Penner's example.

In [Val], Valdivia showed that Penner's sequence and some generalizations have the property that all the mapping classes are the monodromies of a single hyperbolic 3-manifold, and the normalized dilatations are bounded. The results in this paper show further that Penner's sequence (and its generalizations in [Bau] [Val], [Tsa]) can be thought of as a Cauchy sequence in a quotient family, and hence are Cauchy sequences on a one-dimensional linear section of a single fibered face. This information makes it possible to explicitly calculate defining polynomials (Alexander and Teichmüller polynomials) for the homological and geometric dilatations, as well as other properties (see Section 5.1).

1.6. **Outline of paper.** This paper is organized as follows. In Section 2 we review properties of mapping classes on fibered faces. Using covering spaces, we describe the deformations of mapping classes coming from moving around on a fibered face. Theorem B is proved in Section 3 and Theorem C in Section 4. We study examples of quotient families of stable and non-stable type in Section 5. In particular, we show that the mapping classes in Penner's sequence are elements of a quotient family of stable type corresponding to a Cauchy sequence on the parameterizing interval J_{m_0} that converges to 0. Thus, fibered face theory gives a second proof of Penner's boundedness result (see Equation (2) in 1.5), and allows us to compute the dilatations of the mapping classes explicitly as roots of specializations of a Teichmüller polynomial.

Acknowledgements. I am grateful to B. Farb, S. Fenley, V. Gadre, E. Kin, C. Leininger, C. McMullen, H. Sun and J. F. Valdez for helpful conversations and suggestions, and the Tokyo Institute of Technology and University of Tokyo for their hospitality and support in December 2011-July 2012, when the main part of this paper was written.

2. Fibered faces and linear deformations of pseudo-Anosov maps

Let S be a surface of finite type. By the Nielsen-Thurston classification, mapping classes that are not periodic or reducible have the property that for any essential simple closed curve γ on S and any Riemannian metric ω, the growth of the sequence

$$\ell_\omega(\phi^n(\gamma))$$

is exponential, and furthermore the growth rate

$$\lambda = \lim_{n\to\infty} \ell_\omega(\phi^n(\gamma))^{\frac{1}{n}}$$

is independent of γ and ω (see [Thu2], [FM]). Mapping classes that are not periodic or reducible are called *pseudo-Anosov*, and the growth rate $\lambda(\phi) = \lambda$ is called the *dilatation* of ϕ. For more properties of pseudo-Anosov mapping classes see, for example, [Thu2], [FLP], [CB] and [FM].

In this section, we review properties of the monodromies of fibrations of 3-manifolds over the circle. Although we are mainly interested in the case of hyperbolic 3-manifolds, and pseudo-Anosov monodromies, the results of this paper apply in the more general setting of arbitrary mapping classes and their mapping tori.

This exposition focusses on describing the mapping classes in $\mathcal{P}(M,F)$ using abelian unbranched coverings, and expands on the discussion in [McM1] (see Theorem 10.2).

2.1. **Cyclic covering associated to a fibration over the circle.** Let (S,ϕ) be a mapping class, and let M be the mapping torus

$$M = S \times [0,1]/(s,1) \sim (\phi(s),0).$$

Let $\psi \in H^1(M;\mathbb{Z})$ be the element induced by the corresponding fibration $M \to S^1$ over the circle, and let $\overline{\alpha}_\psi$ be the corresponding unit vector in the Thurston norm. Let $M_\psi \to M$ be the cyclic covering over M defined by the epimorphism

$$\psi_* : \pi_1(M) \to \mathbb{Z}$$

induced by ψ.

Lemma 2.1. *There is an identification $M_\psi = S \times \mathbb{R}$ so that $T_\psi : M_\psi \to M_\psi$ defined by*

$$T_\psi(s,t) = (\phi(s), t+1),$$

generates the group of covering automorphisms.

Proof. By the definition of mapping torus, M is the quotient of $S\times[0,1]$ by the identification

$$(s,1) \sim (\phi(s), 0).$$

For $(s',t') \in S \times \mathbb{R}$, let $\overline{(s',t')}$ be its equivalence class in M. Let $M_\psi = S \times \mathbb{R}$, and let $\rho : M_\psi \to M$ be defined by

$$\rho(s,t) = \overline{(\phi^{-n}(s), t-n)},$$

where n is the greatest integer less than t. Then we have

$$\begin{array}{rcl} \rho(T_\psi(s,t)) & = & \rho(\phi(s), t+1) \\ & = & \overline{(\phi^{-(n+1)}(\phi(s)), t+1-(n+1))} \\ & = & \overline{(\phi^{-n}(s), t-n)} \\ & = & \rho((s,t)). \end{array}$$

Thus, T_ψ is a covering automorphism. The covering ρ satisfies the commutative diagram

$$\begin{tikzcd} M_\psi \arrow[r] \arrow[d, "\rho"'] & \mathbb{R} \arrow[d] \\ M \arrow[r, "\psi"'] & S^1. \end{tikzcd}$$

Since the action of T_ψ preserves fibers of the top right arrow, it defines a covering automorphism $\overline{T}_\psi$ of $\mathbb{R}$ over S^1, namely sending j to $j+1$. Furthermore, $\overline{T}_\psi$ sends t to $t-1$, and hence it generates the cyclic group of covering automorphisms on $\mathbb{R}$ over S^1. It follows that T_ψ generates the group of covering automorphisms of M_ψ over M. $\square$

Definition 2.2. A *flow* on a manifold X is a function

$$f : X \times \mathbb{R}_{\geq 0} \to X,$$

such that for $f_v(x) = f(x,v)$, we have $f_0(x) = x$ and $f_{u+v}(x) = f_u(f_v(x))$.

Definition 2.3. A 3-manifold M realized as the mapping torus of a mapping class (S, ϕ) has a natural continuous flow, called the *suspension flow*, defined for $(s,t) \in S \times [0,1]$, $\overline{(s,t)}$ its equivalence class in M, and $0 \leq v < 1$,

$$f_v(\overline{(s,t)}) = \begin{cases} \overline{(s, t+v)} & \text{if } 0 \leq t+v < 1 \\ \overline{(\phi(s), t+v-1)} & \text{if } t+v \geq 1 \end{cases}$$

2.2. **Lifting to an infinite covering.** Let $H = H_1(M;\mathbb{Z})/\mathrm{Tor} \simeq \mathbb{Z}^k$, and let

$$\mathfrak{h} : \pi_1(M) \to H$$

be the Hurewicz map composed with the quotient map $H_1(M;\mathbb{Z}) \to H_1(M;\mathbb{Z})/\mathrm{Tor}$. Then $\mathfrak{h}$ determines a regular $\mathbb{Z}^k$ covering

$$\widetilde{\rho} : \widetilde{M} \to M,$$

of M with group of covering automorphisms H.

The element $\psi \in H^1(M;\mathbb{Z})$ induces a homomorphism, $\psi : \pi_1(M) \to \mathbb{Z}$, which factors through $\mathfrak{h}$. Thus, we have an intermediate covering

$$\begin{tikzcd} \widetilde{M} \arrow[dd, "\widetilde{\rho}"'] \arrow[dr] & \\ & M_\psi \arrow[dl, "\rho_\psi"] \\ M. & \end{tikzcd}$$

Pulling back the product structure on M_ψ to $\widetilde{M}$, we can write $\widetilde{M} = \widetilde{S} \times \mathbb{R}$, where $\widetilde{S}$ is a covering of S. Then the covering map T_ψ lifts to

$$\begin{aligned} \widetilde{T}_\psi : \widetilde{M} &\to \widetilde{M} \\ (s,t) &\mapsto (\widetilde{\phi}(s), t+1), \end{aligned}$$

where $\widetilde{\phi}$ is the lift of ϕ to $\widetilde{S}$.

2.3. **Deformations of ψ via coverings.** The set of fibrations $\Psi(M,F)$ of M corresponding to rational points on a fibered face F, and their associated monodromies $\mathcal{P}(M,F)$ can be described in terms of the suspension flow on M defined by any element of $\Psi(M,F)$ (see [Fri, Thu1]). We explain the relation using the covering $\widetilde{\rho}$ and the flow $\widetilde{f}_\nu$ (cf. [McM1]).

Definition 2.4. Given a 3-manifold M with a flow $f : M \times \mathbb{R} \to M$, a *cross-section* is a subsurface $Y \subset M$ (possibly infinite type) with the property that

(i) for each $y \in \widetilde{Y}$, $f_v(y) \in \widetilde{Y}$ if only if $v = 0$, and
(ii) for all $z \in \widetilde{N}$, there is a $y \in \widetilde{Y}$ and $v \in \mathbb{R}$ such that $t_v(y) = z$.

is a surface $S \subset M$ of finite type so that for some fixed $v_1 > 0$, and all $s \in S$, $f_{v_1}(s) \in S$ and $f_u(s) \notin S$ for all $0 < u < v_1$. Given a cross-section S of a flow f, the *first return map* $\phi : S \to S$, is defined by $\phi(s) = f_{v_1}(s)$.

The following theorem describes the relation between fibered faces and flows on M (see [Thu1, Fri]).

Theorem 2.5. *Let $\psi \in \Psi(M,F)$, and let f be its suspension flow on M. Then $\alpha \in H^1(M;\mathbb{Z})$ is fibered if and only if the dual of α in $H_1(M,\partial M;\mathbb{Z})$ can be represented by a cross-section S_α to the flow f. In this case, the monodromy of α is (S_α,ϕ_α), where ϕ_α is the first return map.*

Take any $\alpha \in \Psi(M,F)$, and let $K_\alpha \subset H$ be the kernel of the induced map

$$\alpha : H \to \mathbb{Z}.$$

Then α determines a free cyclic intermediate covering

$$\begin{tikzcd}
\widetilde{M} \arrow[dd, "\widetilde{\rho}"'] \arrow[dr] & \\
& M_\alpha \arrow[dl, "\rho_\alpha"] \\
M, &
\end{tikzcd}$$

where $M_\alpha = \widetilde{M}/K_\alpha$. Let $\widetilde{T}_\alpha \in H$ be a solution to $\alpha(\widetilde{T}_\alpha) = 1$, and let T_α be the induced map on M_α.

Lemma 2.6. *The map T_α generates the group of covering automorphisms of M_α over M.*

Proof. It suffices to observe that H/K_α is generated by the coset $\widetilde{T}_\alpha K_\alpha$. $\square$

By Theorem 2.5, since $\alpha \in \Psi(M,F)$, there is a cross-section $S_\alpha \subset M$ with respect to the flow f, such that S_α defines a homology class dual to α. Let K_α be the kernel of the induced map

$$\alpha_* : H_1(M;\mathbb{Z}) \to \mathbb{Z}.$$

Then the preimage $\widetilde{S}_\alpha$ in the universal abelian covering $\widetilde{M}$ is preserved under the action of K_α on $\widetilde{M}$ and $S_\alpha = \widetilde{S}_\alpha/K_\alpha$. Let $T_\alpha \in H_1(M;\mathbb{Z})$ be a solution to $\alpha(T_\alpha) = 1$. The flow f defines an identification of $\sigma_\alpha : \widetilde{S}_\alpha \to \widetilde{S}$. Then

$$T_\alpha(s,t) = (\widetilde{\phi}'_\alpha(s), t'),$$

for some $t' \in \mathbb{R}$, and some $\widetilde{\phi}'_\alpha : \widetilde{S} \to \widetilde{S}$. Let $\widetilde{\phi}_\alpha : \widetilde{S}_\alpha \to \widetilde{S}_\alpha$ be the mapping class defined by conjugation by σ_α. The map $\widetilde{\phi}_\alpha$ is equivariant under the action of K_α. Let ϕ_α be the induced map on S_α. Then the pair (S_α,ϕ_α) is the monodromy of α.

To summarize, start with a single mapping class $(S,\phi) \in \mathcal{P}(M,F)$, where $b_1(M) \geq 2$. We think of $\mathcal{P}(M,F)$ as a *neighborhood* of (S,ϕ) in the set of all mapping classes, and the elements of $\mathcal{P}(M,F)$ are *neighbors* of (S,ϕ). Let $\widetilde{M} = \widetilde{S} \times \mathbb{R}$ be the trivialization of the maximal abelian covering of M so that $\widetilde{S}$ is a covering of S and the group of covering automorphisms is generated by $T(s,t) = (\widetilde{\phi}(s),t)$, where $\widetilde{\phi}$ is a lift of ϕ. Then we reconstruct the neighboring monodromies of M as follows.

1. Define from (S, ϕ) the suspension flow $\widetilde{f}$ on $\widetilde{M}$.
2. For each $\alpha \in \Psi(M, F)$, find a surface $\widetilde{S}_\alpha$ in $\widetilde{M}$ that is invariant under the action of the kernel K_α of the map
$$\alpha_* : H \to \mathbb{Z}.$$
3. The monodromy of α is (S_α, ϕ_α), where $S_\alpha = \widetilde{S}_\alpha / K_\alpha$ and ϕ_α can be found by solving $\alpha_*(\widetilde{T}_\alpha) = 1$ and writing $\widetilde{T}_\alpha$ in terms of the trivialization $\widetilde{M} = \widetilde{S} \times \mathbb{R}$.

2.4. **Linear deformations.** Let $\psi \in H^1(M; \mathbb{Z})$ be a fibered element, and let F be the fibered face in $H^1(M; \mathbb{Z})$ containing ψ. Let d be an integer satisfying
$$0 < d \leq b_1(M) - 1.$$
The d-dimensional linear subspaces of F are in one-to-one correspondence with $d+1$ dimensional subspaces of $H^1(M; \mathbb{R})$. For example, let
$$\{\psi, \beta_1, \ldots, \beta_d\} \subset H^1(M; \mathbb{R}),$$
where ψ is induced by a fibration of M over a circle. Let V be the d-dimensional linear section of F cut by the subspace of $H^1(M; \mathbb{R})$ generated by $\{\psi, \beta_1, \ldots, \beta_d\}$. Then V is a linear section of F containing the projection of ψ.

The integral points in the subcone in $H^1(M; \mathbb{R})$ over V can be analyzed by taking $\widetilde{M}$ to be the $\mathbb{Z}^{d+1}$-covering of M determined by the natural map
$$\pi_1(M) \to H_{\psi, \beta_1, \ldots, \beta_d},$$
where
$$H_{\psi, \beta_1, \ldots, \beta_d} = \operatorname{Hom}(\langle \psi, \beta_1, \ldots, \beta_d \rangle, \mathbb{Z}).$$

Definition 2.7. Let $(S, \phi) \in \mathcal{P}(M, F)$ and $\psi : M \to S^1$ be the corresponding fibration. Let $\overline{\psi}$ be the surjection of ψ to F. A *linear deformation* of a (S, ϕ) is the space of monodromies of rational elements in $V \cap U$ where V is a linear section of F passing through $\overline{\psi}$, and U is a neighborhood of $\overline{\psi}$ in F.

in this paper, a linear section will be one-dimensional unless stated otherwise.

2.5. **Quotient families.** Let $(\widetilde{S}, \zeta)$ be a $\mathbb{Z}$-surface with fundamental domain $\Sigma_0 \subset \widetilde{S}$, let $\widehat{\phi} : \widetilde{S} \to \widetilde{S}$ be a mapping class supported on the finite union
$$Y = \Sigma_0 \cup \zeta(\Sigma_0) \cup \cdots \cup \zeta^{m_0}(\Sigma_0),$$
where $m_0 \geq 1$, and let $\mathcal{Q}(\widetilde{S}, \zeta, \widehat{\phi})$ be the quotient family defined by $\widehat{\phi} : \widetilde{S} \to \widetilde{S}$ supported on Y. Let
$$\widehat{\phi}_i = \zeta^i \widehat{\phi} \zeta^{-i}.$$
For integers $m > m_0$ and any i, the maps $\{\widehat{\phi}_{\ell m + i} \ : \ \ell \in \mathbb{Z}\}$ commute with each other. Define, for $m > m_0$ and $i \in \mathbb{Z}$, let

$$\widehat{\phi}_{m\text{-rep}, \ell} \quad = \quad \circ_{i \in \mathbb{Z}} \, \widehat{\phi}_{im + \ell c}, \tag{3}$$

and
$$\widehat{\phi}_{m\text{-rep}} = \widehat{\phi}_{m\text{-rep}, 0}.$$

Lemma 2.8. *For $\ell \in \mathbb{Z}$, we have*

$$\widehat{\phi}_{m\text{-}rep,\ell}\zeta = \zeta\widehat{\phi}_{m\text{-}rep,\ell-1}.$$

Recall the definition of the associated quotient family in Section 1.1.

Lemma 2.9. *The mapping class $(\widetilde{S}, \widetilde{\phi}_c)$ where*

$$\widetilde{\phi}_c = \zeta^a \widehat{\phi}_{m\text{-}rep}$$

is a lifting of (S_c, ϕ_c).

3. Proof of Theorem B

In this section, we construct a 3-manifold M, a linear section V of a fibered face F of M, and an embedding $\overline{\alpha} : J_{m_0} \to V$ so that for all $c \in J_{m_0}$,

$$(S_c, \phi_c) \simeq (S_{\alpha(c)}, \phi_{\alpha(c)}),$$

where $\alpha(c)$ is the primitive integral point in the ray through $\overline{\alpha}(c)$ in $H^1(M; \mathbb{R})$. Here by isomorphism of the pairs (S_c, ϕ_c) and $(S_{\alpha(c)}, \phi_{\alpha(c)})$, we mean a homeomorphism between the surfaces S_c and $S_{\alpha(c)}$ that induces a conjugacy of maps ϕ_c and $\phi_{\alpha(c)}$.

Here is a brief outline of the proof. Fix $c = \frac{k}{m} \in J_{m_0}$. We start by constructing 3-manifolds $\widetilde{M}^{(c)}$ which are setwise equal to $\widetilde{S} \times \mathbb{R}$.

Step 1. Define a projection $\widetilde{S} \times \mathbb{R} \to \mathbb{R}$ with fibers $X_v^{(c)}$. Cut and glue along $X_{\frac{\ell}{m}}^{(c)}$ for $\ell \in \mathbb{Z}$ to obtain a 3-manifold $\widetilde{M}^{(c)}$, a continuous flow $\widetilde{f}^{(c)}$ on $\widetilde{M}^{(c)}$, and a properly discontinuous free abelian group action of rank 2 generated by $Z^{(c)}$ and $T^{(c)}$, where

$$\widetilde{f}_1^{(c)}(s,t) = T^{(c)},$$

and $Z^{(c)}$ commutes with $\widetilde{f}_v^{(c)}$ for all v.

Step 2. Show that there are homeomorphisms $h_{c,c'} : \widetilde{M}^{(c)} \to \widetilde{M}^{(c')}$ that conjugate

$$(\widetilde{M}^{(c)}, \widetilde{f}^{(c)}, Z^{(c)}, T^{(c)}) \quad \text{to} \quad (\widetilde{M}^{(c')}, \widetilde{f}^{(c')}, Z^{(c')}, T^{(c')}).$$

Let $(\widetilde{M}, \widetilde{f}, Z, T)$ denote the equivalence class. Let M be the quotient $M = \widetilde{M}/\langle Z, T\rangle$. Then Z, T define elements of $H_1(M; \mathbb{Z})$, and their duals Z^*, T^* form the basis for a two dimensional linear subspace of $H^1(M; \mathbb{R})$. Define $\alpha : J_{m_0} \to H^1(M; \mathbb{Z})$ by

$$\alpha(c) = kZ^* + mT^*.$$

Step 3. Show that the kernel of $\alpha(c)$ with respect to the representative $(\widetilde{M}^{(c)}, \widetilde{f}^{(c)}, Z^{(c)}, T^{(c)})$ is generated by $(R^{(c)})^m \in \langle Z^{(c)}, T^{(c)}\rangle$, where $R^{(c)}$ is an automorphism of $\widetilde{M}^{(c)}$ that leaves the $X_v^{(c)}$ invariant.

Step 4. Find $a, b \in \mathbb{Z}$ so that $ak + bm = 1$. Then $\alpha(c)((Z^{(c)})^a(T^{(c)})^b) = 1$. By analyzing the action of $(Z^{(c)})^a(T^{(c)})^b$ on $X_0^{(c)}$ we prove Theorem B.

3.1. **Step 1.** Let $\widetilde{M} = \widetilde{S} \times \mathbb{R}$ and let $\mathfrak{p} : \widetilde{S} \to \mathbb{R}$ be a height function such that

(i) $\mathfrak{p}^{-1}(0) = b_0^-$, and

(ii) $\mathfrak{p}(\zeta(s)) = \mathfrak{p}(s) + 1$.

Fix $c = \frac{k}{m} \in J_{m_0}$, and define

$$\begin{array}{rcl} q_v^{(c)} : \widetilde{S} & \to & \widetilde{M} = \widetilde{S} \times \mathbb{R} \\ s & \mapsto & (s, v - c \cdot \mathfrak{p}(s)). \end{array}$$

The images $X_v^{(c)} = q_v^{(c)}(\widetilde{S})$, for $v \in \mathbb{R}$ are the fibers of a projection of $\widetilde{S} \times \mathbb{R}$ to $\mathbb{R}$.

Lemma 3.1. *$Z(q_v^{(c)}(s)) = q_{v+c}^{(c)}(\zeta(s))$.*

Proof. Let $s \in \widetilde{S}$, and let $s' = \zeta(s)$. Then we have

$$\begin{array}{rcl} Z(q_v^{(c)}(s)) & = & Z(s, v - c \cdot \mathfrak{p}(s)) \\ & = & (\zeta(s), v - c \cdot \mathfrak{p}(s)) \\ & = & (s', v - c \cdot \mathfrak{p}(\zeta^{-1}(s'))) \\ & = & (s', v - c(\mathfrak{p}(s') - 1)) \\ & = & (s', v + c - c\mathfrak{p}(s')) \\ & = & q_{v+c}^{(c)}(s') \\ & = & q_{v+c}^{(c)}(\zeta(s)). \end{array}$$

□

As a direct consequence, we have the following.

Corollary 3.2. *$Z(X_v^{(c)}) = X_{v+c}^{(c)}$.*

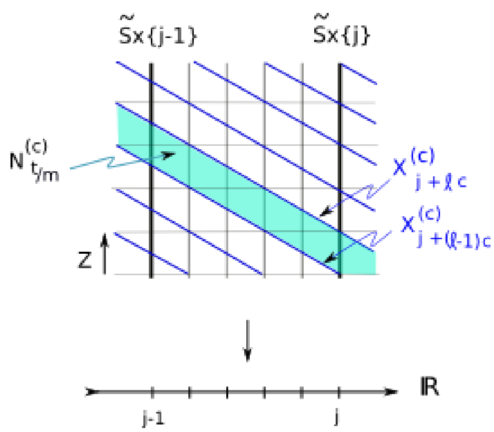

FIGURE 4. The surfaces $X_{\ell c}^{(c)}$ for $c = \frac{2}{5}$, and $t/m = j + \ell c$.

Recall that Y is the support of $\widehat{\phi}$ on $\widetilde{S}$. Let

$$\begin{aligned}\Gamma_{m\text{-rep},\ell} &= \bigcup_{i\in\mathbb{Z}} \zeta^{im+\ell}(Y) \\ &\subset \bigcup_{i\in\mathbb{Z}} \zeta^{im}(\Sigma_\ell \cup \cdots \cup \Sigma_{\ell+m_0-1}) \\ &\subset \widetilde{S}\end{aligned}$$

Then $\Gamma_{m\text{-rep},\ell}$ contains the support of $\widehat{\phi}_{m\text{-rep},\ell}$.

Let $N_{\frac{t}{m}}$ be the region on $\widetilde{S}\times\mathbb{R}$ defined by

$$N_{\frac{t}{m}} = \bigcup_{\frac{t-1}{m}\le v\le\frac{t}{m}} q_v^{(c)}(\widetilde{S}).$$

Then $N_{\frac{t}{m}}$ as the diagonal strip on $\widetilde{S}\times\mathbb{R}$ bounded by $X_{\frac{t-1}{m}}$ and $X_{\frac{t}{m}}$ (see Figure 4).

Let $\widetilde{f}^{(c)}$ be the (semi-continuous) flow on $\widetilde{S}\times\mathbb{R}$ defined, for $s\in\widetilde{S}$, $u\in\mathbb{R}$, and $0\le v<\frac{1}{m}$ by

$$\widetilde{f}_v^{(c)}(q_u^{(c)}(s)) = \begin{cases} q_{u+v}^{(c)}(s) & \text{if } (s,u),(s,u+v)\in N_{\frac{t}{m}} \text{ for some } t\in\mathbb{Z} \\ q_{u+v}^{(c)}(\widehat{\phi}_{m\text{-rep},\ell}(s)) & \text{if } (s,u)\in N_{t/m} \text{ and } (s,u+v)\in N_{(t+1)/m}, \end{cases}$$

where $j,\ell\in\mathbb{Z}$ and $t/m = j+\ell c$.

Lemma 3.3. *The flow is well-defined.*

Proof. The flow is well-defined because given two solutions

$$\frac{t}{m} = j+\ell c = j'+\ell' c,$$

for $j,j',\ell,\ell'\in\mathbb{Z}$, we have

$$m(j-j') = k(\ell'-\ell).$$

Since m and k are relatively prime, m divides $\ell'-\ell$. It follows that $\widehat{\phi}_{m\text{-rep},\ell} = \widehat{\phi}_{m\text{-rep},\ell'}$.
□

Lemma 3.4. *Z commutes with $\widetilde{f}^{(c)}$.*

Proof. For $(s,u),(s,u+v)\in N_{\frac{t}{m}}$ we have, using Lemma 3.1,

$$\begin{aligned}\widetilde{f}_v^{(c)} Z(q_u^{(c)}(s)) &= \widetilde{f}_v^{(c)}(q_{u+c}^{(c)}(\zeta(s))) \\ &= q_{u+v+c}^{(c)}(\zeta(s)) \\ &= q_{u+v+c}^{(c)}(\zeta(s)) \\ &= Z(q_{u+v}^{(c)}(s)) \\ &= Z\widetilde{f}_v^{(c)}(q_u^{(c)}(s)),\end{aligned}$$

and, using Lemma 2.8 and Lemma 3.1, for $(s,u) \in N_{j+\ell c}$ and $(s, u+v) \in N_{j+\ell c+\frac{t}{m}}$, we have

$$
\begin{aligned}
\widetilde{f}_v^{(c)} Z(q_u^{(c)}(s)) &= \widetilde{f}_v^{(c)}(q_{u+c}^{(c)}(\zeta(s))) \\
&= q_{u+v+c}^{(c)}(\widehat{\phi}_{m\text{-rep},\ell+1}\zeta(s)) \\
&= q_{u+v+c}^{(c)}(\zeta\widehat{\phi}_{m\text{-rep},\ell}(s)) \\
&= Z(q_{u+v}^{(c)}(\widehat{\phi}_{m\text{-rep},\ell}(s)) \\
&= Z\widetilde{f}_v^{(c)}(q_u^{(c)}(s)).
\end{aligned}
$$

□

Let $\widetilde{M}^{(c)}$ be defined from $\widetilde{S} \times \mathbb{R}$ as follows. Cut $\widetilde{S} \times \mathbb{R}$ along the $X_{j+\ell c}^{(c)}$, for $j, \ell \in \mathbb{Z}$, to obtain diagonal strips $N_{j,\ell}^{(c)}$ bounded by $(X_{j+(\ell-1)c}^{(c)})^-$ and $(X_{j+\ell c}^{(c)})^+$. Then glue $(X_{j+\ell c}^{(c)})^+$ to $(X_{j+\ell c}^{(c)})^-$ by the map

$$
q_{j+\ell c}^{(c)}(s) \;\mapsto\; q_{j+\ell c}^{(c)}(\widehat{\phi}_{m\text{-rep},\ell}(s)).
$$

Then the induced flow $\widetilde{f}^{(c)}$ on $\widetilde{M}^{(c)}$ is a continuous flow.

Remark 3.5. We could also think of $\widetilde{M}^{(c)}$ as $\widetilde{S} \times \mathbb{R}$ endowed with a new topology (different from the product topology).

Lemma 3.6. *The map Z defines a properly discontinuous self-homeomorphism $Z^{(c)}$ of $\widetilde{M}^{(c)}$.*

Proof. This follows from commutativity of Z and $\widetilde{f}^{(c)}$. □

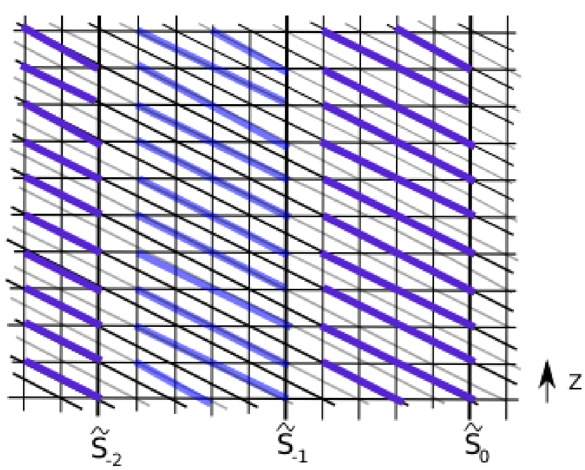

FIGURE 5. The manifold $\widetilde{M}^{(\frac{2}{5})}$ with $m_0 = 2$.

In Figure 5 the diagonal lines indicate the locus of cutting and pasting: the images of $q_{\ell c}^{(c)}$ for $\ell \in \mathbb{Z}$. The images $q_{j+\ell c}^{(c)}(Y_{m\text{-rep},\ell})$ in $\widetilde{M}^{(c)}$ are drawn as thickened segments of the diagonal lines in Figure 5. This is justified by the following.

Lemma 3.7. *Each connected component of the support of $\widehat{\phi}_{m\text{-rep},\ell}$ is sent by $q_{\ell c}^{(c)}$ strictly within a segment of $\widetilde{S} \times \mathbb{R}$ bounded by $\widetilde{S} \times \{j\}$ and $\widetilde{S} \times \{j+1\}$ for some $j \in \mathbb{Z}$.*

Proof. Fix $j, \ell \in \mathbb{Z}$. Then

$$\begin{aligned} q_{j+\ell c}^{(c)}(\Gamma_{m\text{-rep},\ell}) \ \cap \ \widetilde{S} \times [j-1, j] &\subset \ q_{j+\ell c}^{(c)}(\widetilde{S}) \ \cap \ \widetilde{S} \times [j-1, j] \\ &\subset \ q_{j+\ell c}^{(c)}(\Sigma_{\ell c} \cup \cdots \cup \Sigma_{\ell c+m_0-1}). \end{aligned}$$

The second coordinate is bounded from above by the projection of

$$\begin{aligned} q_{j+\ell c}^{(c)}(b_\ell^-) &= b_{\ell m}^- \times \{j + \ell c - c\mathfrak{p}(b_\ell^-)\} \\ &= b_\ell^- \times \{j + \ell c - \ell c\} \\ &= b_\ell^- \times \{j\} \\ &\subset \widetilde{S} \times \{j\}. \end{aligned}$$

It is bounded from below by the projection of

$$\begin{aligned} q_{j+\ell c}^{(c)}(b_{\ell m+m_0-1}^+) &= b_{\ell m}^+ \times \{j + \ell c - c\ \mathfrak{p}(b_{\ell+m_0-1}^+)\} \\ &= b_{\ell m}^+ \times \{j + \ell c - c\ \mathfrak{p}(b_{\ell+m_0}^-)\} \\ &= b_{\ell m}^+ \times \{j + \ell c - c(\ell + m_0)\} \\ &= b_{\ell m}^+ \times \{j - c\ m_0\} \end{aligned}$$

Since

$$c = \frac{k}{m} < \frac{1}{m_0},$$

we have

$$cm_0 < 1.$$

Thus the projection of $q_{j+\ell c}^{(c)}(b_{\ell m+m_0-1}^+)$ to the second coordinate is strictly greater than $j-1$. □

Corollary 3.8. *The cutting and pasting operations on $\widetilde{S} \times \mathbb{R}$ used to obtain $\widetilde{M}^{(c)}$ don't affect $\widetilde{S} \times \{j\}$.*

Proof. The map $\widehat{\phi}$ acts trivially on b_0^-. The rest follows from Lemma 3.7. □

Define $T^{(c)} : \widetilde{M}^{(c)} \to \widetilde{M}^{(c)}$ by

$$T^{(c)}(z) = \widetilde{f}_1^{(c)}(z).$$

From the definitions, and Lemma 3.4, we have the following.

Lemma 3.9. *The maps $T^{(c)}$ and $Z^{(c)}$ generate a rank two free abelian group of properly discontinuous automorphisms of $\widetilde{M}^{(c)}$.*

Proof. By Lemma 3.4 $\widetilde{f}^{(c)}$ and $Z^{(c)}$ commute as maps on $\widetilde{M}^{(c)}$, hence so does the pair $T^{(c)}$ and $Z^{(c)}$. □

3.2. **Step 2.** We prove that $(\widetilde{M}^{(c)}, Z^{(c)}, T^{(c)}, \widetilde{f}^{(c)})$ is independent of c up to isotopy.

Since $c, c' < \frac{1}{m_0}$, $1 - m_0 c, 1 - m_0 c' > 0$. Define a bijection $h_{c,c'} : \widetilde{S} \times \mathbb{R} \to \widetilde{S} \times \mathbb{R}$ by

$$h_{c,c'}(s,t) = \begin{cases} (s, j-1+\frac{1-m_0c'}{1-m_0c}(t-j+1)) & \text{if } j-1 \leq t \leq j-m_0c \\ (s, j+\frac{c'}{c}(t-j)) & \text{if } j-m_0c \leq t \leq j. \end{cases}$$

Lemma 3.10. *For $j, \ell \in \mathbb{Z}$, the subset $Y^{(c)}_{j+\ell c} \subset X^{(c)}_{j+\ell c}$ maps to $Y^{(c')}_{j+\ell c'} \subset X^{(c)}_{j+\ell c'}$ by*

$$h_{c,c'}(q^{(c)}_{j+\ell c}(s)) = q^{(c')}_{j+\ell c'}(s)$$

for $s \in \widetilde{S}$.

Proof. Take $(s,t) = (s, j + c\ell - c\mathfrak{p}(s)) = q^{(c)}_{j+\ell c}(s) \in Y^{(c)}_{j+\ell c}$. Then $\ell \leq \mathfrak{p}(s) \leq \ell + m_0$, and $j - m_0 c \leq j + c\ell - c\mathfrak{p}(s) \leq j$, and we have

$$\begin{aligned} h_{c,c'}(q^{(c)}_{j+\ell c}(s)) &= h(s, j+\ell c - c\mathfrak{p}(s)) \\ &= (s, j+\ell c' - c'\mathfrak{p}(s)) \\ &= q^{(c')}_{j+\ell c'}(s). \end{aligned}$$

□

Proposition 3.11. *For $c, c' \in (0, \frac{1}{m_0})$,*

$$h_{c,c'} : \widetilde{M}^{(c)} \to \widetilde{M}^{(c')}$$

is a homeomorphism satisfying

1. *$h_{c,c'}$ restricted to $\widetilde{S} \times \{j\}$ is the identify map for all $j \in \mathbb{Z}$,*
2. *the flow $h_{c,c'} \circ \widetilde{f}^{(c)} \circ h_{c',c}$ has the same flow-orbits as $\widetilde{f}^{(c')}$, and the flows are isotopic,*
3. *$h_{c,c'} \circ Z^{(c)} = Z^{(c')} \circ h_{c,c'}$, and*
4. *$h_{c,c'} \circ T^{(c)} = T^{(c')} \circ h_{c,c'}$.*

Proof. Since the cutting and pasting used to define $\widetilde{M}^{(c)}$ (resp., $\widetilde{M}^{(c')}$) only occurs at $Y^{(c)}_{j+\ell c}$ (resp., $Y^{(c')}_{j+\ell c'}$) for $j, \ell \in \mathbb{Z}$ and are defined by $\widehat{\phi}_{m\text{-rep},\ell}$ conjugated by $q^{(c)}_{j+\ell c}$ (resp., $q^{(c')}_{j+\ell c'}$), the map $h_{c,c'}$ defines a homeomorphism

$$h_{c,c'} : \widetilde{M}^{(c)} \to \widetilde{M}^{(c')}.$$

For $j \in \mathbb{Z}$, we have $h_{c,c'}(s,j) = (s,j)$ proving (1). Since $h_{c,c'}$ fixes the first coordinate, and is monotone in the second, $h_{c,c'} \circ \widetilde{f}^{(c)} \circ h_{c',c}$ is isotopic to $\widetilde{f}^{(c')}$ proving (2).

Since $h_{c,c'}$ leaves the first coordinate invariant, it commutes with Z, proving (3). Finally, the definition of $h_{c,c'}$ on $\widetilde{S} \times \mathbb{R}$ commutes with the translation map

$$(s,t) \mapsto (s, t+1).$$

For each $j \in \mathbb{Z}$, the gluing maps at the points $q^{(c)}_{j+\ell c}(s)$, $q^{(c)}_{j+1+\ell c}(s)$, and, by Lemma 3.10, their images under $h_{c,c'}$ are the same, namely $\widehat{\phi}_{m\text{-rep},\ell}$. It follows that $h_{c,c'}T^{(c)} = T^{(c')}h_{c',c}$ proving (4). □

We have shown that $\widetilde{M}^{(c)}$ for $c \in J_{m_0}$ defines a single isotopy class of 3-manifold $\widetilde{M}$ with properly discontinuous, rank 2 free abelian group action $G = \langle Z, T\rangle$ and flow $\widetilde{f}$

$$(\widetilde{M}, Z, T, \widetilde{f}) \equiv (\widetilde{M}^{(c)}, Z^{(c)}, T^{(c)}, \widetilde{f}^{(c)}).$$

For $c = \frac{k}{m} \in J_{m_0}$, Let $\alpha(c) = mT^* + kZ^* \in \mathrm{Hom}(G;\mathbb{Z})$. Fix the model $(\widetilde{M}^{(c)}, Z^{(c)}, T^{(c)}, \widetilde{f}^{(c)})$.

3.3. **Step 3.** Define $\widehat{T}^{(c)} : \widetilde{M}^{(c)} \to \widetilde{M}^{(c)}$ by

$$\widehat{T}^{(c)}(z) = \widetilde{f}^{(c)}_{\frac{1}{m}}(z).$$

Then we have for $s \in \widetilde{S}$ and $j + \ell c - \frac{1}{m} < u \leq j + \ell c$, for $j, \ell \in \mathbb{Z}$,

$$\widehat{T}^{(c)}(q^{(c)}_u(s)) = q^{(c)}_{u+\frac{1}{m}}(\widehat{\phi}_{m\text{-rep},\ell}(s)).$$

Lemma 3.12.

$$T^{(c)} = (\widehat{T}^{(c)})^m.$$

Proof.

$$T^{(c)} = \widetilde{f}^{(c)}_1 = (\widetilde{f}^{(c)}_{\frac{1}{m}})^m = (\widehat{T}^{(c)})^m.$$

□

Define

$$R^{(c)} = Z^{(c)}(\widehat{T}^{(c)})^{-k}.$$

Lemma 3.13. *The kernel of $\alpha(c)$ is generated by*

$$(R^{(c)})^m = (Z^{(c)}(\widehat{T}^{(c)})^{-k})^m = (Z^{(c)})^m(T^{(c)})^{-k}$$

Proof. See Lemma 3.12. □

Lemma 3.14. *The map $R^{(c)}$ preserves each $\widetilde{X}^{(c)}_v$ and we have, for $s \in \widetilde{S}$, $j, \ell \in \mathbb{Z}$, and $j + c\ell \leq v < j + c(\ell+1)$,*

$$R^{(c)}(q^{c)}_v(s)) = q^{(c)}_v(\zeta(\widehat{\phi}^{(c)}_{m\text{-}rep,\ell})^{-1}(s)).$$

Proof. By the definitions and Lemma 3.1, we have $q^{(c)}_v(s) \in N^{(c)}_{\frac{t}{m}}$, and

$$\begin{aligned} R^{(c)}(q^{(c)}_v(s)) &= Z(\widehat{T}^{(c)})^{-k}(q^{(c)}_v(s)) \\ &= Zq^{(c)}_{v-c}(\widehat{\phi}^{(c)}_{m\text{-rep},\ell}(s)) \\ &= q^{(c)}_v(\zeta\widehat{\phi}^{(c)}_{m\text{-rep},\ell}(s)). \end{aligned}$$

□

Note that if we had written $R^{(c)}$ as $(\widehat{T}^{(c)})^{-k}Z^{(c)}$ we would have

$$\begin{aligned}
R^{(c)}(q_v^{(c)}(s)) &= (\widehat{T}^{(c)})^{-k}Z(q_v^{(c)}(s)) \\
&= (\widehat{T}^{(c)})^{-k}(q_{v+c}^{(c)}(\zeta(s)) \\
&= q_v^{(c)}(\widehat{\phi}^{(c)}_{m\text{-rep},\ell+1}\zeta(s)) \\
&= q_v^{(c)}(\zeta\widehat{\phi}^{(c)}_{m\text{-rep},\ell}(s)).
\end{aligned}$$

Corollary 3.15. *The fibers $X_v^{(c)} \subset \widetilde{M}^{(c)}$ are preserved by $R^{(c)}$.*

3.4. **Step 4.** Let $a, b \in \mathbb{Z}$ satisfy $ak + bm = 1$. Then $Q^{(c)} = T^b Z^a$ satisfies

$$\alpha(c)(Q^{(c)}) = 1.$$

Then, since $Z^{(c)} = R^{(c)}(\widehat{T}^{(c)})^k$ by definition, and $(\widehat{T}^{(c)})^k = (T^{(c)})^k$ by Lemma 3.12, we have

$$\begin{aligned}
Q^{(c)}(q_0^{(c)}(s)) &= (T^{(c)})^b(Z^{(c)})^a q_0^{(c)}(s) \\
&= (T^{(c)})^b(R^{(c)})^a\widehat{(T^{(c)})}^{(1-bm)}(q_0^{(c)}(s) \\
&= (T^{(c)})^{b-b}(R^{(c)})^a\widehat{T}^{(c)}q_0^{(c)}(s) \\
&= (R^{(c)})^a q^{(c)}_{\frac{1}{m}}(\widehat{\phi}_{m\text{-rep},0}(s)) \\
&= q^{(c)}_{\frac{1}{m}}(\zeta^a\widehat{\phi}_{m\text{-rep},0}(s))
\end{aligned}$$

It follows that $\zeta^a\widehat{\phi}_{m\text{-rep},0}$ is a lift of $(S_{\alpha(c)}\phi_{\alpha(c)})$.

Define

$$\begin{aligned}
\overline{\alpha} : (0, \frac{1}{m_0}) &\to H^1(M;\mathbb{R}) \\
c &\mapsto \frac{\alpha(c)}{||\alpha(c)||}.
\end{aligned}$$

By Lemma 2.9, this competes the proof of Theorem B.

4. Stability of quotient families and proof of Theorem C

Let H be the group of covering automorphisms of $\widetilde{M}$ over M. This is generated by T and Z. Extend the map $\overline{\alpha}$ defined in Section 3 to the map

$$\begin{aligned}
\overline{\alpha} : (0, \frac{1}{m_0}) &\to H^1(M;\mathbb{R}) \\
\frac{k}{m} &\mapsto kZ^* + mT^*
\end{aligned}$$

As we have shown in Section 3, the image of $J_{m_0} \subset (0, \frac{1}{m_0})$ under the map α lies on a fibered face of the Thurston norm ball for M, and, in particular, for $c \in J_{m_0}$, we have

$$\overline{\alpha}(c) = \frac{\psi_{\alpha(c)}}{||\psi_{\alpha(c)}||}$$

for some fibered element $\psi_{\alpha(c)} \in H^1(M;\mathbb{Z})$, with monodromy $(S_{\alpha(c)}, \phi_{\alpha(c)})$.

4.1. **Case 1.** Assume $(\widetilde{S}, \zeta, \widehat{\phi})$ is of stable type.

Fix $m > m_0$. Then we have

$$\begin{aligned} \widehat{\phi}_{-m+1} \cdots \widehat{\phi}_0 &= \zeta^{-m+1}\widehat{\phi}\zeta^{m-1}\zeta^{-m+2}\widehat{\phi}\zeta^{m-2}\cdots\widehat{\phi} \\ &= \zeta^{-m}(\zeta\widehat{\phi})^m \\ &= \zeta^{-m_0}(\zeta\widehat{\phi})^{m_0} \\ &= \widehat{\phi}_{-m_0+1}\cdots\widehat{\phi}_0. \end{aligned}$$

For $i \in \mathbb{Z}$, by conjugating by ζ^i, we have

$$\widehat{\phi}_{i-m+1}\cdots\widehat{\phi}_i = \widehat{\phi}_{i-m_0+1}\cdots\widehat{\phi}_i.$$

Define $\widetilde{\phi} : \widetilde{S} \to \widetilde{S}$ so that for $s \in \Sigma_i$,

$$\widetilde{\phi}(s) = \widehat{\phi}_{i-m+1}\cdots\widehat{\phi}_i(s). \tag{4}$$

This is well-defined since, if $s \in b_i^+ = b_{i+1}^-$, then $\widehat{\phi}_{i+1}(s) = s$, and

$$\begin{aligned} \widehat{\phi}_{i-m+2}\cdots\widehat{\phi}_{i+1}(s) &= \widehat{\phi}_{i-m+2}\cdots\widehat{\phi}_i(s) \\ &= \widehat{\phi}_{i-m_0+1}\cdots\widehat{\phi}_i(s). \end{aligned}$$

Lemma 4.1. *The maps ζ and $\widetilde{\phi}$ commute as functions on $\widetilde{S}$.*

Proof. Let $s \in \Sigma$. Then we have

$$\begin{aligned} \zeta\widetilde{\phi}(s) &= \zeta\widehat{\phi}_{-m_0+1}\widehat{\phi}_{-m_0+1}\cdots\widehat{\phi}_0(s) \\ &= \widehat{\phi}_{-m_0+2}\circ\cdots\circ\widehat{\phi}_0\circ\widehat{\phi}_1(\zeta(s)) \\ &= \widetilde{\phi}(\zeta(s)). \end{aligned}$$

□

The limit of the projections $\overline{\alpha}(\frac{1}{m})$ of

$$\alpha(\frac{1}{m}) = Z^* + mT^*$$

as $m \to \infty$ on F is the projection of T^* on F, and we have

$$T(s,t) = \widetilde{f}_1(s,t) = (\widetilde{\phi}(s), t+1)$$

for all $s \in \widetilde{S}$.

Let $S = \widetilde{S}/\zeta$. Since ζ and $\widetilde{\phi}$ commute, there is a well defined mapping class $\phi : S \to S$ on the quotient surface, Thus, T^* defines a fibered element with monodromy (S,ϕ), and T^* projects to an interior point on F.

Thus, (S,ϕ) is the monodromy associated to ψ.

4.2. **Case 2.** Suppose $(\widetilde{S}, \zeta, \widehat{\phi})$ is of unstable type. Then for each integer $m \geq 1$, there exists an $s \in \widetilde{S}$ such that

$$\widehat{\phi}_{m+1}\widehat{\phi}_m \cdots \widehat{\phi}_0(s) \neq \widehat{\phi}_m \cdots \widehat{\phi}_0(s).$$

In this case $\widetilde{S}_0$ is not a cross-section of $\widetilde{f}_v$. Thus, T^* lies on the boundary of the cone over F. By Theorem 1.13,

$$\lim_{\alpha \in F,\ \alpha \to 0} L_F(S_\alpha, \phi_\alpha) = \infty.$$

This completes the proof of Theorem C.

Remark 4.2. For irrational $\alpha \in (0, \frac{1}{m_0})$, we can still define cross-sectional surfaces $\widetilde{Y}_v^{(\alpha)}$ and a transversal flow $\widetilde{f}_v$ on $\widetilde{M}$, however, in this case the cross-sectional surfaces project to dense leaves in the quotient M (see [McM1] for further discussion).

5. Examples

In this section we study some explicit quotient families of stable and non-stable type.

5.1. **Quotient familiy of stable type.** The stable quotient families of mapping classes described in Section 1.1 are generalizations of examples of Penner in [Pen] (see also, [Bau], [Tsa], and [Val]). Penner showed that under certain extra conditions, the mapping classes are pseudo-Anosov and have bounded normalized dilatations, by analyzing the transition matrices. In [Val], Valdivia proved that certain sequences of mapping classes generalizing Penner's examples are the monodromy of a single 3-manifold. Our results in Section 1.1 imply that we can obtain Penner's sequence, and its generalizations as the monodromy of $\alpha(\frac{1}{m})$ for some triple $(\widetilde{S}, \zeta, \widehat{\phi})$ of stable type.

The explicit sequence of mapping classes constructed by Penner in [Pen] are of the form $(S_g, \phi_g) \in \mathcal{P}$ where S_g are closed surfaces of genus g and the normalized dilatations

$$\lambda(\phi_g)^g$$

are bounded. He used this example together with a lower bound from properties of Perron-Froebenius matrices to show that

$$\log(\lambda(\phi_g)) \asymp \frac{1}{g}.$$

For any simple closed curve γ, let δ_γ be the right Dehn twist along γ. Let S_g^0 be an oriented genus g surface with 2 boundary components, and let

$$\phi_g = r_g \delta_{c_g} \delta_{b_g}^{-1} \delta_{a_g}$$

where the curves a_g, b_g and c_g are as shown in Figure 3, and r_g is a rotation of order g centered at the two boundary components (one of which is hidden in the picture).

The surface (S, ϕ) is shown in Figure 6, where ϕ is the mapping class on the torus with two boundary components given by the product of Dehn twists $\delta_c \circ \delta_b^{-1} \circ \delta_a$ centered at the curves a, b and c, and d is the path connecting the two boundary components. By Theorem A, we have the following.

Proposition 5.1. *Penner's sequence of mapping classes ϕ_g satisfies*

$$\lim_{g\to\infty} L(S_g^0, \phi_g) = L(S, \phi) \approx 46.9787.$$

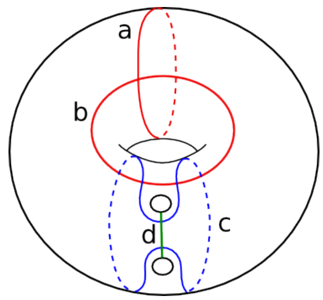

FIGURE 6. The minimal mapping class in the quotient family associated to Penner's sequence.

The action of ϕ on the first homology $H^1(S, \mathbb{Z})$ is given by the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and hence has a 1-dimensional invariant subspace. Thus, the mapping torus M has $b_1(M) = 2$. The cyclic covering $\widetilde{S} \to S$ defined by d is drawn in Figure 7. Let ζ generate the group of covering automorphisms. Then $\zeta \times \{\mathrm{id}\}$ and $T_{\widetilde{\phi}}$ define generators for $H_1(M; \mathbb{Z})$. Let Z^* be the dual of $\zeta \times \mathrm{id}$, or, equivalently, the extension of the map $\pi_1(S) \to \mathbb{Z}$ given by intersecting closed loops with the relative closed curve d. Let T^* be the fibration map dual to ϕ.

Let $u, t \in H_1(M; \mathbb{Z})$ be duals to Z^* and T^* respectively. Let τ be the train track for ϕ given by smoothing the union of a, b and c at the intersections (see [Pen]). The Teichmüller polynomial is the characteristic polynomial for the action of the lift of ϕ on the cyclic covering of S defined by τ on the lift $\widetilde{\tau}$ of τ, or more precisely on the space of allowable measures on $\widetilde{\tau}$. Using the switch conditions, we can replace the space of allowable measures with the space of labels on the lifts of the curves a, b and c. Then the Teichmüller polynomial of the fibered face defined by ϕ is a factor of the characteristic polynomial of the *twisted transition matrix*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1+t \\ 1+t^{-1} & 2(1+t^{-1}) & 1+(1+t)(1+t^{-1}). \end{bmatrix},$$

and is given by

$$\Theta(u, t) = u^2 - u(5 + t + t^{-1}) + 1.$$

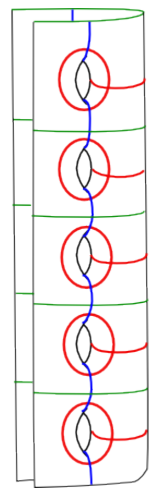

FIGURE 7. The simultaneous cyclic covering of Penner's examples.

Remark 5.2. This definition differs slightly from that of McMullen, where he divides this polynomial by a cyclotomic factor defined by the action of ϕ on vertices of τ. Since this does not affect the computation of dilatation.

The Alexander polynomial Δ is the characteristic polynomial of the action of the lift $\widetilde{\phi}$ of ϕ on the first homology of $\widetilde{S}$. The lifts of a, b and c generate $H_1(\widetilde{S}; \mathbb{Z})$ as a $\mathbb{Z}[t, t^{-1}]$ module, and the action of $\widetilde{\phi}$ on these generators is given by

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1-t \\ 1-t^{-1} & 2(1-t^{-1}) & 1+(1-t)(1-t^{-1}) \end{bmatrix}.$$

We thus have

$$\Delta(u,t) = \Theta(u,-t) = u^2 - u(5 - t - t^{-1}) + 1. \tag{5}$$

5.2. **Normalized dilatations.** By the relation between the Alexander and Thurston norms [McM2], it follows that the fibered cone C_ψ in $H^1(M;\mathbb{R})$ containing ψ is given by elements $a\psi + b\mu$, where

$$a > |b|,$$

and the Thurston norm is given by

$$||(a,b)||_T = \max\{2|a|, 2|b|\}.$$

The dilatation $\lambda(\phi_{(a,b)})$ corresponding to primitive integral points (a,b) in C_ψ is the largest solution of the polynomial equation

$$\Theta(x^a, x^b) = 0.$$

In particular, Penner's examples (S_g, ϕ_g) correspond to the points $(g, 1) \in C_\psi$, and we have the following.

Proposition 5.3. *The dilatation of ϕ_g is given by the largest root of the polynomial*

$$\Theta(x^g, x) = x^{2g} - x^{g+1} - 5x^g - x^{g-1} + 1.$$

The symmetry of Θ with respect to $x \mapsto -x$ and convexity of L on fibered faces implies the minimum normalized dilatation realized in $\mathcal{P}(M, F)$ must occur at $(a, b) = (1, 0)$. Thus, we have the following.

Proposition 5.4. *The minimum normalized dilatation for the monodromies in C_ψ is given by $L(S, \phi) \approx 46.9787$.*

5.3. **Orientabilitiy.** A pseudo-Anosov mapping class is *orientable* if it has orientable invariant foliations, or equivalently the geometric and homological dilatations are the same, and the spectral radius of the homological action is realized by a real (possibly negative) eigenvalue (see, for example, [LT] p. 5). Given a polynomial f, the largest complex norm amongst its roots is called the *house of* f, denoted $|f|$. Thus, ϕ_g is orientable if and only if

$$|\Delta(x^g, x))| = |\Theta(x^g, x)|. \tag{6}$$

Proposition 5.5. *The mapping classes (S_g, ϕ_g) are orientable if and only if g is even.*

Proof. By Equation (5), the homological dilatation of ϕ_g is the largest complex norm amongst roots of

$$\Delta(x^g, x) = x^{2g} + x^{g+1} - 5x^g + x^{g-1} + 1.$$

Let λ be the real root of $\Delta(x^g, x)$ with largest absolute value. Plugging λ into $\Theta(x^g, x)$ gives

$$\Theta(\lambda^g, \lambda) = -2\lambda^{g+1} - 2\lambda^{g-1} \neq 0.$$

while for $-\lambda$ we have

$$\Theta(-\lambda^g, -\lambda) = (-\lambda)^{g+1} - (\lambda)^{g+1} + (-\lambda)^{g-1} - (\lambda^{g-1}).$$

It follows that $|\Delta(x^g, x)| = \lambda = |\Theta(x^g, x)|$ if and only if g is even. $\square$

5.4. **Quotient families of unstable type.** We give two examples. The first example $\widehat{\phi}$ is defined by $T_{\widetilde{c}_1} T_{\widetilde{b}} T_{\widetilde{a}} T_r^{-1}$, where T_γ is a positive Dehn twist along γ, and a, b, c_1, r are the curves on $\widetilde{S}$ drawn in Figure 8. This example also appears in [GHKL]. The techniques in this paper lead to an explicit calculation of the dilatations of members of the corresponding quotient family.

As before, let (S_m, ϕ_m) be the mapping classes associated to $\alpha(\frac{1}{m})$. For example, when $m = 2$, S_m is a closed surface of genus 3, and $\phi_3 = \rho T_{c_0} T_{b_1} T_{a_0} T_{r_0}^{-1}$, where the curves $a_0, b_0, c_0, r_0, a_1, b_1, c_1, r_1$ are as drawn in Figure 9.

We will show the following.

Proposition 5.6. *The mapping classes (S_m, ϕ_m) are associated to the image of $\frac{1}{m}$ under an embedding $\alpha : (0,1) \to F$, where F is a fibered face. The corresponding quotient family is of unstable type, and as m goes to infinity, $L(S_m, \phi_m)$ behaves asymptotically as*

$$L(S_m, \phi_m) \asymp \frac{\log(m)}{m}.$$

From Penner's semi-group criterion, we have the following.

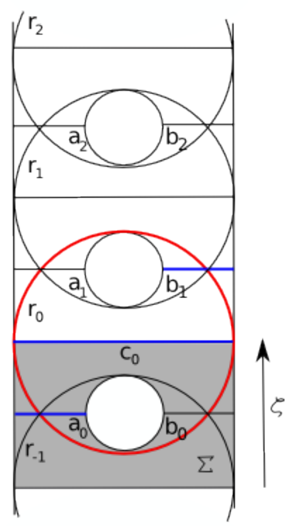

FIGURE 8. Infinite covering $\widetilde{S}$ and compactly supported map $\widehat{\phi}$.

Lemma 5.7. *The map ϕ_2 is pseudo-Anosov.*

A train track τ for ϕ_2 can be obtained by turning right on intersections with a, b, c, ta, tb, tc and turning left on intersections with r. The curves $a_0, b_0, c_0, a_1, b_1, c_1, r_0, r_1$ span the vector space of allowable weights on the edges of τ.

The transition matrix with respect to these vectors can be written as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2 & 0 & 0 & 2 & 0 & 5 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 2 & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

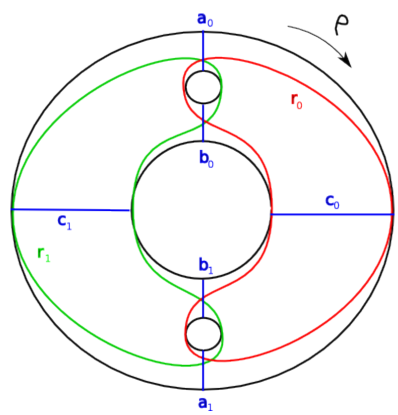

FIGURE 9. genus 3 case

The characteristic polynomial is

$$\Theta(u) = (u-1)^2(1+u)^2(1-2u-10u^2-2u^3+u^4),$$

and its house $|\Theta|$, is approximately 4.37709.

We can compute the Teichmuller polynomial restricted to the corresponding linear section of the fibered face of the mapping torus of (S_2, ϕ_3) by considering the induced transition matrix on the lifted train track.

$$\begin{bmatrix} 0 & t & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & t & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2t & 0 & 0 & 2 & 0 & 5 & 2 \\ 0 & t & 0 & 0 & 1 & 0 & 2 & 1 \\ t & t & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 2t & 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & t & t & 0 & 0 & 0 \end{bmatrix}.$$

The characteristic polynomial is

$$\Theta(u,t) = (t-u^2)^2(-t^2+(1+t)tu+10tu^2+(1+t)u^3-u^4).$$

For convenience of notation, we change variables:

$$\theta(x,y) = \frac{x^{10}\Theta(x^{-1},x^{-2}y)}{(y-1)^2} = 1-x-y(10+x+x^{-1})+(1-x^{-1})y^2.$$

Thus, for all $m \geq 2$, the dilatation of ϕ_m is the house of the polynomial

$$\theta(z,z^m) = 1-z-z^m(10+z+z^{-1})+(-z^{-1}+1)z^{2m}.$$

Proposition 5.8. *The minimum normalized dilatation for ϕ_m occurs at ϕ_2, and is approximately* 367.064.

Proof. Let $\overline{\alpha} : (0,\frac{1}{2}) \to F$ be the associated parameterization into a fibered face. One observes that for this example $\zeta\widehat{\phi}$ is conjugate to $\zeta^{-1}\widehat{\phi}$. Thus, the map $\overline{\alpha}$ extends to $(0,1)$, and (S_α,ϕ_α) is conjugate to $(S_{1-\alpha},\phi_{1-\alpha})$. It follows that $L(S_\alpha,\phi_\alpha) = L(S_{1-\alpha},\phi_{1-\alpha})$, and by convexity of L, the minimum of L must occur at $\overline{\alpha} = \frac{1}{2}$. □

Remark 5.9. In this case, the minimum normalized dilatation for the linear section defined by α occurs at the mapping class with smallest topological Euler characteristic (in absolute value). This is not typical as we see in the variation below.

We now consider the family of digraphs associated to the transition matrices for the train tracks of (S_m,ϕ_m). The following is a general property of Perron-Frobenius (PF)-digraphs.

Proposition 5.10. *Let Γ be a PF-digraph, with m vertices, such that for some constants c and d, the digraph Γ has one self-loop and all other cycles have length greater than $\frac{m}{d}-c$, where m is large compared to k and c. Then the spectral radius $\lambda(\Gamma)$ of the PF matrix associated to Γ satisfies*

$$m^{\frac{1}{2m}} \leq \lambda(\Gamma) \leq m^{\frac{3}{m}}.$$

The digraph associated to (S_2,ϕ_2) is shown in Figure 10, where the dotted edge is considered as a solid edge. The digraphs associated to (S_m,ϕ_m) are gotten by subdividing each dotted edge using $m-2$ vertices. One observes that there is a self-loop at a_1 independent of m.

Applying Proposition 5.10, it follows that as m approaches infinity, we have

$$\log L(S_{\frac{1}{m}},\phi_{\frac{1}{m}}) \asymp \frac{\log m}{m}.$$

This completes the proof of Proposition 5.6.

5.5. **Variation.** The second example is the quotient family associated to $(\widetilde{S},\zeta,\widehat{\phi})$, where $\widetilde{S}$ and ζ are as above, and

$$\widehat{\phi} = T_{c_0}T_{b_1}T_{a_0}^2.$$

The mapping classes (S_m,ϕ_m) are the same as before, except that

$$\phi_m = \rho \circ T_{c_0}T_{b_1}T_{a_0}^2T_{r_0}^{-1}.$$

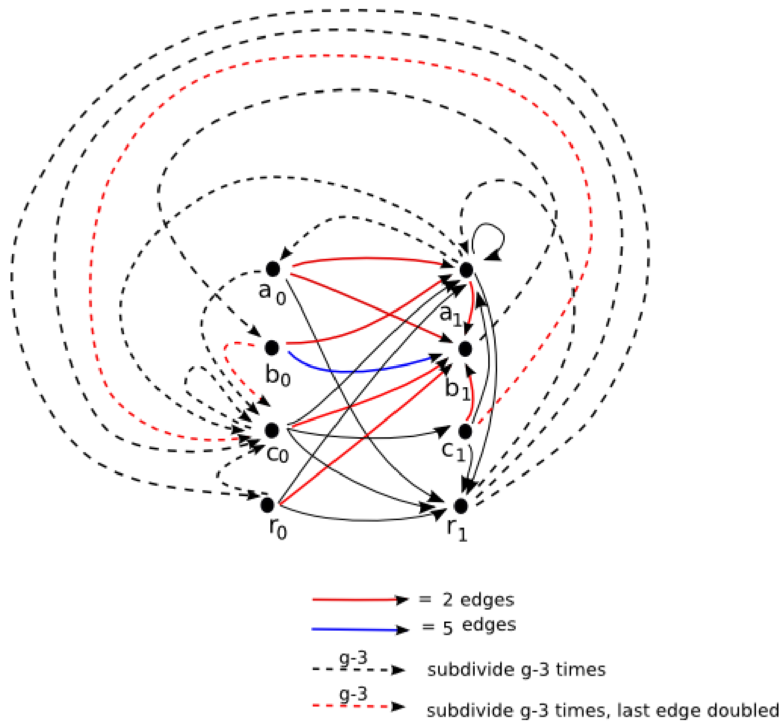

FIGURE 10. Digraph associated to (S_m, ϕ_m).

In this case, the twisted transition matrix is

$$\begin{bmatrix} 0 & t & 0 & 0 & 3 & 0 & 2 & 1 \\ 0 & t & 0 & 0 & 1 & 1 & 2 & 1 \\ 0 & 2t & 0 & 0 & 4 & 0 & 5 & 2 \\ 0 & t & 0 & 0 & 2 & 0 & 2 & 1 \\ t & t & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 2t & 0 & 0 & 2 & 0 & 2 & 1 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & t & 0 & t & 2t & 0 & 0 & 0 \end{bmatrix}.$$

and has characteristic polynomial

$$\Theta(u,t) = (t-u^2)^2(t^2 - tu - 2t^2u - 12tu^2 - 2u^3 - tu^3 + u^4).$$

Let

$$\theta(x,y) = \frac{x^4\Theta(x^{-1}, x^{-2}y)}{(y-1)^2} = 1 - 2x - y(12 + x + x^{-1}) + (1 - 2x^{-1})y^2.$$

The map α extends to $\alpha : (0,1) \to F$, and as $\alpha \in (0,1)$ approaches 0 or 1, $A(\alpha)$ approaches the boundary of F, and the dilatation of $(S_{\frac{k}{m}}, \phi_{\frac{k}{m}})$ is given by $|\theta(z^k, z^m)|$.

Lemma 5.11. *The behavior of* $\lambda(\alpha(c)$ *as* c *approaches* 0 *is given by*

$$\lim_{c\to 0} \lambda(\alpha(c)) = 2.$$

Computation shows that the smallest normalized dilatation for rational points in $\alpha(0,1)$ with denominator less than 70 occurs at $\alpha(\frac{3}{5})$, and

$$\lambda(\alpha(3/5)) \approx 1.93964.$$

Thus, in this case, the minimum normalized dilatation (if it exists) does not occur at a point where the topological Euler characteristic has smallest absolute value.

6. How common are quotient families?

Quotient families have special structure, and any quotient family can be reconstructed from any single element of the family.

Proposition 6.1. *Let* (S,ϕ) *be a mapping class. Then* (S,ϕ) *belongs to a quotient family if and only if* $\phi = r \circ \eta$, *where*

(i) r *is periodic of order* $m \geq 2$ *with fundamental domain* Σ *with bounded by* b_- *and* $b_+ = \zeta b_-$,
(ii) η *has support*

$$Y \subset \Sigma \cup \zeta\Sigma \cup \cdots \cup \zeta^{m-1}\Sigma.$$

Proof. Let $\widetilde{S}$ and ζ by taking the cyclic covering of S corresponding to the map $H_1(S;\mathbb{Z}) \to \mathbb{Z}$ given by intersection number with b_-. Let Σ' be a lift of Σ. Then η determines a map $\widehat{\phi}$ with support contained in

$$\Sigma' \cup \zeta\Sigma' \cup \cdots \cup \zeta^{m-1}\Sigma'.$$

Thus (S,ϕ) lies in the quotient family defined by $(\widetilde{S},\zeta,\widehat{\phi})$. □

Question 6.2. *Are there one-dimensional linear sections of fibered faces that do not contain any quotient family?*

Definition 6.3. Quotient families are examples of families that are strongly quasi-periodic. A mapping class on S is *quasi-periodic with support* $Y \subset S$ if there are mapping classes $r, \eta : S \to S$ such that

1. $\phi = r \circ \eta$,
2. r is supported on a subsurface $X \subset S$ and is periodic relative to the boundary of X, and
3. η is supported on Y.

A mapping class (S,ϕ) is *strongly quasi-periodic with support* Y if r is periodic on all of S, i.e., $X = S$.

A family of mapping classes $\mathcal{F} \subset \mathcal{P}$ is a *(strongly) quasi-periodic family* if for some κ,

$$\phi_\alpha = r_\alpha \circ \eta_\alpha$$

is (strongly) quasi-periodic with support Y_α, where

$$|\chi(Y_\alpha)| \leq \kappa.$$

Penner-type sequences and quotient families of mapping classes are strongly quasi-periodic.

Given $\ell > 1$, let $\mathcal{P}_\ell \subset \mathcal{P}$ be the elements with normalized dilatation less than ℓ.

Question 6.4 (Quasi-periodicity question)**.** *Is $\mathcal{P}_\ell$ a quasi-periodic family for all ℓ? It is strongly quasi-periodic?*

The strong version is known as the *symmetry question* and was posed by Farb, Leininger and Margalit in unpublished work.

Explicit quasi-periodic families are found in [HK], [Hir1], [Hir2] (cf. [AD] and [KT]) including examples whose normalized dilatations L converge to the smallest known accumulation point of L, namely

$$L_0 = \left(\frac{3+\sqrt{5}}{2}\right)^2,$$

which is the normalized dilatation of the simplest hyperbolic braid. It is not known whether or not these families are also strongly quasi-periodic.

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