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Proposed by Michael Woltermann, Washington & Jefferson College, Washington, PA. A block fountain of coins is an arrangement of \( n \) identical coins in rows such that the coins in the first row form a contiguous block, and each row above that forms a contiguous block. If \( a_n \) denotes the number of block fountains with exactly \( n \) coins in the base, then \( a_n = F_{2n-1} \), where \( F_k \) denotes the \( k \)th Fibonacci number. (Wilf, generatingfunctionology, 1994.)

How many block fountains are there if two fountains that are mirror images of each other are considered to be the same?

Let \( B_n \) denote the set of block fountains that have exactly \( n \) coins in the base and that possess mirror symmetry. We prove \( |B_n| = F_{n+1} \) for \( n \geq 1 \) by strong induction. The case \( n = 1 \) is trivial. Assume \( |B_n| = F_{n+1} \) for \( 1 \leq n < k \). A contiguous block of \( k \) coins is an element of \( B_k \), and centering an element of \( B_j \) atop a contiguous block of \( k \) coins, for \( 1 \leq j < k \) with \( j \) and \( k \) of opposite parity, forms an element of \( B_k \). Conversely, deleting the base from an element of \( B_k \) reveals either the empty block fountain or an element of \( B_j \), for \( j \) as above. It follows that

\[
|B_k| = 1 + \sum_{j=1}^{\lfloor k/2 \rfloor} |B_{k-2j+1}| = 1 + \sum_{j=1}^{\lfloor k/2 \rfloor} F_{k-2(j-1)} = F_{k+1}.
\]

Now the set of block fountains that have exactly \( n \) coins in the base and that do not possess mirror symmetry may be partitioned into sets of mirror pairs. We conclude that \( a_n = |B_n| + \frac{F_{2n-1} - |B_n|}{2} = \frac{F_{n+1} + F_{2n-1}}{2} \).