LENGTH ONE IDEAL EXTENSIONS AND THEIR ASSOCIATED GRADED RINGS

Sam Huckaba

Dedicated to Wolmer V. Vasconcelos, in honor of his sixty fifth birthday.

Abstract. Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring. Given $\mathfrak{m}$-primary ideals $J \subset I$ of R such that I is contained in the integral closure of J and $\lambda(I/J) = 1$, we compare $\operatorname{depth} G(J)$ and $\operatorname{depth} G(I)$. For example, if J has reduction number one, $JI = I^2$, and $\mu(J) \leq d+1$, we prove that $\operatorname{depth} G(I) \geq d-1$. If, in addition, $\mu(I) = d+1$, we show that I has reduction number one, and hence $G(I)$ is Cohen-Macaulay. These results, besides leading to statements comparing depths of associated graded rings along a composition series, make visible the possibility of studying powers of an ideal by using reductions that are not minimal reductions.

1. Introduction

This paper studies the following question: If $(R, \mathfrak{m})$ is a local ring (at least Cohen-Macaulay, sometimes Gorenstein or regular) and $J \subset I \subset \overline{J}$ (where $\overline{J}$ denotes the integral closure of J) are $\mathfrak{m}$-primary ideals of R such that $\lambda(I/J) = 1$, what is the relationship between $\operatorname{depth} G(I)$ and $\operatorname{depth} G(J)$? The expression $\lambda(N)$ means the length of the R-module N, $G(I)$ denotes the associated graded ring of I, and $\operatorname{depth} G(I)$ is the depth of the local ring $G(I)_M$, where M is the homogeneous maximal ideal $\mathfrak{m}/I \oplus I/I^2 \oplus I^2/I^3 \oplus \ldots$ of $G(I)$.

As partial motivation, consider an $\mathfrak{m}$-primary ideal J of the Cohen-Macaulay local ring $(R, \mathfrak{m})$ and let $J = J_0 \subset J_1 \subset \cdots \subset J_k = \overline{J}$ be a chain of ideals such that $\{J_{i+1}/J\}$ is a composition series for the R-module $\overline{J}/J$. Our interest is in understanding changes that may occur among blowup algebras associated to J_i, as i increases from 0 to k. For example, if R is a two-dimensional regular local ring and J_0 is generated by a regular sequence, then both J_0 and $\overline{J}$ have Cohen-Macaulay associated graded rings. However, a saturated sequence of ideals leading from J_0 to $\overline{J}$ may contain ideals whose associated graded rings have depth zero. What features of the sequence lead to intermediate associated graded rings having depth 0, 1, or 2? What is the significance of the existence of a sequence, all of whose ideals have Cohen-Macaulay associated graded rings? More generally, what features of $\overline{J}$, $G(J)$ or $G(\overline{J})$ are visible within the associated graded rings $G(J_i)$. This paper contributes to understanding questions like these by focusing on length one extensions, as described in the first paragraph.

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Another inspiration for the topics contained here is a better understatnding of depth properties of associated graded rings, and the significance of those properties. Research has established that if depth $G(I)$ is sufficiently high, then the nature of the filtration $\{I^n\}$ is accessible. For example, it is often possible to describe explicitly the Hilbert-Samuel polynomial $P_I(x)$ of I, which by definition returns the value $\lambda(R/I^n)$ when evaluated at n for $n \gg 0$, whenever depth $G(I)$ is at least $\dim G(I) - 1$ (see [Huc], [HM], [M], [RV], [S1], [S2], [S3]). Further, the Cohen-Macaulay property of $G(I^k)$ is equivalent (in the proper context) to the vanishing of cohomology arising in a desingularization $X = \operatorname{Proj} R[It]$ of $\operatorname{Spec} R$ (by invoking versions of the Grauert-Riemenschnieder vanishing theorem — see [SS] and [L]). These and other reasons add substance to the task of identifying associated graded rings (and Rees algebras) of high depth.

One useful approach to understanding depth $G(I)$ is to consider a minimal reduction J of I (see Section 2 for the definition) and take advantage of properties of the filtration $\{J^n\}$ that transfer, via the reduction equation $JI^r = I^{r+1}$, to the filtration $\{I^n\}$. If r in the equation can be chosen to be sufficiently small then this technique works reasonably well, provided that $\{J^n\}$ is adequately well-understood. For example, if R is Cohen-Macaulay and J is a complete intersection ideal of R, then $G(J)$ is Cohen-Macaulay (it is isomorphic to a polynomial ring over R/J). In this case, the condition $r = 1$ forces $G(I)$ to be Cohen-Macaulay (see, e.g. [Vas, Proposition 5.1.2]).

In our setting of a Cohen-Macaulay local ring $(R, \mathfrak{m})$ and $\mathfrak{m}$-primary ideals $J \subset I \subset \overline{J}$ such that $\lambda(I/J) = 1$, J is a reduction of I but only in extreme cases is it a minimal reduction of I. Because we wish to study the situation generally, we must address the technical challenge of transfering information from $\{J^n\}$ to $\{I^n\}$ when J is not a minimal reduction of I. We will show that the condition $JI = I^2$ fulfills our needs for interesting cases.

In the context of the question stated in the first paragraph, here is one of our results.

> If $\lambda(I/J) = 1$, $JI = I^2$, $\mu(J) \leq d + 1$, and J has reduction number one, then depth $G(I) \geq d-1$. Further, if $\mu(I) = \dim R + 1$, then $G(I)$ is Cohen-Macaulay.

This appears in section 3 along with our other results. Section 2 consists of background material.

2. PRELIMINARIES

Suppose $(R, \mathfrak{m})$ is a local ring. If I is an ideal of R, a *reduction* of I is an ideal $J \subseteq I$ that satisfies $JI^k = I^{k+1}$ for all $k \gg 0$ (see [NR]). A *minimal reduction* of I is a reduction J_0 of I that is minimal with respect to inclusion among all reductions of I. It turns out that J is a reduction of I if and only if $J \subseteq I \subseteq \overline{J}$, where $\overline{J}$ is the integral closure of J. In general, if $R/\mathfrak{m}$ is infinite, then every minimal reduction of I is generated by a elements, where $a = \dim R[It]/\mathfrak{m}R[It]$ is the analytic spread of I. Furthermore, $\operatorname{ht} I \leq a \leq \dim R$, and hence $a = \dim R$ if I is $\mathfrak{m}$-primary. Therefore, if R is Cohen-Macaulay and I is $\mathfrak{m}$-primary, then every minimal reduction of I is a complete intersection ideal.

Suppose that I is an $\mathfrak{m}$-primary ideal of R and that $d = \dim R$. The Hilbert-Samuel function of I is defined by $H_I(n) = \lambda(R/I^n)$. It follows from the theory of Hilbert functions applied to the associated graded ring of I that there is an integer-valued polynomial $P_I(x)$ of degree d with rational coefficients such that $P_I(n) = H_I(n)$ for all $n \gg 0$. $P_I(x)$ is called the *Hilbert-Samuel polynomial* of I. Writing $P_I(n)$, to emphasize that its length interpretation is valid only for positive integers, it can be expressed in the form

$$P_I(n) = e_0(I)\binom{n+d-1}{d} - e_1(I)\binom{n+d-2}{d-1} + \cdots + (-1)^{d-1}e_{d-1}(I)n + (-1)^d e_d(I).$$

The proofs of our main results use a partial characterization of $\operatorname{depth} G(I)$ in terms of the coefficient $e_1(I)$ of $P_I(n)$. For convenience we state this separately.

Theorem 2.1. *(*[HM]*, Theorem 4.7)Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring, I an $\mathfrak{m}$-primary ideal of R, and J_0 a minimal reduction of I. Then*

$$\sum_{n \geq 0} \lambda((I^{n+1} + J_0)/J_0) \leq e_1(I) \leq \sum_{n \geq 0} \lambda(I^{n+1}/J_0 I^n).$$

Furthermore,

$$\sum_{n \geq 0} \lambda((I^{n+1} + J_0)/J_0) = e_1(I) \text{ if and only if } G(I) \text{ is Cohen-Macaulay}$$

and

$$\sum_{n \geq 0} \lambda(I^{n+1}/J_0 I^n) = e_1(I) \text{ if and only if } \operatorname{depth} G(I) \geq d-1.$$

There is a standard technique for reducing to the case of an infinite residue field; replace $(R, \mathfrak{m})$ with $(S, \mathfrak{n})$, where $S = R[x]_{(x)}$ and $\mathfrak{n} = (\mathfrak{m}, x)R[x]_{(x)}$.

Remark 2.2. Let $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ be as above, let I be an ideal of R, and let $K = IS$. Then $\operatorname{depth} G(I) = \operatorname{depth} G(K)$.

Remark 2.2 will be used below to insure the existence of a d-generated minimal reduction of an $\mathfrak{m}$-primary ideal, and also of superficial sequences. Recall that an element $x \in I \backslash I^2$ is said to be *superficial* (of order one) for I if there is a positive integer c such that $(I^n : x) \cap I^c = I^{n-1}$ for all $n > c$. By the Artin-Rees lemma, if x is R-regular and superficial for I, then $(I^n : x) = I^{n-1}$ for all $n \gg 0$. Superficial elements for I exist if R/m is infinite ([N] or [ZS]). If $y_1, \ldots, y_s \in I$ such that the image of y_i in $I/(y_1, \ldots, y_{i-1})$ is superficial for $I/(y_1, \ldots, y_{i-1})$ for each i, then $\{y_1, \ldots, y_s\}$ is called a *superficial sequence* for I. The importance of generating a minimal reduction by a superficial sequence for I is that doing so sets up an induction procedure for calculating $\operatorname{depth} G(I)$ (see [HM, Lemma 2.2]).

3. RESULTS

We start with a basic reduction number calculation.

Lemma 3.1. *Let* $(R, \mathfrak{m})$ *be a local ring, and let* $J_0 \subseteq J \subset I \subseteq \overline{J_0}$ *be* $\mathfrak{m}$*-primary ideals of* R *such that* $\lambda(I/J) = 1$*,* $J_0 J = J^2$*, and* $JI = I^2$*. If the generators of* J_0 *form part of a minimal generating set of* I *and* $\mu(I) = \mu(J_0) + 1$*, then* $J_0 I = I^2$*.*

Proof. Write $J = J_0 + \mathfrak{a}$ where $\mathfrak{a}$ is an ideal of R. Using that the generators of J_0 form part of a minimal generating set of I and $\lambda(I/J) = 1$, we may write $I = J_0 + (u) = J + (u)$, where $u\mathfrak{m} \subset J$. Then $I^2 = J_0 I + (u^2)$. By the assumption $JI = I^2$, $u^2 \in JI = (J_0 + \mathfrak{a})(J_0 + (u)) = J_0 I + u\mathfrak{a}$. It follows that $I^2 = J_0 I + u\mathfrak{a}$.

We claim that $\mathfrak{a} \subset J_0 + u\mathfrak{m}$. Let $y \in \mathfrak{a}$ be arbitrary and (using that $\mathfrak{a} \subset I$) write $y = x + \alpha u$, where $x \in J_0$ and $\alpha \in R$. If $\alpha \notin \mathfrak{m}$, then $u \in (y, x) \subset J$, a contradiction because $u \in J$ forces $J = I$. Thus $\alpha \in \mathfrak{m}$ and $y = x + \alpha u \in J_0 + u\mathfrak{m}$, proving the claim.

Using the claim,

$$I^2 = J_0 I + u\mathfrak{a} \subseteq J_0 I + u^2\mathfrak{m} \subseteq J_0 I + \mathfrak{m} I^2.$$

By Nakayama's Lemma, $I^2 = J_0 I$. □

Remark. If J_0 is assumed to be a minimal reduction in Lemma 3.1, then the condition that its generators form part of a minimal generating set of I is automatically satisfied.

As an immediate corollary we obtain conditions for passing the Cohen-Macaulay property of the associated graded ring along adjacent ideals.

Corollary 3.2. *Let* $(R, \mathfrak{m})$ *be a* d*-dimensional Cohen-Macaulay local ring and* J_0 *an ideal generated by a system of parameters of* R*. Let* I *be an* $\mathfrak{m}$*-primary ideal contained in* $\overline{J_0}$ *such that* $\lambda(I/J_0) = k$*, and let*

$$J_0 \subset J_1 \subset \cdots \subset J_{k-1} \subset J_k = I$$

be a saturated chain of ideals. If $\mu(J_i) = d + 1$ *and* $J_{i-1} J_i = J_i^2$ *for* $1 \leq i \leq k$*, then* $J_0 J_i = J_i^2$ *for* $1 \leq i \leq k$*. In particular,* $G(J_i)$ *is Cohen-Macaulay for* $0 \leq i \leq k$*.*

Proof. By the assumptions, J_0 is a minimal reduction of I and is generated by an R-regular sequence. Lemma 3.1 and induction on k show that $J_0 J_i = J_i^2$ for $1 \leq i \leq k$. Each $G(J_i)$, $1 \leq i \leq k$, is therefore Cohen-Macaulay by [VV, Proposition 3.1]. □

Lemma 3.1 is false, even in a regular local ring, if the assumption $\mu(I) = \mu(J_0) + 1$ is removed.

Example 3.3. Let k be a field and $R = k[x, y]_{(x,y)}$. Set $J_0 = (x^4, y^4)$, $J = (x^4, y^4, x^2y^2)$, and $I = (x^4, y^4, x^2y^2, x^3y)$. Then $J_0 J = J^2$ and $JI = I^2$, but $J_0 I \neq I^2$. Both $G(J_0)$ and $G(J)$ are Cohen-Macaulay, but $G(I)$ is not Cohen-Macaulay.

It turns out that $\operatorname{depth} G(I) = 1$ $(= d - 1)$ in Example 3.3, which means that *relatively* high depth is preserved in going from $G(J)$ to $G(I)$, even though the Cohen-Macaulay property does not survive. Our next result provides conditions on J and I (in a Cohen-Macaulay local ring) under which this "almost maximal depth" property persists in $G(I)$. The result covers Example 3.3.

Theorem 3.4. *Let $(R, \mathfrak{m})$ be a d-dimensional Cohen-Macaulay local ring and let J and I be $\mathfrak{m}$-primary ideals of R such that $J \subset I \subset \overline{J}$, $\lambda(I/J) = 1$, $r(J) \leq 1$, and $JI = I^2$.*

(a) *If $\mu(J) \leq d+1$, then* $\operatorname{depth} G(I) \geq d-1$.

(b) *If $\mu(J) \leq d+1$ and $\mu(I) = d+1$, then $G(I)$ is Cohen-Macaulay.*

(c) *If $\mu(J) = d+2$ and $e_2(I) \neq 0$, then* $\operatorname{depth} G(I) \geq d-1$.

Proof. By Remark 2.2 we may assume that $R/\mathfrak{m}$ is infinite. Let J_0 be a common minimal reduction of J and I and choose generators for J_0 that form a superficial sequence for both J and I. $G(J)$ is Cohen-Macaulay, because $r(J) \leq 1$, and therefore the reduction number $r(J)$ is independent of the chosen minimal reduction. It follows that $J_0 J = J^2$. By application of [HM, Lemma 2.2], we may assume that $d = 2$. The length one assumption on I/J implies that $I = J + (w)$ for some $w \in I$ satisfying $w\mathfrak{m} \subset J$.

We first prove (a). Note that $J_0 I^2 = J_0 J I = J^2 I = I^3$. Therefore, if we show that $\lambda(I^2/J_0 I) \leq 1$, then it will follow by [S1, Lemma 2.3] that $\operatorname{depth} G(I) \geq 1$. By assumption, $2 \leq \mu(J) \leq 3$. The case $\mu(J) = 2$ is well-known; J must equal J_0, which forces $G(I)$ to be Cohen-Macaulay.

If $\mu(J) = 3$, then writing $J = J_0 + (u)$ and using the assumptions implies that

$$I^2 = JI = J^2 + wJ = J_0 I + (uw).$$

Further, $uw\mathfrak{m} \subset uJ \subset J^2 = J_0 J \subset J_0 I$. Therefore $\lambda(I^2/J_0 I) \leq 1$, completing the proof of (a).

The statement in (b) follows from Lemma 3.1, because the generators of J_0 form part of a minimal generating set for I.

We now proof (c). Assume that $\mu(J) = 4$ and write $J = J_0 + (u, v)$. By Theorem 2.1, it suffices to show that

$$e_1(I) = \sum_{m \geq 0} \lambda(I^{m+1}/J_0 I^m),$$

and because $J_0 I^2 = I^3$, the sum on the right collapses to $\lambda(I/J_0) + \lambda(I^2/J_0 I)$.

To complete the proof we need to calculate $\lambda(I^2/J_0 I)$. Using the assumptions $JI = I^2$, $J_0 J = J^2$, $J = J_0 + (u, v)$, and $I = J + (w)$, note that $I^2 = J_0 I + (uw, vw)$. Further, $w\mathfrak{m} \subset J$ implies that $(uw, vw)\mathfrak{m} \subset J^2 \subset J_0 I$. It follows that $\lambda(I^2/J_0 I) \leq 2$. If $\lambda(I^2/J_0 I) = 1$, then by applying [S1, Lemma 2.3] again, $\operatorname{depth} G(I) \geq 1$. Therefore, we assume for the rest of the proof that

$$\lambda(I^2/J_0 I) = 2. \tag{3.5}$$

Making use of Huneke's formulas [Hun, 2.7, 2.8] and (3.5),

$$e_1(I) = \lambda(I/J_0) + 2 - \sum_{m \geq 1} \lambda((I^m : J_0)/I^{m-1}, \tag{3.6}$$

and

$$e_2(I) = 2 - \sum_{m \geq 2} m \lambda((I^m : J_0)/I^{m-1}.$$

Recalling that $e_2(I)$ is non-negative [Na] and $e_2(I) \neq 0$ (by assumption),

$$0 < e_2(I) = 2 - \sum_{m \geq 2} m\lambda((I^m : J_0)/I^{m-1}.$$

Hence

$$(I^m : J_0) = I^{m-1} \text{ for } m \geq 1,$$

and therefore

$$e_1(I) = \lambda(I/J_0) + 2 = \sum_{m \geq 0} \lambda(I^{m+1}/J_0 I^m),$$

by (3.6). This completes the proof of (c). □

We can apply Theorem 3.4, as in (3.2), to pass the almost maximal depth property through an appropriate composition series.

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and J_0 an ideal generated by a system of parameters of R. Let I be an $\mathfrak{m}$-primary ideal contained in $\overline{J_0}$ such that $\lambda(I/J_0) = k$, and let*

$$J_0 \subset J_1 \subset \cdots \subset J_{k-1} \subset J_k = I$$

be a saturated chain of ideals. If $\mu(J_i) = d + 1$ for $1 \leq i \leq k - 1$, and $J_{i-1}J_i = J_i^2$ for $1 \leq i \leq k$, then $G(J_i)$ is Cohen-Macaulay for $0 \leq i \leq k - 1$ and $\operatorname{depth} G(I) \geq d - 1$.

Proof. By Corollary 3.2, each $G(J_i)$ is Cohen-Macaulay for $0 \leq i \leq k-1$, and Theorem 3.4(a) implies that $\operatorname{depth} G(I) \geq d - 1$. □

Notice that if k is equal to 2 in the statement of Corollary 3.7, then the assumption $\mu(J_i) = d + 1$, $1 \leq i \leq k - 1$, is automatically satisfied. We record this separately.

Corollary 3.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring and J_0 an ideal generated by a system of parameters of R. Let I be an $\mathfrak{m}$-primary ideal contained in $\overline{J_0}$ such that $\lambda(I/J_0) = 2$, and let*

$$J_0 \subset J_1 \subset J_2 = I$$

be a saturated chain of ideals. If $J_0 J_1 = J_1^2$ and $J_1 I = I^2$, then $\operatorname{depth} G(I) \geq d - 1$.

By using the formulas appearing in [Huc, Corollary 2.11] we can explicitly display the Hilbert-Samuel polynomials of the ideals in the statement of Corollary 3.7.

Corollary 3.9. *Let $(R, \mathfrak{m})$, J_i, and I be as in Corollary 3.7. Then*

$$P_{J_i}(n) = \lambda(R/J_0)\binom{n+d-1}{d} - i\binom{n+d-2}{d-1}$$

for $0 \leq i \leq k - 1$, and

$$P_I(n) = \lambda(R/J_0)\binom{n+d-1}{d} - (k + \lambda(I^2/J_0 I))\binom{n+d-2}{d-1}.$$

Proof. For $0 \leq i \leq k-1$, $J_0 J_i = J_i^2$ by Corollary 3.2, and $J_0 I^2 = I^3$ from the proof of Theorem 3.4(a). The result follows from [Huc, Corollary 2.11] because $\operatorname{depth} G(J_i) \geq d-1$ in all cases and $\lambda(J_i/J_0) = i$. □

We end by considering the case where $(R, \mathfrak{m})$ is a d-dimensional Gorenstein local ring. There is added structure available that we will use to study the first few steps of a saturated chain ascending from a paramater ideal J_0. First of all, note that there is exactly one extenstion ideal J_1 such that $\lambda(J_1/J_0) = 1$, namely $J_1 = (J_0 : \mathfrak{m})$. This holds because any possible J_1 must be contained in $(J_0 : \mathfrak{m})$, and, because R/J_0 is Gorenstein, $\lambda((J_0 : \mathfrak{m})/J_0) = 1$.

Work of Corso, Polini, and Vasconcelos is relevant to studying the first step of a saturated chain. In particular, a special case of [CPV, Theorem 2.1] implies that $(J_0 : \mathfrak{m})$ is contained in $\overline{J}_0$ and has reduction number one (here we assume that if R is regular then $d \geq 2$ and at least two generators of J_0 are contained in $\mathfrak{m}^2$; see [CVP]). Therefore, the first step of a saturated chain produces an ideal J_1 satisfying $J_0 J_1 = J_1^2$. We can use Lemma 3.1 and Theorem 3.4 to push this out a bit further.

Corollary 3.10. *Let $(R, \mathfrak{m})$ be a d-dimensional Gorenstein local ring with $d \geq 1$ and let J_0 be an $\mathfrak{m}$-primary ideal generated by d elements. In case R is regular assume that $d \geq 2$ and at least two generators of J_0 are contained in $\mathfrak{m}^2$. Suppose*

$$J_0 \subset J_1 \subset J_2 \subset J_3 \subset \overline{J}_0$$

are ideals satisfying the conditions $\lambda(J_1/J_0) = \lambda(J_2/J_1) = \lambda(J_3/J_2) = 1$, $J_1 J_2 = J_2^2$, and $J_2 J_3 = J_3^2$. Then the following are true.

(a) $r(J_1) = r(J_2) = 1$*; in particular, $G(J_1)$ and $G(J_2)$ are Cohen-Macaulay.*

(b) $\operatorname{depth} G(J_3) \geq d-1$.

(c) *If $\mu(J_3) = d+1$, then $J_0 J_3 = J_3^2$ and in particular $G(J_3)$ is Cohen-Macaulay.*

Proof. From the discussion above, the ideal J_1 must equal $(J_0 : \mathfrak{m})$, and hence $J_0 J_1 = J_1^2$ by [CPV, Theorem 2.1]. Write $J_2 = J_1 + (u_2)$ where $u_2 \mathfrak{m} \subset J_1$. Since R/J_0 is Gorenstein, every nonzero ideal of R/J_0 contains the (one-dimensional) socle of R/J_0, which equals J_1/J_0. A consequence is that $J_0 + (u_2)$ contains J_1, and hence $J_2 = J_0 + (u_2)$. Therefore $\mu(J_2) = d+1$.

The statements (a), (b), and (c) now follow quickly from Lemma 3.1, Theorem 3.4(a), and Lemma 3.1, respectively. □

The following example illustrates that Corollary 3.10(c) is false without the assumption $\mu(J_3) = d+1$.

Example 3.11. Let k be a field and $R = k[x,y,z]_{(x,y,z)}$. Set

$$J_0 = (x^2, y^2, z^2), J_1 = (x^2, y^2, z^2, xyz), J_2 = (x^2, y^2, z^2, xy), \text{ and } J_3 = (x^2, y^2, z^2, xy, xz).$$

Then

$$\lambda(J_1/J_0) = \lambda(J_2/J_1) = \lambda(J_3/J_2) = 1, J_0 J_1 = J_1^2, J_1 J_2 = J_2^2, \text{ and } J_2 J_3 = J_3^2.$$

However, $J_0 J_3 \neq J_3^2$ and $G(J_3)$ is not Cohen-Macaulay.

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Department of Mathematics, Florida State University, Tallahassee, FL 32306-4510
E-mail address: `huckaba@math.fsu.edu`