Desingularization of linear difference operators with polynomial coefficients

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Abstract

We consider the following two problems related to linear difference equations with polynomial coefficients:

1. Let E be the shift operator defined on sequences by $E(c_n) = c_{n+1}$ and defined on functions of n as $E(f(n)) = f(n+1)$. Consider a numeric sequence $c = \{c_l, c_{l+1}, \ldots\}$ where l is an integer. Suppose $T_1(c) = T_2(c) = 0$, where $T_1, T_2 \in \mathbb{C}[n, E]$, and the coefficients of E^0 in T_1, T_2 do not vanish for $n \geq l-1$. Both T_1 and T_2 allow to determine a corresponding value of c_{l-1}. Do the two values equal?

2. Let $F \in \mathbb{C}[n, E]$ and let its coefficient of E^0 vanish for some values of n. Does there exist $T \in \mathbb{C}[n, E]$ right divisible (over $\mathbb{C}(n)$, i.e. $T = G \circ F$ for some $G \in \mathbb{C}(n)[E]$) by F such that its coefficient of E^0 does not vanish for integer values of n? If so, how can one find it?

Solutions of these and of some more general problems are proposed.

1 Introduction

A recurrence relation is a tool to define sequences. If we have a linear recurrence operator $f_d(n)E^d + \cdots + f_1(n)E + f_0(n)$ and a sequence $\{c_l, c_{l+1}, \ldots, c_{l+d-1}\}$ where l is an integer, then we can try to extend this sequence to an infinite double-sided sequence $\{\ldots, c_{l-1}, c_l, c_{l+1}, \ldots, c_{l+d-1}, c_{l+d}, \ldots\}$. The integer roots of $f_d(n)$ and $f_0(n)$ present obstacles to this process.

In this paper an operator $F \in \mathbb{C}[n, E]$ is called a right divisor of an operator $T \in \mathbb{C}[n, E]$ if $T = V \circ F$ for some $V \in \mathbb{C}(n)[E]$, so we will allow rational functions in n as coefficients for the left hand factor V. If the original operator F is a right divisor of another operator $T \in \mathbb{C}[n, E]$ such that its coefficient $t_0(n)$ of E^0 has few (ideally: no) common roots with $f_0(n)$ then one can use T to determine the next term to the left left at the points where F does not determine the next term. Something similar can be done for extending sequences to the right, then one must consider the roots of $f_d(n-d)$ instead of $f_0(n)$, c.f. section 6.2. This method of continuing sequences raises the following question: if F is a right-hand factor of two operators T_1 and T_2, and we can not extend the sequence by F but we can extend the sequence by T_1 and T_2 (because $f_0(n)$ has an integer root at that point but the coefficients of E^0 of T_1 and T_2 do not) will the result be the same if we use T_1 or T_2? The answer to this question is yes (a more general result is given in Corollary 1 in section 3), and furthermore we will give another method for extending this sequence in section 5, which will give the same result as well. Acknowledgment: We would like to thank Ha Q. Le for comments on an earlier draft.

Example 1. Let $F = (n-1)E^2 + (1/2 - n^2)E + n(2n-1)/4$ and let $c_1 = 2$, $c_2 = 1$. Then for c_0 (take $n = 0$) we find the equation $-c_2 + \frac{1}{2} \cdot c_1 + 0 \cdot c_0 = 0$ which does not determine c_0. For c_3 (take $n = 1$) we find the equation $0 \cdot c_3 - \frac{1}{2} \cdot c2 + \frac{1}{4} \cdot c_1 = 0$ from which we can not determine c_3. Suppose we take for example $c_3 = 0$. Even though the recurrence still does not determine c_0, given the values of c_1, c_2 and c_3 there is a canonical way of choosing a value for c_0, as follows: F is a right divisor of $T = E^3 - (5/2+n)E^2 + (3/4+n)E - n/4 + 1/8$, namely $T = 1/n \cdot (E - 1/2) \circ F$. This recurrence gives a non-trivial equation for c_0 in terms of c_1, c_2 and c_3. The difference operator T is called a *desingularization* of F. We find $c_3 - \frac{5}{2}c_2 + \frac{3}{4}c_1 + \frac{1}{8}c_0 = 0$ so we find the value 8 for c_0, the same value as the ϵ-method in section 5 would give.

2 Matrices related to difference operators

Let

$$T = t_m(n)E^m + \cdots + t_0(n) \tag{1}$$

belong to $\mathbb{C}[n, E]$ and u be a positive integer. Assume $t_m \neq 0$, so the order of T is m. We define the matrix $T^{(u)}$ as

$$\begin{pmatrix} E^{u-1}t_m(n) & \ldots & E^{u-1}t_0(n) & & \\ & E^{u-2}t_m(n) & \ldots & E^{u-2}t_0(n) & \\ & \ddots & & \ddots & \\ & & t_m(n) & \ldots & t_0(n) \end{pmatrix} \tag{2}$$

We will use such matrices as blocks to construct other matrices. The blocks will be arranged vertically, one above the other. If necessary we add columns with zeros on the left of narrower blocks in order to make the widths of all blocks the same:

$$\begin{pmatrix} 0 & \ldots & 0 & E^{u-1}t_m(n) & \ldots & E^{u-1}t_0(n) & \\ & & & \ddots & & \ddots & \\ 0 & \ldots & 0 & & t_m(n) & \ldots & t_0(n) \end{pmatrix} \tag{3}$$

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This construction allows one to define the matrix $\mathrm{mat}(T_1^{(s_1)}, \ldots, T_k^{(s_k)})$:

$$\begin{pmatrix} T_1^{(s_1)} \\ \hline T_2^{(s_2)} \\ \hline \vdots \\ \hline T_k^{(s_k)} \end{pmatrix} \tag{4}$$

for $T_1, \ldots, T_k \in \mathbb{C}[n, E]$ and positive integers $s_1, \ldots, s_k$. Block (2) has $m + u$ columns. For block (3) one has

$$v - u - m = w,$$

where v is the total number of columns and w is the number of additional zero columns. If

$$\mathrm{ord}\, T_i = m_i, \; i = 1, \ldots, k, \tag{5}$$

and

$$v = \max_{i=1}^{k} (m_i + s_i), \tag{6}$$

then matrix (4) is a polynomial $(s_1 + \cdots + s_k) \times v$-matrix.

Lemma 1 *Let $F, T_1, \ldots, T_k \in \mathbb{C}[n, E]$ be such that $T_1, \ldots, T_k$ are right divisible over $\mathbb{C}(n)$ by F. Let $M, s_1, \ldots, s_k$ be positive integers such that*

$$\mathrm{ord}\, F + M \geq \max_{i=1}^{k} (\mathrm{ord}\, T_i + s_i). \tag{7}$$

Let $A = \mathrm{mat}(F^{(M)}, T_1^{(s_1)}, \ldots, T_k^{(s_k)})$. Then $\mathrm{rank}_{\mathbb{C}(n)} A = M$.

Proof: Because of (7) and the right divisibility of $T_1, \ldots, T_k$ by F any row of A is a $\mathbb{C}(n)$-linear combination over of the first-block rows, which are linearly independent over $\mathbb{C}(n)$. □

Recall that by right divisible we mean right divisible over $\mathbb{C}(n)$, i.e. the coefficients of the left hand factor are not restricted to $\mathbb{C}[n]$ but may be in $\mathbb{C}(n)$. If $l \in \mathbb{Z}$ and A is a matrix with elements from $\mathbb{C}[n]$ then denote by $A|_{n=l}$ the matrix obtained by substituting l for n in A.

Lemma 2 *Let the hypothesis of Lemma 1 hold. Let $l \in \mathbb{Z}$. Then $\mathrm{rank}_{\mathbb{C}} (A|_{n=l}) \leq M$.*

Proof: By Lemma 1 the determinant of any minor of size $M+1$ (i.e. a $M+1$ by $M+1$ submatrix) of A is zero. This still holds after the substitution. □

Definition 1 *Let $g_1, \ldots, g_u$ be elements of a ring, $u \geq 1$. The size of the row $(g_1, \ldots, g_u)$ is the integer number* $\max(\{0\} \cup \{i \,|\, 1 \leq i \leq u, \, g_i \neq 0\})$.

Definition 2 *Let A be a polynomial matrix, $l \in \mathbb{Z}$ and $B = A|_{n=l}$. A row of B is regular w.r.t. A if it is of the same size as the original row of A.*

Theorem 1 *Let $F, T_1, \ldots, T_k \in \mathbb{C}[n, E]$ be such that $T_1, \ldots, T_k$ are right divisible by F. Let M and s be positive integers such that*

$$\mathrm{ord}\, F + M \geq \max_{i=1}^{k} \mathrm{ord}\, T_i + s.$$

Let $A = \mathrm{mat}(F^{(M)}, T_1^{(s)}, \ldots, T_k^{(s)})$. For an integer $l \in \mathbb{Z}$ assume that $B = A|_{n=l}$ contains M rows that are regular w.r.t. A and of pairwise different sizes. Then

i) the set of these rows includes the first $M - s$ rows of B;

ii) the intersection of this set of rows and of the last M columns of B defines a basic minor of B (i.e., a minor, of maximal size, with non-zero determinant).

iii) the sizes of these rows are $d+1, \ldots, d+M$.

Proof: There are at most M possibilities for the size of a regular row of B (so they must all occur), namely the sizes of the rows of A. If $d = \mathrm{ord}\, F$ then these are $d+1, \ldots, d+M$, so *iii)* follows. Only the first $M - s$ rows of B can be regular rows of sizes $d+1, \ldots, d+M-s$, hence *i)* follows.

The determinant of the minor is nonzero because the minor is a lower triangular matrix with nonzero diagonal elements. By Lemma 2 this minor is of maximal size, so *ii)* follows. □

3 Extending P-recursive sequences

A sequence

$$c = \{c_v, c_{v+1}, \ldots\}, \tag{8}$$

$v \in \mathbb{Z}$, is *P-recursive* if it satisfies a linear recurrence $R(c) = 0, R \in \mathbb{C}[n, E]$. It is easy to see that sequence (8) satisfies a recurrence $R(c) = 0$ iff for any $\nu, \nu \geq \mathrm{ord}\, R + v$,

$$(c_\nu, c_{\nu-1}, \ldots, c_v) \tag{9}$$

satisfies the system of homogeneous linear equations with coefficient matrix $R^{(N)}|_{n=v}$, where $N = \nu - v + 1 - \mathrm{ord}\, R$.

Theorem 2 *Let $F, T_1, \ldots, T_k \in \mathbb{C}[n, E]$ be such that $T_1, \ldots, T_k$ are right divisible by F, $d = \mathrm{ord}\, F > 0$. Let $f_0(n), t_{1,0}(n), \ldots, t_{k,0}(n)$ be the coefficients of E^0 in, resp., $F, T_1, \ldots, T_k$ and let $l, v \in \mathbb{Z}$, $l < v$, be such that*

i) for all $\alpha \in \mathbb{Z}$, $\alpha \geq v$, each of the values $f_0(\alpha), t_{1,0}(\alpha), \ldots, t_{k,0}(\alpha)$ is not 0,

ii) for all $\alpha \in \mathbb{Z}$, $l \leq \alpha < v$, at least one of the values $f_0(\alpha), t_{1,0}(\alpha), \ldots, t_{k,0}(\alpha)$ is not 0.

Let a sequence c of the form (8) satisfy recurrences

$$F(c) = T_1(c) = \cdots = T_k(c) = 0. \tag{10}$$

Then the sequence c can be uniquely extended to a sequence

$$\{c_l, c_{l+1}, \ldots, c_{v-1}, c_v, c_{v+1}, \ldots\} \tag{11}$$

satisfying recurrences (10).

Proof: Set $s = v - l$, $M = \max_{i=1}^{k} \mathrm{ord}\, T_i + s - d$. Then all the assumptions of Theorem 1 are satisfied. Let the matrix A be as described in Theorem 1. Due to Theorem 1, the system S of homogeneous linear equations with coefficient matrix $A|_{n=l}$ has the following property. We can take the values of the first $d = \mathrm{ord}\, F$ unknowns arbitrarily and then determine (uniquely) the values of the remaining M

unknowns using equations corresponding to the chosen M regular rows (whose sizes are pairwise different). In doing so the remaining equations of S are satisfied.

We can take $c_{(M+l)+d-1}, c_{(M+l)+d-2}, \ldots, c_{M+l}$ as the values of the first d unknowns and then uniquely determine the values of the remaining M unknowns. Denote them as $c'_{M+l-1}, c'_{M+l-2}, \ldots, c'_l$. Then the sequence

$$\{c'_l, \ldots, c'_{M+l-1}, c_{M+l}, c_{M+l+1}, \ldots\} \tag{12}$$

satisfies (10). But due to *i)* of Theorem 1 we have $c'_{M+l-1} = c_{M+l-1}, \ldots, c'_{s+l} = c_{s+l}$. Observe finally that $s + l = v$. So (12) is an extension of (11). Setting $c_l = c'_l, c_{l+1} = c'_{l+1}, \ldots, c_{v-1} = c'_{v-1}$ we get (11). Let $\nu \geq \operatorname{ord} T_i + l$ where $1 \leq i \leq k$. Then

$$(c_\nu, c_{\nu-1}, \ldots, c_l) \tag{13}$$

satisfies (as we proved immediately above) the last $v - l$ equations of the linear algebraic system whose matrix is $T_i^{(N)}|_{n=l}$, $N = \nu - l - \operatorname{ord} T_i + 1$. Additionally (13) satisfies the remaining initial equations of the system, since (8) satisfies the recurrence $T_i(c) = 0$. Therefore (11) satisfies the recurrence $T_i(c) = 0$. The recurrence $F(c) = 0$ can be similarly investigated. □

Corollary 1 *Let $T_1, \ldots, T_k \in \mathbb{C}[n, E]$. Let a sequence $c = \{c_w, c_{w+1}, \ldots\}$, $w \in \mathbb{Z}$, satisfy recurrences (10). Let $l \in \mathbb{Z} \bigcup \{infty\}$ be such that $l < w$ and the coefficients $t_{1,0}(n), \ldots, t_{k,0}(n)$ of E^0 in $T_1, \ldots, T_k$ do not vanish simultaneously for $n \geq l$. Then the sequence c can be uniquely extended to a sequence $\{c_l, \ldots, c_{w-1}, c_w, c_{w+1}, \ldots\}$ satisfying recurrences (10).*

Proof: Denote the greatest common right divisor of a set of operators by GCRD. Let $F = \mathrm{GCRD}(T_1, \ldots, T_k) \in \mathbb{C}[n, E]$ and let $v \geq w$ be such that the coefficients of E^0 in $T, F_1, \ldots, F_k$ do not vanish for $n \geq v$; by Theorem 2 we get what is claimed. □

Example 2. Consider

$$T_1 = E^2+2E+1 = \left(\frac{1}{n-1}E + \frac{1}{n-1}\right)\circ((n-2)E + (n-1)),$$

$$T_2 = E^2+nE+n = \left(\frac{1}{n-1}E + \frac{n}{n-1}\right)\circ((n-2)E + (n-1)).$$

The sequence

$$c_n = (-1)^n(n-2),\ n = 1, 2, \ldots,$$

satisfies both recurrences $T_1(c) = 0, T_2(c) = 0$. According to Corollary 1 the sequence can be uniquely extended for all n. A direct check shows that the sequence

$$c_n = (-1)^n(n-2),\ -\infty < n < \infty, \tag{14}$$

satisfies $T_1(c) = 0, T_2(c) = 0$.

Example 3. Consider the operator

$$T = (n-2)E + (n-1).$$

Although the coefficient of E^0 has an integer root, one can still extend sequences using the fact that T is a right divisor of the first operator considered in Example 2.

Example 4. Consider the operator

$$T = E - n.$$

The sequence

$$c_n = (n-1)!,\ n = 1, 2, \ldots,$$

satisfies the recurrence $T(c) = 0$. This sequence can not be extended for $n = 0$ in such a way that the recurrence would still hold: by $T(c) = 0$ we would have $c_1 = 0$. But $c_1 = 1$. So we can conclude that T is not the right divisor of an element of $\mathbb{C}[n, E]$ whose coefficient of E^0 does not vanish for $n = 0$.

4 Singularities of linear difference operators with polynomial coefficients

Let $F \in \mathbb{C}[n, E]$

$$f_d(n)E^d + \cdots + f_0(n) \tag{15}$$

with $f_0(n) \neq 0$. Let $f_0(n)$ have at least one integer root. Consider the following problem: is there $T \in \mathbb{C}[n, E]$ of the form (1) which is right divisible by F and such that $t_0(n)$ has no integer root? First we give a simple necessary condition.

Definition 3 *An operator of the form (15) is bordered if*
i) $f_d(n)$ and $f_0(n)$ both have integer roots,
ii) the maximal integer root of $f_d(n)$ is greater than any integer root of $f_0(n)$.

Lemma 3 *Let F be of the form (15) with $\gcd(f_0(n), \ldots, f_d(n)) = 1$. Let $f_0(n)$ have at least one integer root. Suppose F is a right divisor of an operator $T \in \mathbb{C}[n, E]$ whose coefficient of E^0 has no integer root. Then F is a bordered operator.*

Proof: Assume F is not bordered; let l be the maximal integer root of $f_0(n)$ and suppose $f_d(n) \neq 0$ for all $n > l$. Choose $c_{l+1}, \ldots, c_{l+d} \in \mathbb{C}$ such that

$$f_d(l)c_{l+d} + \cdots + f_1(l)c_{l+1} \neq 0. \tag{16}$$

Applying the recurrence $F(c) = 0$ we extend $(c_{l+1}, \ldots, c_{l+d})$ to an infinite sequence

$$c = \{c_{l+1}, \ldots, c_{l+d}, c_{l+d+1}, \ldots\}.$$

By (16) there is no $c_l \in \mathbb{C}$ such that $c' = \{c_l, c_{l+1}, \ldots, c_{l+d}, c_{l+d+1}, \ldots\}$ satisfies $F(c') = 0$. But due to Theorem 2 this contradicts the existence of T. □

In the remainder of this section we assume

- F to be a bordered operator of the form (15), $\gcd(f_0(n), \ldots, f_d(n)) = 1$;
- α to be the maximal integer root of $f_d(n)$;
- $f_0(n)$ to have a nonempty set $Z = \{l_1, \ldots, l_\tau\}$ of integer roots, $l_1 > \cdots > l_\tau$; the multiplicities of the integer roots $l_1, \ldots, l_\tau$ are positive integers $\gamma_1, \ldots, \gamma_\tau$.

For a matrix A the matrix $A|_{n=l}$ is $A \ \mathbf{mod}\,(n-l)$, i.e. A with l substituted for n. More generally, the matrix $A \ \mathbf{mod}\,(n-l)^\gamma$ where γ is a positive integer is the image of A in the ring of matrices over $\mathbb{C}[n]/(n-l)^\gamma$. An element $q \in \mathbb{C}[n]/(n-l)^\gamma$ can be represented by a polynomial in n of degree less than γ. In this ring one has $q^\gamma = 0$ iff $q(l) = 0$ and q is invertible iff $q(l) \neq 0$.

Lemma 4 *Let C be the matrix $F^{(\alpha-l_1+1)}$, $D = C \,\mathbf{mod}\,(n-l_1)^{\gamma_1}$. Assume that the last row of D is a $\mathbb{C}[n]/(n-l_1)^{\gamma_1}$ linear combination of the other rows. Then there exists $U \in \mathbb{C}[n,E]$ such that*

i) U is right divisible by F;

ii) $\mathrm{order}(U) \le \alpha - l_1 + \mathrm{order}(F)$

iii) The coefficient of E^0 in U is $\frac{f_0(n)}{(n-l_1)^{\gamma_1}}$.

Proof We can construct such a nontrivial $\mathbb{C}[n]/(n-l_1)^{\gamma_1}$ linear relation of the rows of D that the coefficient of the last row of D in this relation is 1. If we take this linear relation and replace the coefficients in $\mathbb{C}[n]/(n-l_1)^{\gamma_1}$ by representatives in $\mathbb{C}[n]$ of degree $< \gamma_1$ then we find a $\mathbb{C}[n]$ linear combination of the rows of C, in which the last row occurs with coefficient 1, and this linear combination is 0 modulo $(n-l)_1^{\gamma}$, i.e. it is divisible by $(n-l)_1^{\gamma}$. This linear combination corresponds to an operator V with polynomial coefficients of degree $< \gamma_1$. The coefficients of operator $V \circ F$ are polynomials divisible by $(n-l_1)^{\gamma_1}$. The operator

$$\frac{1}{(n-l_1)^{\gamma_1}} V \circ F$$

belongs to $\mathbb{C}[n,E]$ and its coefficient of E^0 is equal to $f_0(n)/(n-l_1)^{\gamma_1}$. □

Lemma 5 *Let $T \in \mathbb{C}[n,E]$ be right divisible by F and let its coefficient of E^0 have no integer root $\ge l_1$. Then there exists $U \in \mathbb{C}[n,E]$ such that i), ii), iii) formulated in Lemma 4 hold.*

Proof: Consider the matrix $A = \mathrm{mat}(F^{(M)}, T^{(1)})$ where $\mathrm{ord}\, T = m$, $M = \max\{\alpha - l_1 + 1, m - d + 1\}$. Set $B = A \,\mathbf{mod}\,(n-l_1)^{\gamma_1}$. We will denote elements of B by b_{ij}, $i = 1,\dots,M+1$, $j = 1,\dots,d+M+1$. The rows of B will be denoted by $b_1,\dots,b_{M+1}$. In both matrices B and $A|_{n=l_1}$ the rows with number $1,\dots,M-1,M+1$ have sizes $d+1,\dots,d+M-1,d+M$. It implies in particular that elements $b_{1,d+1}, b_{2,d+2},\dots,b_{M-1,d+M-1}$ of B are invertible in $\mathbb{C}[n]/(n-l_1)^{\gamma_1}$. Using the invertibility we can find (by the Gaussian elimination) $\lambda_1,\dots,\lambda_{M-1} \in \mathbb{C}[n]/(n-l_1)^{\gamma_1}$ such that

$$\lambda_1 b_1 + \dots + \lambda_{M-1} b_{M-1} - b_M \qquad (17)$$

is a row of size $\le \mathrm{ord}\, F$. If this size is positive then the matrix B has a minor of size $M+1$ whose determinant is the product of M invertible elements and a non-zero element of row 17. Hence this determinant is non-zero, but by Lemma 1 the corresponding minor of A has determinant zero, and it remains zero after reducing $\mathbf{mod}\,(n-l_1)^{\gamma_1}$. Therefore (17) is the zero row.

If $M = \alpha - l_1 + 1$ then we are done, otherwise $M > \alpha - l_1 + 1$ and we can apply the following: Since b_{11} is invertible and $b_{M1} = 0$ we get $\lambda_1 = 0$. Similarly $\lambda_2 = \ldots = \lambda_{M-(\alpha-l_1+1)} = 0$. By Lemma 4 we get what was claimed. □

4.1 Algorithm ds and DS.

How can one determine if there exists $T \in \mathbb{C}[n,E]$ of the form (1) which is right divisible by F and such that $t_0(n)$ has no integer root? And if it exists, how can one find such operator? Using the idea of Lemmas 4, 5 we can give an algorithm that does this step by step. It computes $U_1,\dots,U_\tau$ such that the coefficient of E^0 in U_i is equal to $f_0(n)/(n-l_i)^{\gamma_i}$, $1 \le i \le \tau$. We call this algorithm **ds**. If the algorithm fails then such operator T does not exist.

Set

$$M = \alpha - l_\tau + 1, \quad C_0 = F^{(M)} \qquad (18)$$

and

$$k_1 = \alpha - l_1 + 1, \dots, k_\tau = \alpha - l_\tau + 1. \qquad (19)$$

Together with the construction of $U_1, U_2, \dots$ such that $\mathrm{ord}\, U_j \le d + \alpha - l_j$, the algorithm **ds** transforms the matrix C_0 into $C_1, C_2, \dots$ and, resp.,

$$D_1 = C_0 \,\mathbf{mod}\,(n-l_\tau)^{\gamma_1}, D_2 = C_1 \,\mathbf{mod}\,(n-l_\tau)^{\gamma_2}, \dots$$

The steps of constructing $D_1, C_1, U_1; D_2, C_2, U_2; \dots$ (steps $1,2,\dots$ of **ds**) are as follows. We find D_1 and represent its k_1-th row as a linear combination over $\mathbb{C}[n]/(n-l_\tau)^{\gamma_1}$ of the preceding $k_1 - 1$ rows. Let $\delta_1,\dots,\delta_{k_1-1} \in \mathbb{C}[n]/(n-l_\tau)^{\gamma_1}$ be the coefficients of this combination and let $\Delta_1,\dots,\Delta_{k_1-1} \in \mathbb{C}[n]$ be such that $\delta_i = \Delta_i \,\mathbf{mod}\,(n-l_\tau)^{\gamma_1}$, $i = 1,\dots,k_1-1$. Then construct the linear combination of the first $k_1 - 1$ rows of C_0 with coefficients $\Delta_1,\dots,\Delta_{k_1-1}$ and subtract it from the k_1-th row of this matrix. Now all the polynomials which are the elements of k_1-th row are divisible by the polynomial $(n-l_\tau)^{\gamma_1}$. By taking the quotients we get the matrix C_1. Its k_1-th row contains (in the first $d + k_1 - 1$ positions) the coefficients of $E^{M-k_1} \circ U_1$. Set $D_2 = C_1 \,\mathbf{mod}\,(n-l_\tau)^{\gamma_2}$. After this we consider the k_2-th row of the matrix D_2 and represent it as a linear combination of the preceding $k_2 - 1$ rows and so on. In the end we get the matrix C_i whose k_j-th row, $j = 1,\dots,i$, contains (in the first $d + k_j - 1$ positions) the coefficients of $E^{M-k_j} \circ U_j$ and, either $i = \tau$ or $i < \tau$ and it is not possible to construct the corresponding linear combination of the matrix D_i rows.

Observe that for D_i we actually consider only its first k_i rows, other rows are not involved in the computation.

The algorithm **ds** can stop after the i-th step, $0 \le i \le \tau$, if after constructing D_i we are not able to construct the corresponding linear combination of its rows.

Theorem 3 *An operator $T \in \mathbb{C}[n,E]$ right divisible by F and such that its coefficient of E^0 has no integer root out of the set $\{l_{i+1},\dots,l_\tau\}$ exists iff the algorithm **ds** does not stop before D_i, C_i are completely constructed (i.e., the i-th step of the performance is terminated).*

Proof: To prove the necessity we use induction on i.

1. $i = 1$. This case was considered in Lemma 5.

2. $i > 1$. The proof is similar to the proof of Lemma 5: we can consider the matrix

$$A = \mathrm{mat}(F^{(M)}, T^{(1)}, (E^{M-k_1} \circ U_1)^{(1)}, \dots, (E^{M-k_{i-1}} \circ U_{i-1})^{(1)}) \qquad (20)$$

(instead of $\mathrm{mat}(F^{(M)}, T^{(1)})$).

The proof of the sufficiency is in the description of the process of constructing the matrices D_1, C_1; D_2, C_2; ... □

Lemma 6 *Let $l_1,\dots,l_i$ be pairwise different integers, $i \ge 1$. Then there exist $\mu_1,\dots,\mu_i \in \mathbb{C}$ such that the polynomial*

$$\sum_{j=1}^{i} \mu_j (n-l_1)^{\gamma_1} \cdots (n-l_{j-1})^{\gamma_{j-1}} (n-l_{j+1})^{\gamma_{j+1}} \cdots (n-l_i)^{\gamma_i} \qquad (21)$$

has no integer root. (We assume (21) to be equal to μ_1 in the case $i = 1$.)

Proof: The case $i = 1$ is obvious. Otherwise the roots of (21) are continuous non-constant functions of $\mu_1, \ldots, \mu_i$. It implies what was claimed. (It is also possible to describe an algorithm to construct suitable $\mu_1, \ldots, \mu_i$.) $\square$

Theorem 4 *If the algorithm* **ds** *does not stop before the i-th step is done then there exists $W_i \in \mathbb{C}[n, E]$, $\operatorname{ord} W_i \leq d + \alpha - l_i$, right divisible by F and such that the coefficient of E^0 has no integer root out of the set $\{l_{i+1}, \ldots, l_\tau\}$.*

Proof: Compute $U_1, \ldots, U_i$ by **ds** and, additionally $\mu_1, \ldots, \mu_i \in \mathbb{C}$ such that (21) has no integer root. Then the coefficient of E^0 in the operator $\mu_1 U_1 + \cdots + \mu_i U_i$ is equal to

$$\frac{f_0(n)}{(n-l_1)^{\gamma_1} \cdots (n-l_i)^{\gamma_i}} \sum_{j=1}^{i} \frac{\mu_j}{(n-l_j)^{\gamma_j}} \prod_{k=1}^{i} (n-l_k)^{\gamma_k}.$$

The claim follows from this. $\square$

Factually we have described algorithm **DS** computing the maximal i, $0 \leq i \leq \tau$, such that there exists $W \in \mathbb{C}[n, E]$ right divisible by F and whose coefficient of E^0 has no integer root out of the set $\{l_{i+1}, \ldots, l_\tau\}$. Algorithm **DS** constructs a concrete W of order $\leq d + \alpha + l_i$ as well. If $i = \tau$ then we get W such that its coefficient of E^0 has no integer root.

Example 5. Let $F = (n-3)(n-2)E + (n-1)^2 n$. Then $l_1 = 1, l_2 = 0, \alpha = 3, M = 4, k_1 = 3, k_2 = 4, C_0 = F^{(4)} =$

$$\begin{pmatrix} n(n+1) & (n+2)^2(n+3) & 0 & 0 & 0 \\ 0 & (n-1)n & (n+1)^2(n+2) & 0 & 0 \\ 0 & 0 & (n-2)(n-1) & n^2(n+1) & 0 \\ 0 & 0 & 0 & (n-3)(n-2) & (n-1)^2 n \end{pmatrix},$$

$$D_1 = C_0 \textbf{ mod } n^2 = \begin{pmatrix} n & 16n+12 & 0 & 0 & 0 \\ 0 & -n & 5n+2 & 0 & 0 \\ 0 & 0 & -3n+2 & 0 & 0 \\ 0 & 0 & 0 & -5n+6 & n \end{pmatrix}.$$

First we represent the third row of D_1 as a linear combination of the preceding two rows (using the equalities $(5n+2)^{-1} = \frac{1}{4}(-5n+2)$, $(16n+12)^{-1} = \frac{1}{144}(-16n+12)$ in $\mathbb{C}[n]/n^2$). We get the coefficients $\frac{n}{12}$, $1-4n$ for this linear combination. Then construct the combination of the first two rows of C_0 with polynomial coefficients $\frac{n}{12}$, $1-4n$ and subtract it from the third row of this matrix. After division by n^2 we will get C_1:

$$\begin{pmatrix} n(n+1) & (n+2)^2(n+3) & 0 & 0 & 0 \\ 0 & (n-1)n & (n+1)^2(n+2) & 0 & 0 \\ \frac{-n-1}{12} & \frac{-n^2+41n}{12} - \frac{19}{3} & 4n^2+15n+17 & n+1 & 0 \\ 0 & 0 & 0 & (n-3)(n-2) & (n-1)^2 n \end{pmatrix}$$

We compute the operator U_1 such that the coefficient of E^i in $E \circ U_1$ equals the i'th entry in row 3 in C_1. So the coefficients of C_1 are shifts of the entries in this row. This operator

$$U_1 = -\frac{n}{12}E^3 + \left(-\frac{1}{12}n^2 + \frac{43}{12}n - \frac{59}{6}\right)E^2 + (4n^2+7n+6)E + n \tag{22}$$

is divisible by F and its coefficient of E^0 vanishes only at 0.

In the next step we compute

$$D_2 = C_1 \textbf{ mod } n = \begin{pmatrix} 0 & 12 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ -\frac{1}{12} & -\frac{19}{3} & 17 & 1 & 0 \\ 0 & 0 & 0 & 6 & 0 \end{pmatrix}$$

and try to represent the last row of D_2 as a linear combination over $\mathbb{C}$ of the preceding three rows. But such representation is not possible. Thus using algorithm **DS** we make sure that there is no operator from $\mathbb{C}[n, E]$ right divisible by F and whose coefficient of E^0 has no integer root. We get operator (22) right divisible by F and whose coefficient of E^0 has the one unique root 0 along with it.

5 The ϵ-criterion

Example 1, continued. Let F_ϵ be the operator obtained from the operator F in the example in section 1 by substituting $n+\epsilon$ for n. If we compute modulo ϵ then the operator has not changed, but we will compute modulo ϵ^2. Expressions modulo ϵ^2 can be conveniently expressed with the O notation. If we want to extend the sequence $c_3 = 0$, $c_2 = 1$ to the left then we take $c_3 = 0 + O(\epsilon^2)$, $c_2 = 1 + O(\epsilon^2)$ and use with the recurrence operator F_ϵ to find $c_1 = 2 + 2\epsilon + O(\epsilon^2)$ and $c_0 = 8 + O(\epsilon)$ (since we started with accuracy $O(\epsilon^2)$, and we have to divide by ϵ to find c_0, we can only find c_0 with accuracy $O(\epsilon)$, which is sufficient). By substituting $\epsilon = 0$ we find in a natural way, starting with $c_3 = 0$, $c_2 = 1$, the value 8 for c_0, even though the recurrence F does not actually define the value of c_0. Theorem 6 below then says that then F can be desingularized. Such a desingularization T is given in the example in section 1. Given $c_3 = 0$, $c_2 = 1$, we can use F to find $c_1 = 2$, and then use T to find $c_0 = 8$. So starting from $c_3 = 0$, $c_2 = 1$, we have two methods to find a value for c_0, and both give the same result.

In this section we assume the suppositions about F formulated after Lemma 3. Let

$$N = \alpha - l_1 + 1, C = F^{(N)}, D = C \textbf{ mod } (n-l_1)^{\gamma_1}.$$

Consider the system $Cz = 0$ of linear equations and investigate when this system has the following property:

P. For any $z_1, \ldots, z_d \in \mathbb{C}[[n-l_1]]$ there exist $z_{d+1}, \ldots, z_{d+N} \in \mathbb{C}[[n-l_1]]$ such that $(z_1, \ldots, z_{d+N})^T$ is a solution of $Cz = 0$.

Lemma 7 *The system $Cz = 0$ has property* **P** *iff the last row of D is a linear combination over $\mathbb{C}[n]/(n-l_1)^{\gamma_1}$ of its preceding rows.*

Proof: Let $z_1, \ldots, z_d \in \mathbb{C}[[n-l_1]]$. The invertibility in $\mathbb{C}[[n-l_1]]$ of the elements of C with indices $(1, d+1), (2, d+2), \ldots, (N-1, d+N-1)$ implies that $z_{d+1}, \ldots, z_{d+N-1} \in \mathbb{C}[[n-l_1]]$ are uniquely determined by the first $N-1$ equations of the system $Cz = 0$. If we substitute the $z_1, \ldots, z_{d+N-1}$ into the N-th equation then we get the equality

$$z_{d+N} = \frac{s}{f_0(n-l_1)} \tag{23}$$

for some $s \in \mathbb{C}[[n-l_1]]$.

Let the N-th row of D be a linear combination over $\mathbb{C}[n]/(n-l_1)^{\gamma_1}$ of its preceding rows. Then one can substract from the last row in C a $\mathbb{C}[[n-l_1]]$-linear combination of the other rows, and obtain a row that is divisible by $(n-l_1)^{\gamma_1}$. Dividing this row by $(n-l_1)^{\gamma_1}$ makes the entry $C_{N,d+N}$ invertible, and so this row allows one to find $z_{d+N} \in \mathbb{C}[[n-l_1]]$. The converse is proved in a similar way. $\square$

So if the system has property **P** then its last equation can be replaced by an equation whose last coefficient does not vanish at $n = l_1$ (the process of building such an equation was described in Lemma 4). The new system $C'z = 0$ allows, starting with $z_1, \ldots, z_d \in \mathbb{C}[[n - l_1]]$, to determine uniquely the $z_{d+1}, \ldots, z_{d+N}$ as solutions of the system $Cz = 0$. But now the system with the matrix $C' \; \mathbf{mod} \, (n - l_1)^{\gamma_1}$ allows to find in particular $z_{d+N} \; \mathbf{mod} \, (n - l_1)^{\gamma_1}$. Using this line of reasoning and considering $l_2, \ldots, l_\tau$ we get the following theorem on the system with matrix C_0 of the form (18) (this theorem is an analogue of Theorem 3):

Theorem 5 *The algorithm* **ds** *does not stop before* D_i, C_i *are completely constructed iff for any* $z_1, \ldots, z_d \in \mathbb{C}[[n - l_\tau]]$ *there exist* $z_{d+1}, \ldots, z_{d+\alpha-l_i+1} \in \mathbb{C}[[n - l_\tau]]$ *such that* $z_1, \ldots, z_{d+\alpha-l_i+1} \in \mathbb{C}[[n - l_\tau]]$ *satisfy the first* $\alpha - l_i + 1$ *equations of the system* $C_0 z = 0$. □

Below we will reformulate this theorem in a more convenient form. First we make two remarks.

R1. It is sufficient to consider the d following possibilities for $(z_1, \ldots, z_d)$:

$$(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1). \tag{24}$$

R2. For each vector of (24) the corresponding $z_{d+1}, z_{d+2}, \ldots$ belong to $\mathbb{C}(n)$. The solvability of the system $C_0 z = 0$ in $\mathbb{C}[[n - l_\tau]]$ is equivalent to the fact that these rational functions do not have a pole at $n = l_\tau$.

Consider the algorithm $\mathbf{H}_\epsilon$ which can be applied to $F, (z_1, \ldots, z_d), k$, where F is an operator of the form (15), $z_1, \ldots, z_d \in \mathbb{C}$ and k is non-negative integer:

1. Construct the operator F_ϵ by replacing n by $n + \epsilon$ in the coefficients of F (ϵ is an additional variable).

2. Taking $c_{\alpha+d} = z_1, c_{\alpha+d-1} = z_2, \ldots, c_{\alpha+1} = z_d$ and using the recurrence $F_\epsilon(c) = 0$ to compute $c_\alpha, \ldots, c_{\alpha+k-1} \in \mathbb{C}(\epsilon)$.

Theorem 6 (The ϵ-criterion.) *The algorithm* **ds** *does not stop before* D_i, C_i *are completely constructed (i.e., before the i-th step of the algorithm is terminated) and, as a consequence there exists a right divisible by* F *operator* $T \in \mathbb{C}[n, E]$ *whose coefficient of* E^0 *has no integer root out of* $\{l_{i+1}, \ldots, l_\tau\}$*, iff applying* $\mathbf{H}_\epsilon$ *to any* $(z_1, \ldots, z_d)$ *of the form (24) and to* $k = \alpha - l_i + 1$ *gives rational functions which do not have poles at* $\epsilon = 0$.

Proof: Taking into account remarks R1, R2 we see that this theorem is a reformulation of Theorem 5 when the notation $\epsilon = n - l_\tau$ is used. □

Example 6. Let $F = (n-3)(n-2)E + n(n-1)$. Here $d = 1$, hence in (24) we have one unique vector with single element 1. $F_\epsilon = (n+\epsilon-3)(n+\epsilon-2)E + (n+\epsilon-1)(n+\epsilon)$, i.e.,

$$c_n = -\frac{(n+\epsilon-3)(n+\epsilon-2)}{(n+\epsilon-1)(n+\epsilon)} c_{n+1}.$$

Setting $c_4 = 1$ we get

$$c_3 = -\frac{\epsilon(\epsilon+1)}{(\epsilon+3)(\epsilon+2)}, \; c_2 = \frac{(\epsilon-1)\epsilon^2}{(\epsilon+2)^2(\epsilon+3)},$$

$$c_1 = -\frac{(\epsilon-2)(\epsilon-1)^2\epsilon}{(\epsilon+1)(\epsilon+2)^2(\epsilon+3)}, \; c_0 = \frac{(\epsilon-3)(\epsilon-2)^2(\epsilon-1)}{(\epsilon+1)(\epsilon+2)^2(\epsilon+3)}.$$

We can desingularize F since neither c_1 nor c_0 have the pole at $\epsilon = 0$. Using **DS** we get

$$W = (n+1)E^4 + (-17n+29)E^3 + (-17n-56)E^2 + (7n-11)E - 1.$$

Example 5, continued. Consider again F from Example 5. Here $d = 1$, $F_\epsilon = (n+\epsilon-3)(n+\epsilon-2)E + (n+\epsilon-1)^2(n+\epsilon)$ and

$$c_n = -\frac{(n+\epsilon-3)(n+\epsilon-2)}{(n+\epsilon-1)^2(n+\epsilon)} c_{n+1}.$$

Setting $c_4 = 1$ we get

$$c_3 = -\frac{\epsilon(\epsilon+1)}{(\epsilon+3)(\epsilon+2)^2}, \; c_2 = \frac{(\epsilon-1)\epsilon^2}{(\epsilon+1)(\epsilon+2)^2(\epsilon+3)},$$

$$c_1 = -\frac{(\epsilon-2)(\epsilon-1)^2\epsilon}{(\epsilon+1)^2(\epsilon+2)^2(\epsilon+3)}, \; c_0 = \frac{(\epsilon-3)(\epsilon-2)^2}{\epsilon(\epsilon+1)^2(\epsilon+2)^2(\epsilon+3)}.$$

We can desingularize F only partially since c_0 has a pole at $\epsilon = 0$ (as was found before).

As illustrated in example 1 continued, one can also compute modulo ϵ^{a+1} where a is the number of times we have to divide by ϵ. This is usually more efficient. Note that the ϵ-method of extending sequences is used in [2] to determine pole orders of rational solutions. It is used in [3] to determine valuation growths and extension maps $E_{p,r}$ which can be used for computing hypergeometric solutions.

6 Extended algorithms

6.1 Decreasing of the root multiplicity

If it is impossible to eliminate an integer root of the coefficient of E^0 in a given operator F, then one can try to decrease the multiplicity (in case it is > 1). Actually if the corresponding linear algebra problem is not solvable modulo $(n-l)^\gamma$ then it is still possible that the problem is solvable modulo $(n-l)^{\gamma'}, \gamma' < \gamma$. Hence the algorithms **ds**, **DS** can be extended in such a way that in case if it is impossible to perform the next-in-turn step and, this step is connected with computations modulo $(n-l)^\gamma$ then we try to perform it modulo $(n-l)^{\gamma-1}$ and so on. Only after this additional step the running of extended versions of **ds**, **DS** are terminated.

One can observe the following. The proof of Lemma 3 shows that if F is bordered then the decreasing of the multiplicity of l_1 is possible iff any sequence $\{c_{l_1+1}, c_{l_1+2}, \ldots\}$ satisfying the equation $F(y) = 0$ can be extended by an element c_{l_1} so that $\{c_{l_1}, c_{l_1+1}, c_{l_1+2}, \ldots\}$ satisfies the same equation.

6.2 Roots of the leading coefficient

The desingularization problem can be considered as being applied to the leading coefficient of the operator F. We will use analogous (with respect to the already considered problem) algorithms if $F = f_0(n) + f_{-1}(n)E^{-1} + \cdots + f_{-d}(n)E^{-d}$ and we are looking for an operator of the form

$$S = s_0(n) + s_{-1}(n)E^{-1} + \cdots + s_{-m}(n)E^{-m} \tag{25}$$

which is right divisible by F and such that $s_0(n)$ has, say, no integer root (the only distinction is that we have to consider integer roots in increasing order). However if S has the form (25) then $T = E^m \circ S$ has the form (1). If F has the form (15) we can consider the integer roots $l_1 < \ldots < l_\tau$ (with multiplicities $\gamma_1, \ldots, \gamma_\tau$) of the polynomial $f_d(n-d)$; it is possible to describe the algorithm $\mathbf{ds}^+$ that constructs operators $T_1, \ldots, T_i$ right divisible by F such that $T_j = t_{j,m_j}(n)E^{m_j} + \ldots + t_{j,0}(n)$ and

$$t_{j,m_j}(n-m_j) = \frac{f_d(n-d)}{(n-l_j)^{\gamma_j}},$$

$j = 1, \ldots, i$, for the largest possible $i < \tau$. Similarly we can describe $\mathbf{DS}^+$ which constructs T of the form (1) such that $t_m(n-m)$ has no integer root out of the set $\{l_{i+1}, \ldots, l_\tau\}$.

6.3 Non-integer roots

The described algorithms can be applied to eliminate non-integer roots. Instead of $\mathbb{Z}$ we can consider an arbitrary set $c + \mathbb{Z}$ where c is a fixed complex number. The idea of the greatest factorial factorization [5] is quite convenient for this. This, combined with corollary 1, can be used for bounding the set of poles of rational solutions (c.f. [1, 2]) of systems of linear difference equations.

In conclusion of this section remark that the ϵ-criterion can be extended for all the cases considered in subsections 6.1, 6.2, 6.3.

7 An application

We call the *order* of a P-recursive sequence (8) the order of the minimal operator which annihilates a sequence $c' = \{c_N, c_{N+1}, \ldots\}$ where N is an integer $\geq v$. For example, the order of any hypergeometric sequence is equal to 1. Let T of the form (1) have solution c of order d and F of the form (15) be the corresponding minimal annihilator. If we want to know a concrete N for the sequence c' satisfying both equalities $T(c') = 0, F(c') = 0$ we at the first glance should take into account the form of F. But by Corollary 1 we can clam that the equation $T(y) = 0$ has a solution of order d with $N = n_0 + 1$, where n_0 is the largest integer root of $t_0(n)$ if such roots exist and $n_0 = -\infty$ otherwise. Furthemore, let F be the minimal annihilator for a sequence c of the form (8) and as before F be a right divisor of T. Then there exists a sequence $d = \{d_{n_0+1}, d_{n_0+2}, \ldots\}$ satisfying $T(d) = F(d) = 0$ such that $c_i = d_i$ for all $i \geq \max\{n_0 + 1, v\}$.

Example 7. Going back to Example 2 we see that $T = T_1$ is right divisible by $F = (n-2)E + (n-1)$. Let $c_n = (-1)^{n+1}(n-2)$ for $n = 2, 3, \ldots$ Then defining additionally c_0, c_1 as, resp.,

$$c_0 = c_1 = 0;\ c_0 = 2, c_1 = -1;\ c_0 = -2, c_1 = 1$$

we get three sequences of first order c', c'', c''' that satisfy the equation $F(y) = 0$. At the same time the sequence c of the form (14) satisfies the equation $T(y) = 0$ and this sequence coincides with the sequences c', c''' for $n \geq 2$ and coincides with c'' for $n \geq 0$.

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