HOMEWORK 1 **DIFFERENTIAL TOPOLOGY** **MTG 5932-02**

due Wednesday, September 10

Note: Problems 1-3 are taken from Lang's book *Analysis I*, p. 339.

1. (Attributed by Lang to John Tate.) Suppose a map $f : R^n \to R^m$ is continuous and has the following property: There exists a real number $C > 0$ such that for all $x, y \in R^n$,

$$\|f(x+y) - f(x) - f(y)\| \leq C.$$

Prove that there exists a unique linear map $g : R^n \to R^m$ such that $g - f$ is bounded in the sup norm. (Hint: Show that for all x,

$$g(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$$

exists – to do this, show that the sequence is Cauchy.)

2. Let $\bar{B}_r$ denote the closed ball of radius r in R^n centered at 0. Suppose $f : \bar{B}_r \to R^n$ satisfies:
(a) $\|f(x) - f(y)\| \leq b\|x - y\|$, with $0 < b < 1$.
(b) $\|f(0)\| \leq r(1 - b)$.
Show that there exists a unique $x \in \bar{B}_r$ such that $f(x) = x$.

3. Assume the situation of the previous exercise. In addition, let c be a positive real number, and let $g : \bar{B}_r \to R^n$ be a map satisfying $\|g(x) - f(x)\| \leq c$ for all $x \in \bar{B}_r$. Assume g has a fixed point x_2, and let x_1 be the fixed point of f (whose existence you proved in the previous exercise). Prove that $\|x_2 - x_1\| \leq c/(1 - b)$.

4. Define $f : L(R^n, R^n) \to L(L(R^n, R^n), L(R^n, R^n))$ by $f(u) = \tau_u$, where $\tau_u(v) = u \circ v$. Prove that f is continuous with respect to the operator norm defined in class.