Summary of the PhD Defense Presentation

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Notation:

\( \partial \)
\( \frac{d}{dx} \)
\( L_{inp} \) Input differential operator; a second order linear differential operator
\( H_{c,x}^{a,b} \) Gauss hypergeometric differential operator
\( S(x) = 2F_1(a, b; c \mid x) \) Gauss hypergeometric function; a solution of \( H_{c,x}^{a,b} \)
\( (e_0, e_1, e_\infty) \) Exponent differences of \( H_{c,x}^{a,b} \) at \( (0, 1, \infty) \)
\( f \rightarrow_c \) Change of variables
\( r_0, r_1 \rightarrow_G \) Gauge transformation
\( r \rightarrow_E \) Exponential product
\([g_0, g_1, \ldots, g_k] \) \( k \)-constellation; \( g_i \in S_n \) (details in the slides)

Given:
A second order linear differential operator \( L_{inp} \) with rational function coefficients (see slides for motivation);
1. irreducible and has no Liouvillian solutions,
2. has five regular singularities where at least one of the singularities is logarithmic, and
3. has arithmetic monodromy group.

Goal:
Solve \( L_{inp} \) in terms of \( 2F_1(a, b; c \mid f) \), i.e, find a solution \( y \) of the following form (also called the closed form solution):

\[
y = \exp(\int r \, dx) \left( r_0 S(f) + r_1 S(f)' \right) \neq 0
\]

where \( S(x) = 2F_1(a, b; c \mid x) \), \( r, r_0, r_1, f \in \mathbb{C}(x) \).

Problem Discussion:
For the solver program to be complete we need the complete tables of Belyi, Belyi-1 and Belyi-2 maps. We use the following correspondence to prove the completeness:

- Belyi maps \( \leftrightarrow \) dessins, i.e, equivalence classes of 3-constellations \([g_0, g_1, g_\infty] \) mod conjugation
- Belyi-1 maps \( \leftrightarrow \) near dessins, i.e, equivalence classes of 4-constellations \([g_0, g_1, g_1, g_\infty] \) mod conjugation
- Belyi-2 maps \( \leftrightarrow \) we have algorithms to compute such maps

- We compute Belyi, Belyi-1 maps solving polynomial equations and using other techniques.
- We compute dessins, near dessins using combinatorial search plus various techniques to prevent computational explosion.

Proof: Compare!
Dessin:

“Labelled dessin”, i.e, 3-constellation:

Read off permutations:

| black vertices: | $x = -\sqrt{3}, x = 0, x = \sqrt{3}$ | $g_0 = (1) (2 4 6 3) (5)$ |
| white vertices: | $x = -1, x = 1$ | $g_1 = (1 2 3) (4 5 6)$ |
| faces: | $x = -\frac{1}{\sqrt{3}}, x = \frac{1}{\sqrt{3}}, x = \infty$ | $g_{\infty} = (2) (6) (1 3 5 4)$ |

(follow outside face, $x = \infty$, clockwise)

Note:

Since the labelling was done arbitrarily, we should expect the result from “algcurves[monodromy]” to match only up to conjugation (same dessin, not the same 3-constellation). We have a program that can find the conjugation.