# EXPONENTIAL MARTINGALES AND TIME INTEGRALS OF BROWNIAN MOTION

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ABSTRACT. We find a simple expression for the probability density of $\int \exp(B_s - s/2)ds$ in terms of its distribution function and the distribution function for the time integral of $\exp(B_s + s/2)$. The relation is obtained with a change of measure argument where expectations over events determined by the time integral are replaced by expectations over the entire probability space. We develop precise information concerning the lower tail probabilities for these random variables as well as for time integrals of geometric Brownian motion with arbitrary constant drift. In particular, $E[\exp(\theta/\int \exp(B_s)ds)]$ is finite iff $\theta < 2$. We present a new formula for the price of an Asian call option.

## 1. INTRODUCTION

Time integrals of one-dimensional geometric Brownian motion have appeared in financial models where certain expected values are the computed prices of 'Asian options'. Approximate values for some of these expectations were obtained in [RS]. Some fundamental work in [Y] and [GY] focuses on the distribution and density functions of time integrals.

We present new relationships between the density functions and distribution functions for these random variables. Our results provide a precise description of the lower tail of these distributions and we settle several moment questions involving an exponent which is the reciprocal of a time integral. We also consider other moments where the exponent includes further exponential terms involving Brownian motion. A surprising result shows that certain of these double exponential moments are indeed finite.

We let $B_t$ denote a one-dimensional Brownian motion. Our notation for a time integral of exponential Brownian motion is somewhat nonstandard. We let $M_t$ denote the simple exponential martingale

$$M_t = \exp(B_t - \frac{t}{2})$$

and we define its time integral as

$$A_t = \int_0^t M_s ds.$$

In [Y] and [BTW] the authors use $A_t$ to denote the time integral of exponential Brownian motion without drift, or an integral with a drift other than $-1/2$. We will obtain some distributional properties of the random variable $A_t$. In particular

we present a formula for its probability density and we obtain a sharp lower tail estimate for its distribution.

Our method of argument uses a delicate change of measure, where the process

$$(1.1) \qquad \tilde{B}_t := B_t - 2\log(1 - \frac{y}{2}A_t)$$

becomes a standard Brownian motion. Here, $y$ is a positive constant. Of course, this path translation has a singularity at the random time $\tau_\infty$ defined by

$$(1.2) \qquad A_{\tau_\infty} = \frac{2}{y}$$

Therefore, we will consider sample paths only for $t$ strictly less than $\tau_\infty$.

## 2. The Change of Measure

We follow the Girsanov formalism: A sufficient condition for a process

$$\tilde{B}_t := B_t - \int_0^t \theta_s ds$$

to be a Brownian motion over a compact time interval $[0, T]$ is that the process

$$(2.1) \qquad \Lambda_t := \exp\left(\int_0^t \theta_s dB - \frac{1}{2}\int_0^t \theta_s^2 ds\right)$$

be a martingale over the time interval (see [KS]). $\tilde{B}_t$ is a standard Brownian motion w.r.t. the measure $Q$ defined by

$$(2.2) \qquad \frac{dQ}{dP} = \Lambda_T$$

In view of equation (1.1), we consider as our choice of $\theta_t$ the process

$$(2.3) \qquad R_t = 2\frac{d}{dt}\log(1 - \frac{y}{2}A_t) = -\frac{M_t}{y^{-1} - \frac{1}{2}A_t}$$

Notice that $R_0 = -y$ and that $R_t$ is defined up to the random time $\tau_\infty$. The process $R_t$ has the following convenient and remarkable property:

**Lemma 2.1.** *The process given in equation (2.3) satisfies the SDE*

$$(2.4) \qquad dR_t = R_t dB_t - \frac{1}{2}R_t^2 dt$$

*for $t < \tau_\infty$.*

*Proof.* This is a simple calculation using the fact that $dM_t = M_t dB_t$.      □

**Remark.** *By writing equation (2.4) in its integral form,*

$$R_t = -y + \int_0^t R_s dB - \frac{1}{2}\int_0^t R_s^2 ds,$$

*we see that $R_t$ is essentially the exponent of the Girsanov density process it gener-ates. This unusual property of $R_t$ allows us to analyze the behavior of $A_t$ through a change of measure.*

**Definition 2.2.** *For each $n = 1, 2, \ldots$ let $\tau_n$ denote the stopping time given by*

$$\tau_n = \inf\{t : R_t \le -n\}$$

Although each stopping time, and $\tau_\infty$ as well, depends on the choice of $y$ , we will omit mentioning their dependence on this parameter unless we explicitly change the value of $y$. We see from equation (1.2) and the fact that $R_t$ is negative that

$$\tau_n < \tau_\infty \quad \text{and also} \quad \lim_{n \to \infty} \tau_n = \tau_\infty \quad \text{a. s.}$$

**Proposition 2.3.** *For each $n = 1, 2, \ldots$ the process*

(2.5)
$$\Lambda_t^{(n)} = \exp[\ y + R_{t \wedge \tau_n}\ ]$$

*forms a martingale. Moreover, the process*

$$B_t - 2\log(1 - \frac{y}{2} A_{t \wedge \tau_n})$$

*is a standard Brownian motion for $t \le T$ w.r.t. the probability measure*

$$dQ = \Lambda_T^{(n)} dP$$

*Proof.* We choose a Girsanov density process as in equation (2.1) by setting

$$\theta_s = R_s 1_{\{s < \tau_n\}}$$

With this choice, $\theta_s$ is a bounded adapted process and hence satisfies a Novikov condition (see Corollary 5.13 of [KS]). The Novikov condition is sufficent for $\Lambda_t$ to be a martingale. In our case,

$$\Lambda_t = \exp\left(\int_0^{t \wedge \tau_n} R_s dB - \frac{1}{2}\int_0^{t \wedge \tau_n} R_s^2 ds\right)$$

It follows from Lemma 2.1 that the exponent above is precisely $R_{t \wedge \tau_n} + y$. That is,

$$\Lambda_t = \exp(y + R_{t \wedge \tau_n})$$

This proves the martingale assertion of the proposition. Moreover, the calcula-tion in equation (2.3) shows that

$$\int_0^t \theta_s ds = 2\log(1 - \frac{y}{2} A_{t \wedge \tau_n})$$

and so the Girsanov theorem implies that the process

$$B_t - 2\log(1 - \frac{y}{2} A_{t \wedge \tau_n})$$

is a standard Brownian motion on compact time intervals with the change of measure given by $\Lambda_T$.

$\square$

## 3. The Correspondence Between $B_t$ and $\tilde{B}_t$

**Definition 3.1.** *For fixed $n = 1, 2, \ldots$ and $y > 0$ we let $\tilde{B}_t$ denote the process appearing in Proposition 2.3. That is*

$$\tilde{B}_t := B_t - 2\log(1 - \frac{y}{2}A_{t \wedge \tau_n}) \tag{3.1}$$

*We also define*

$$\tilde{M}_t = \exp(\tilde{B}_t - t/2)$$

*and*

$$\tilde{A}_t = \int_0^t \tilde{M}_s ds$$

We note that all quantities in this definition depend on our choice for $n$ and $y$.

**Proposition 3.2.**

$$\tilde{M}_t = \frac{M_t}{(1 - \frac{y}{2}A_{t \wedge \tau_n})^2} \tag{3.2}$$

*and*

$$1 + \frac{y}{2}\tilde{A}_{t \wedge \tau_n} = \frac{1}{1 - \frac{y}{2}A_{t \wedge \tau_n}} \tag{3.3}$$

*and*

$$R_{t \wedge \tau_n} = -\frac{\tilde{M}_{t \wedge \tau_n}}{y^{-1} + \frac{1}{2}\tilde{A}_{t \wedge \tau_n}} \tag{3.4}$$

*Proof.* From Definition 3.1 we have

$$\tilde{M}_t = \exp(B_t - t/2 - 2\log(1 - \frac{y}{2}A_{t \wedge \tau_n}))$$
$$= \frac{M_t}{(1 - \frac{y}{2}A_{t \wedge \tau_n})^2}$$

Now if $t < \tau_n$ then

$$\frac{d}{dt}(1 + \frac{y}{2}\tilde{A}_t) = \frac{y}{2}\tilde{M}_t = \frac{yM_t}{2(1 - \frac{y}{2}A_t)^2} = \frac{d}{dt}\left(\frac{1}{1 - \frac{y}{2}A_t}\right)$$

Hence

$$1 + \frac{y}{2}\tilde{A}_t = \frac{1}{1 - \frac{y}{2}A_t}$$

Finally, for $t$ in this same range,

$$R_t = -\frac{M_t}{y^{-1} - \frac{1}{2}A_t}$$

$$= -\frac{\tilde{M}_t(1 - \frac{y}{2}A_t)^2}{y^{-1} - \frac{1}{2}A_t}$$

$$= -y\tilde{M}_t(1 - \frac{y}{2}A_t)$$

$$= -\frac{y\tilde{M}_t}{1 + \frac{y}{2}\tilde{A}_t}$$

These equalities hold up to the time $\tau_n$, and these are the assertions (3.3) and (3.4) in the proposition. $\qquad\square$

**Proposition 3.3.** *If $f(x,z)$ is a nonnegative Borel-measurable function and $y > 0$ then*

(3.5) $$E[f(M_t, R_t) \; ; \; A_t < 2/y]$$

$$= E[f(\frac{M_t}{(1 + \frac{y}{2}A_t)^2}, \frac{-M_t}{y^{-1} + \frac{1}{2}A_t}) \exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t} - y)]$$

*Proof.* For fixed $n$ and $y > 0$ we consider

$$E[f(M_t, R_t) \; ; \; \tau_n > t]$$

Now

$$f(M_t, R_t)1_{\{\tau_n > t\}} = f(M_t, R_t)\exp(-R_t - y)\exp(R_t + y)1_{\{\tau_n > t\}}$$

Since this function vanishes for $t \geq \tau_n$, each time parameter may be replaced by $t \wedge \tau_n$. This allows us to apply Proposition 2.3 where we take

$$\Lambda_t^{(n)}1_{\{\tau_n > t\}} = \exp(R_t + y)1_{\{\tau_n > t\}}$$

We obtain the identity

(3.6) $$E[f(M_t, R_t) \; ; \; \tau_n > t] = E_Q[f(M_t, R_t)\exp(-R_t - y)1_{\{\tau_n > t\}}]$$

Each term in the r.h. expected value can be expressed in terms of the Brownian motion $\tilde{B}_t$. We use the identities in Proposition 3.2 to see that

$$f(M_t, R_t) \exp(-R_t - y) 1_{\{\tau_n > t\}} =$$

$$f(\tilde{M}_t (1 - \frac{y}{2} A_t)^2, -\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t}) \exp(\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t} - y) 1_{\{\tau_n > t\}}$$

$$= f(\frac{\tilde{M}_t}{(1 + \frac{y}{2}\tilde{A}_t)^2}, -\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t}) \exp(\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t} - y) 1_{\{\tau_n > t\}}$$

Moreover, the event $\tau_n > t$ equals the event

$$\min_{s \le t} R_s > -n$$

which in turn equals the event

$$\max_{s \le t} \frac{\tilde{M}_s}{y^{-1} + \frac{1}{2}\tilde{A}_s} < n$$

The r.h. expected value in equation (3.6) is then

$$E_Q[f(\frac{\tilde{M}_t}{(1 + \frac{y}{2}\tilde{A}_t)^2}, -\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t}) \exp(\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t} - y) 1_{\{\max_{s \le t} \frac{\tilde{M}_s}{y^{-1} + \frac{1}{2}\tilde{A}_s} < n\}}]$$

But since the integrand is nonnegative we may take the limit as $n \to \infty$ and obtain the limiting value

$$E_Q[f(\frac{\tilde{M}_t}{(1 + \frac{y}{2}\tilde{A}_t)^2}, -\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t}) \exp(\frac{\tilde{M}_t}{y^{-1} + \frac{1}{2}\tilde{A}_t} - y)]$$

In addition, since

$$\lim_{n \to \infty} \tau_n = \tau_\infty$$

as $\tau_\infty$ is defined in equation (1.2) we see that the limit of the l.h. side of equation (3.6) is

$$E[f(M_t, R_t) \; ; \; A_t < 2/y]$$

This establishes the identity (3.5) of the proposition.

$\square$

**Remark.** *Proposition 3.3 is quite similar to Theorem 1 of* [WH]. *The authors study Girsanov density processes and develop necessary and sufficient conditions for a Girsanov process to be a martingale. Theorem 1 shows that, in great generality, the expected value of a Girsanov density equals the tail probability for a certain stopping time.*

*Our arguments proving our proposition are similar to those in* [WH]. *In our case, we extend their result to include expected values of a function of the Brownian motion process multiplied by a Girsanov density. The choice $f(x, z) \equiv 1$ is a special case of the theorem in* [WH]. *The reader may see this by making the choice*

$$X(t) = \frac{2M_t}{a + A_t}$$

*as required by Theorem 1. To define the correct stopping time, one should consider the process*

$$\int_0^t Y(u)du := -2\log(1 - \frac{1}{a}A_t)$$

*One may verify directly that $Y(t)$ satisfies the functional equation mentioned in Proposition 1 of* [WH].

## 4. THE DISTRIBUTION OF $A_t$

**Theorem 4.1.** *For $a > 0$ the distribution of $A_t$ is given by*

$$(4.1) \qquad \Pr\{A_t \le a\} = e^{-\frac{2}{a}} E[\exp\left(\frac{2M_t}{a + A_t}\right)]$$

*Moreover, the random variable $A_t$ has a continuous, positive probability density function $g_t(a)$ which is simply related to the distribution functions of $A_t$ and $\frac{A_t}{M_t}$:*

$$(4.2) \qquad g_t(a) = \frac{2}{a^2} \Pr\{A_t \le a\} - \frac{2}{a^2} \Pr\{\frac{A_t}{M_t} \le a\}$$

*Proof.* The first assertion of the Theorem follows from Proposition 3.3 by making the simple choice

$$f(x, z) \equiv 1$$

Identity (3.5) becomes in this case

$$\Pr\{A_t < 2/y\} = E[\exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t} - y)]$$

$$= E[\exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t})]e^{-y}$$

This identity has the rather surprising corollary that *the expected value above is finite.* The integrand involves a *double exponential of Browian motion.* Notice that the integrand is a monotone function of $y$.

If $y$ varies over some positive interval

$$(y_0, y_1)$$

then each integrand is dominated by the integrable random variable

$$\exp(\frac{M_t}{y_1^{-1} + \frac{1}{2}A_t})$$

If a sequence $\{y_k\}$ converges to $\tilde{y}$ in this interval, we apply the dominated convergence theorem to prove that the expected value converges to its value for $\tilde{y}$. Hence, the expected value is a continuous function of $y$. It follows that the distribution function of $A_t$ is continuous and we may write the identity as

$$(4.3) \qquad \Pr\{A_t \le 2/y\} = E[\exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t})]e^{-y}$$

This establishes equation (4.1).

We show the probability density exists by proving that the right hand expression in equation (4.1) is differentiable w.r.t. $a$. To see this we consider the case of Proposition 3.3 for

$$f(x, z) = x$$

Identity (3.5) becomes

$$E[M_t; A_t \le 2/y] = E[\frac{M_t}{(1 + \frac{y}{2}A_t)^2} \exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t})]e^{-y}$$

$$= y^{-2}E[\frac{M_t}{(y^{-1} + \frac{1}{2}A_t)^2} \exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t})]e^{-y}$$

As in the previous case, the expected value

$$E[\frac{M_t}{(y^{-1} + \frac{1}{2}A_t)^2} \exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t})]$$

is necessarily finite and again the integrand is monotone in $y$. Therefore, the expression is a continuous function of $y$ as we argued in proving (4.1). However, this same expected value arises by formally differentiating the expected value

$$(4.4) \qquad E[\exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t})]$$

on the r.h. side of (4.1) w.r.t. $y$. Now, as $y$ varies over some positive interval $(y_0, y_1)$ each integrand

$$y^{-2}\frac{M_t}{(y^{-1} + \frac{1}{2}A_t)^2} \exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t})$$

is dominated by the integrable random variable

$$y_0^{-2}\frac{M_t}{(y_1^{-1} + \frac{1}{2}A_t)^2} \exp(\frac{M_t}{y_1^{-1} + \frac{1}{2}A_t})$$

So, the $y-$integral from $y_0$ to $y_1$, which equals

$$\exp(\frac{M_t}{y_1^{-1} + \frac{1}{2}A_t}) - \exp(\frac{M_t}{y_0^{-1} + \frac{1}{2}A_t})$$

is dominated by the product of $y_1 - y_0$ and an integrable function. Therefore, the difference quotient for the expected value in (4.4) converges as $y_0 \to y_1$ and the limit is the entire expression for

$$(4.5) \qquad -E[M_t; A_t \le 2/y]e^y$$

The same argument applies to the case $y_1 \to y_0$ so that expression (4.4) has a derivative which equals the expression (4.5). Since the distribution function is the product of $e^{-y}$ and expression (4.4) we conclude that the distribution function of $A_t$ has a continuous probability density. We differentiate terms in the identity (4.3) to obtain

(4.6) $$-2y^{-2}g_t(2/y) = -\Pr\{A_t \le 2/y\} + E[M_t; A_t \le 2/y]$$

Now, the expected value

$$E[M_t; A_t \le 2/y]$$

is another distribution function. Using the change of measure induced by the factor $M_t$, we see that

$$B_s - s = W_t$$

is a standard Brownian motion and so the expected value equals

$$\Pr\{\int_0^t \exp(W_s + s - s/2)ds \le 2/y\}$$

We substitute this expression into equation (4.6) to obtain

$$g_t(a) = \frac{2}{a^2}\left(\Pr\{A_t \le a\} - \Pr\{\int_0^t \exp(B_s + s/2)ds \le a\}\right)$$

We see that the expression for $g_t(a)$ is strictly positive since

$$\int_0^t \exp(B_s + s/2)ds$$

is strictly larger than the random variable $A_t$ for each sample path. Finally we note that the random variable $\frac{A_t}{M_t}$ has the same distribution as the time integral above.

$$\frac{A_t}{M_t} = \int_0^t \exp(B_s - B_t + t/2 - s/2)ds$$

has the same distribution as

$$\int_0^t \exp(W_{t-s} + t/2 - s/2)ds$$

where $W_s$ denotes a standard Brownian motion; we may change variables in the time integral to obtain the time integral of geometric Brownian motion with positive drift. This establishes assertion (4.2) of the theorem.

$$\square$$

**Remark.** *We can rewrite the density formula (4.2) by combining the two probabilities. The density equals*

(4.7) $$g_t(a) = \frac{2}{a^2}\Pr\{\int_0^t \exp(B_s - s/2)ds \le a < \int_0^t \exp(B_s + s/2)ds\}$$

*This expresses the probability density for $A_t$ in terms of a single condition on Brownian motion sample paths up to time t.*

*The existence of a continuous probability density for $A_t$ is a corollary of Proposition 2 of Yor [Y]. In the proposition a conditional density for $A_t$ is given as an integral transform of various transcendental functions. No explicit connection is made between the density and the distributions of $A_t$ and $A_t/M_t$.*

*In Dufresne [D] nice formulas are obtained for the density of a reciprocal of a time integral for some values of a drift parameter in the Brownian motion. Our*

*choice corresponds to the choice of $\mu = -1$, and the density is given as an integral transform in* [D].

**Remark.** *The density $g_t(a)$ is a solution of the PDE derived in [BTW, equation (26)] where it is shown that time integrals of more general geometric Brownian motions have smooth densities. In particular, the random variable*

$$\frac{A_t}{M_t}$$

*has a smooth density since it has the same distribution as the time integral of $\exp(B_s + s/2)$. The PDE, equation (26), has a simple form in the case of the density of $A_t$:*

$$\frac{\partial g}{\partial t} = \frac{\partial^2}{\partial a^2}\{\frac{a^2 g}{2}\} - \frac{\partial g}{\partial a}$$

*But, equation (4.2) shows that $a^2 g/2$ is the difference of two distribution functions. Therefore, the PDE becomes*

$$\frac{\partial g}{\partial t} = \frac{\partial}{\partial a}\{g - \frac{\partial}{\partial a}\Pr\{\frac{A_t}{M_t} \leq a\}\} - \frac{\partial g}{\partial a}$$

$$= -\frac{\partial^2}{\partial a^2}\Pr\{\frac{A_t}{M_t} \leq a\}$$

*That is, the time derivative of the density is obtained by differentiating the density of $A_t/M_t$.*

**Corollary 4.2.** *For each $y > 0$ the process*

$$Z_t = \exp\left(\frac{M_t}{y^{-1} + \frac{1}{2}A_t}\right)$$

*is a supermartingale. In particular,*

(4.8)                    $$E[Z_t] = e^y \Pr\{A_t \leq 2/y\}$$

*so that $E[Z_t]$ is a strictly decreasing function of $t$.*

*Proof.* The process

$$Y_t = \frac{M_t}{y^{-1} + \frac{1}{2}A_t}$$

also satisfies the SDE that appears in Lemma 2.1. A simple calculation shows that

(4.9)                    $$dY_t = Y_t dB_t - \frac{1}{2}Y_t^2 dt$$

Therefore, the remark following Lemma 2.1 applies to the process $Y_t$:

Since

$$Y_t - y = \int_0^t Y_s dB - \frac{1}{2}\int_0^t Y_s^2 ds,$$

$Y_t - y$ is the exponent of the Girsanov density process which $Y_t$ generates. A simple stopping time argument, similar to one in the proof of Proposition 2.3, shows that

$$Z_t = \exp(Y_t)$$

is a positive local martingale. However, the integral identity (4.1) of the Theorem implies that

$$E[\exp(Y_t)]$$

is decaying function of $t$. Consequently, $Z_t$ is a local martingale but it is not a martingale. $\qquad\square$

**Remark.** *The process $Y_t$ appears implicitly in Lemma 2.1 of* [BTW]. *In order to derive PDE's for certain expected values involving $A_t$, the authors compute a diffusion equation for processes slightly more general than*

$$(Y_t^{-1}, M_t).$$

*The PDE identities do not apply to equation (4.8) since $Z_t$ is not a homogeneous function of $Y_t^{-1}$.*

*    One can derive the corollary from Theorem 1 of* [WH], *but the argument here connects the result directly to the behavior of time integrals of geometric Brownian motion.*


**Corollary 4.3.**

$$(4.10) \qquad\qquad E[\exp\big(\frac{2}{A_t}\big)] = \infty$$


*Proof.* For each $a > 0$

$$E[\exp\big(\frac{2}{A_t}\big) \; ; A_t \le a] \ge \exp(2/a)\Pr\{A_t \le a\}$$

and equation (4.1) of the theorem implies that the r.h. expression equals

$$E[\exp\big(\frac{2M_t}{a + A_t}\big)]$$

But, this quantity increases as $a \to 0$ and therefore the random variable

$$\exp\big(\frac{2}{A_t}\big)$$

is not integrable. $\qquad\square$

## 5. Geometric Brownian Motion with Drift

**Definition 5.1.** *For each $\nu \in \mathbb{R}$ we let $A_t^{(\nu)}$ denote the time integral*

$$(5.1) \qquad A_t^{(\nu)} = \int_0^t \exp(B_s + \nu s - s/2) ds$$

The random variable $A_t$ in the previous sections is $A_t^{(0)}$ with this notation.

**Theorem 5.2.** *For $a > 0$ the distribution of $A_t^{(\nu)}$ is given by*

$$(5.2) \qquad \Pr\{A_t^{(\nu)} \leq a\} = a^{2\nu} e^{-\frac{2}{a}} E[(a + A_t^{(\nu)})^{-2\nu} \exp\big(\frac{2\exp(B_t + \nu t - t/2)}{a + A_t^{(\nu)}}\big)]$$

*Proof.* For $y > 0$ we apply Proposition 3.3 to evaluate

$$E[(M_t)^\nu \ ; \ A_t \leq 2/y]$$

The choice of $f(x, z) = x^\nu$ in the proposition gives the expected value

$$E[\frac{M_t^\nu}{(1 + \frac{y}{2} A_t)^{2\nu}} \exp(\frac{M_t}{y^{-1} + \frac{1}{2} A_t} - y)]$$

Next, we multiply these expected values by $\exp(\nu t/2 - \nu^2 t/2)$ so that

$$M_t^\nu \exp(\nu t/2 - \nu^2 t/2) = \exp(\nu B_t - \nu^2 t/2)$$

We use this exponential martingale factor to change measure in each integral so that the process

$$\tilde{B}_s = B_s - \nu s$$

is a standard Brownian motion for $s \leq t$. We see that

$$E[M_t^\nu \exp(\nu t/2 - \nu^2 t/2) \ ; \ A_t \leq 2/y] = \Pr\{A_t^{(\nu)} \leq 2/y\}$$

while the other expected value equals

$$E[(1 + \frac{y}{2} A_t^{(\nu)})^{-2\nu} \exp(\frac{\exp(B_t + \nu t - t/2)}{y^{-1} + \frac{1}{2} A_t^{(\nu)}} - y)]$$

This establishes identity (5.2). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Corollary 5.3.** *As $a \downarrow 0$ the function*

$$\frac{\exp(2/a)}{a} \Pr\{A_t^{(1/2)} \leq a\}$$

*increases.*

*Proof.* For the case that $\nu = 1/2$, the formula in (5.2) for the distribution function becomes

$$(5.3) \qquad \Pr\{A_t^{(1/2)} \le a\} = ae^{-\frac{2}{a}} E[\frac{1}{a + A_t^{(1/2)}} \exp\big(\frac{2\exp(B_t)}{a + A_t^{(1/2)}}\big)]$$

The integrand of the expected value in (5.3) increases as $a \downarrow 0$.

$\square$

**Corollary 5.4.** *The following expected value is infinite.*

$$(5.4) \qquad E[\exp\big(\frac{2}{\int_0^t \exp(B_s)ds}\big)] = \infty$$

*Proof.* The corollary states that $E[\exp\big(\frac{2}{A_t^{(1/2)}}\big)] = \infty$. Let $F(a)$ denote the distribution function of $A_t^{(1/2)}$ and consider

$$E[\exp\big(\frac{2}{A_t^{(1/2)}}\big) \; ; \; A_t^{(1/2)} \le 1]$$

We use a standard argument that justifies integration by parts:

$$e^{\frac{2}{a}} = \int_a^\infty \frac{2}{x^2} e^{\frac{2}{x}} dx$$

so that

$$E[\exp\big(\frac{2}{A_t^{(1/2)}}\big) \; ; \; A_t^{(1/2)} \le 1] = \int_0^1 \int_a^\infty \frac{2}{x^2} e^{\frac{2}{x}} dx dF(a)$$

$$= \int_0^\infty \frac{2}{x^2} e^{\frac{2}{x}} \int_0^{x \wedge 1} dF(a) dx$$

$$= \int_0^\infty \frac{2}{x^2} e^{\frac{2}{x}} F(x \wedge 1) dx$$

But, Corollary 5.3 implies that

$$\frac{1}{x^2} e^{\frac{2}{x}} F(x) \ge \frac{k}{x}$$

for $x < 1$ where the constant $k > 0$. Hence, the integral is infinite.

$\square$

## 6. Finite Exponential Moments

**Lemma 6.1.** *For any $t > 0$*

(6.1)
$$E[\exp\big(\frac{1}{2\int_0^t \exp(B_s)ds}\big)] < \infty$$

*Proof.* The expected value in (6.1) is

$$E[\exp\big(\frac{1}{2A_t^{(1/2)}}\big)]$$

with the notation of Section 5. And, an upper bound for the expected value is

$$e + \int_e^\infty \Pr\{\exp\big(\frac{1}{2A_t^{(1/2)}}\big) \geq x\}dx$$

Let $x = e^s$ to obtain the following form of the integral above:

(6.2)
$$\int_1^\infty \Pr\{\frac{1}{2A_t^{(1/2)}} \geq s\}e^s ds$$

$$= \int_1^\infty \Pr\{\frac{1}{2s} \geq A_t^{(1/2)}\}e^s ds$$

It suffices to prove that the integral from $k$ to $\infty$ is finite where $k$ is chosen so that on the interval of integration

$$\frac{4}{s} \leq t$$

For any $s$ in the interval we have

$$A_t^{(1/2)} = \int_0^t \exp(B_u)du \geq \int_0^{4/s} \exp(B_u)du$$

$$= \frac{4}{s}\cdot\frac{s}{4}\int_0^{4/s} \exp(B_u)du \geq \frac{4}{s}\exp\big(\frac{s}{4}\int_0^{4/s} B_u\big)du$$

by Jensen's inequality. Since the random variable

$$\frac{s}{4}\int_0^{4/s} B_u du$$

is normal with standard deviation

$$\frac{2}{\sqrt{3s}}$$

we may replace it with

$$\frac{2}{\sqrt{3s}}Z$$

where $Z$ denotes a standard normal random variable. The integrand in equation (6.2) is dominated by

$$\Pr\{\frac{1}{2s} \geq \frac{4}{s}\exp\big(\frac{2}{\sqrt{3s}}Z\big)\}e^s$$

$$= \Pr\{\frac{1}{8} \geq \exp\big(\frac{2}{\sqrt{3s}}Z\big)\}e^s$$

$$< \Pr\{-2 \geq \frac{2}{\sqrt{3s}} Z\} e^s$$

since $\log(1/8) < -2$. We may write this as

$$\Pr\{-\sqrt{3s} > Z\} e^s$$

Hence the integral in equation (6.2),

$$\int_1^\infty \Pr\{ \frac{1}{2s} \geq A_t^{(1/2)} \} e^s ds,$$

is dominated by

$$e^k + \int_k^\infty \Pr\{-\sqrt{3s} > Z\} e^s ds$$

$$\approx e^k + \int_k^\infty \exp(-\frac{3s}{2}) e^s ds < \infty$$

$\square$

**Theorem 6.2.** *For any $\theta < 2$ and $t_0 > 0$ the process*

$$\exp\left(\frac{\theta M_t}{A_t}\right)$$

*is a supermartingale for $t \geq t_0$. In particular,*

(6.3) $$E[\exp\left(\frac{\theta M_t}{A_t}\right)] < \infty$$

*and the expected value is a strictly decreasing function of $t$.*

*Proof.* We have seen (proof of Theorem 4.1) that $A_t/M_t$ has the same distribution as

$$\int_0^t \exp(B_s + s/2) ds$$

Since this random variable is larger than $A_t^{(1/2)}$, Lemma 6.1 implies that

$$E[\exp\left(\frac{M_t}{2A_t}\right)] < \infty$$

Now let

(6.4) $$U_t = \frac{e^{ct}}{4} \frac{M_t}{A_t}$$

for any fixed $c > 0$. If $t_0 > 0$ is sufficiently small, so that $e^{ct_0}/4 \leq 1/2$, we have

$$E[\exp(U_{t_0})] < \infty$$

We define $t_1$ by

$$e^{ct_1}/4 = \theta$$

and we claim that
$$E[\exp(U_{t_1})] < \infty$$
The integrabliity of $\exp\left(\frac{\theta M_t}{A_t}\right)$ will follow because the choice of $c$ is arbitrary. We first consider the SDE for $U_t$:

$$dU_t = cU_t dt + U_t dB_t - \frac{e^{ct}}{4}\frac{M_t}{A_t^2}M_t dt$$

$$= U_t dB_t + cU_t dt - \frac{4}{e^{ct}}U_t^2 dt$$

$$= U_t dB_t + U_t\{c - 4e^{-ct}U_t\}dt$$

Next, we compute the SDE for the process $\exp(U_t)$.

$$d\exp(U_t) = U_t \exp(U_t)dB_t + \exp(U_t)U_t\{c - 4e^{-ct}U_t\}dt + \frac{1}{2}\exp(U_t)U_t^2 dt$$

In order to compute an expected value, we introduce the stopping times

$$\tau_n = \inf\{t \ge t_0 : U_t \ge n\}$$

and we obtain

$$E[\exp(U_{\tau_n \wedge t_1})] - E[\exp(U_{t_0})]$$

$$= E[\int_{t_0}^{\tau_n \wedge t_1} \exp(U_t)U_t\{c + \frac{1}{2}U_t - 4e^{-ct}U_t\}\,dt]$$

$$\le E[\int_{t_0}^{\tau_n \wedge t_1} \exp(U_t)U_t\{c + \frac{1}{2}U_t - 4e^{-ct_1}U_t\}\,dt]$$

(6.5) $$= E[\int_{t_0}^{\tau_n \wedge t_1} \exp(U_t)U_t\{c + \frac{1}{2}U_t - \theta^{-1}U_t\}\,dt]$$

Notice that if
$$U_t \ge b$$
where
$$b := c(\theta^{-1} - \frac{1}{2})^{-1}$$
then the integrand in equation (6.5) is negative. It follows that

$$E[\exp(U_{\tau_n \wedge t_1})] - E[\exp(U_{t_0})]$$

$$\le E[\int_{t_0}^{\tau_n \wedge t_1} \exp(U_t)U_t\{c + \frac{1}{2}U_t - \theta^{-1}U_t\}1_{\{U_t \le b\}}\,dt]$$

Now we take the limit as $n \to \infty$ and, using the condition that $U_t \le b$ in the integrand, we see that

$$E[\exp(U_{t_1})] < \infty$$

This establishes that the expected value in equation (6.3) is finite. It is a strictly decreasing function of $t$ because the random variable $\frac{A_t}{M_t}$ has the same distribution as $A_t^{(1)}$ which is a strictly increasing function of $t$. It remains to establish the supermartingale property. Corollary 4.2 implies that for any $\alpha < 1$ the process

$$\left\{\exp\left(\frac{2M_t}{a+A_t}\right)\right\}^\alpha$$

is a supermartingale because it is a concave function of a supermartingale. Now as $a \to 0$ the pointwise limit of this process is

$$\left\{\exp\left(\frac{2M_t}{A_t}\right)\right\}^\alpha = \exp\left(\frac{2\alpha M_t}{A_t}\right)$$

That is, the process in the statement of the theorem is integrable and is the limit of non-negative supermartingales. Hence, it is also a supermartingale. $\qquad\square$

**Remark.** *It follows from Theorem 5.2 that the distribution function of $\frac{M_t}{A_t}$ has the form*

$$a^2 \exp(-2/a)K_a$$

*where the function $K_a$ increases as $a \downarrow 0$. So, if $\theta > 2$ in equation (6.3), the expected value is infinite. It is unclear for the case $\theta = 2$ if the expected value is finite.*

**Corollary 6.3.** *For any $\theta < 2$ and $t > 0$*

(6.6) $$E\left[\exp\left(\frac{\theta}{\int_0^t \exp(B_s)ds}\right)\right] < \infty$$

*Proof.* Since the expected value in (6.6) is a decreasing function of $t$, it suffices to show the expected value is finite for $t$ arbitrarily small. Since $\frac{M_t}{A_t}$ has the same distribution as

$$\int_0^t \exp(B_s + s/2)ds,$$

Theorem 6.2 implies that for any $\tilde\theta < 2$

$$E\left[\exp\left(\frac{\tilde\theta}{\int_0^t \exp(B_s + s/2)ds}\right)\right] < \infty$$

For a given $\theta < 2$, choose $t$ so small that

$$\tilde\theta := \theta \exp\left(\frac{t}{2}\right) < 2$$

so that

$$E\left[\exp\left(\frac{\theta}{e^{-t/2}\int_0^t \exp(B_s + s/2)ds}\right)\right] < \infty$$

We see that the expected value above is larger than

$$E\left[\exp\left(\frac{\theta}{\int_0^t \exp(B_s)ds}\right)\right]$$

and so this expected value is finite. $\qquad\square$

**Remark.** *Some related work on exponential moments of $A_t$ is mentioned in Yor* [Y]. *Equation (1.e) of the article states that*

$$E\left[ \frac{1}{\sqrt{A_t}} \exp\left(-\frac{u^2}{2A_t}\right) \right] = \frac{1}{\sqrt{(1+u^2)t}} \exp\left(-\frac{1}{2t}(\sinh^{-1} u)^2\right)$$

*However, a difference in notation requires that $A_t$ be expressed with our notation as $\frac{1}{4}A_{t/4}^{(1/2)}$. By writing $t' = t/4$ we see that the formula in Equation (1.e) is an expression for*

$$E\left[ \frac{2}{\sqrt{A_{t'}^{(1/2)}}} \exp\left(-\frac{2u^2}{A_{t'}^{(1/2)}}\right) \right]$$

*Equation (6.6) implies that the expected value is finite for all complex values of $u$ such that $Re(u^2) > -1$ and is analytic on this region. Since*

$$\sinh^{-1} u = \log(u + \sqrt{1 + u^2})$$

*one sees that the right hand expression also has an analytic extension. The formula has a singularity at the value $u = i$ which corresponds to the infinite exponential moment of Corollary 5.4.*

*In addition, Theorem 4.1 of* [D] *provides an integral formula for the density of $A_t^{(1/2)}$ and one can show that the expected value is finite (for $\theta < 2$) using Theorem 4.1. The corollary provides a probabilistic proof of this fact. The result that the moment is infinite for $\theta = 2$ is new.*

An expected value considered in [Y], [RS], and in other works concerning time integrals in financial mathematics is the 'price' for an Asian call option. We present a new expected value for the simplest option price and indicate how to derive a corresponding formula for arbitrary constant drift and volatility values.

**Proposition 6.4.** *For any $a > 0$*

$$(6.7) \qquad E[\ (A_t - a)^+\ ] = t - a + a^2 E[(a + A_t)^{-1} \exp\left(\frac{2M_t}{a + A_t} - \frac{2}{a}\right)]$$

*Proof.* The notation $(A_t - a)^+$ involves the indicator function of the event $A_t \geq a$ so it is natural to consider

$$E[\ A_t - 2/y\ ;\ A_t < 2/y\ ]$$
$$= -\frac{2}{y} E[\ 1 - \frac{y}{2} A_t\ ;\ A_t < 2/y\ ]$$

We apply Proposition 3.3 where we make the choice

$$f(x, z) = \frac{x}{-z}$$

Then

$$-2f(M_t, R_t) = -2M_t \frac{y^{-1} - \frac{1}{2}A_t}{M_t}$$
$$= -\frac{2}{y}\left(1 - \frac{y}{2} A_t\right)$$

Equation (3.5) shows that the expected value over $A_t < 2/y$ is given by

$$-\frac{2}{y}E[(1 + \frac{y}{2}A_t)^{-1}\exp(\frac{M_t}{y^{-1} + \frac{1}{2}A_t} - y)]$$

On the other hand,

$$E[\ A_t - \frac{2}{y}\ ] = t - \frac{2}{y}$$

so we take the difference of these expected values and let $\frac{2}{y} = a$ to obtain the result. $\qquad\square$

**Remark.** *The proposition requires the Brownian motion to have drift $-1/2$. To derive an expectation formula for arbitrary drift one can apply Proposition 3.3 to the expected value*

$$E[(M_t)^{\nu}(\ A_t - 2/y)\ ;\ A_t < 2/y\ ]$$

*The factor of $(M_t)^{\nu}$ is relevant for a change of measure (see the proof of Theorem 5.2) so that the time integral will contain an arbitrary drift. To incorporate a volatility factor in the Brownian motion, one can make a simple time scale change. We let $s = \sigma^2 s'$ so that*

$$\frac{1}{\sigma^2}\int_0^t \exp(B_s)ds = \int_0^{t'} \exp(B_{\sigma^2 s'})ds'$$

*This random variable has the same distribution as the time integral of $\exp(\sigma B_s)$.*

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