HOMEWORK 3 **DIFFERENTIAL TOPOLOGY** **MTG 5932-02**

due Monday, September 29

1. (a) Prove that 0 is a regular value of the function $f : R^2 \to R$ defined by

$$f(x, y) = y^2 - x(x + 1)(x + 2).$$

Conclude that the curve $y^2 = x(x + 1)(x + 2)$ in R^2 is a smooth 1-dimensional manifold. Sketch this curve.

(b) Is 0 a regular value of the function $g : R^2 \to R$ defined by $g(x, y) = y^2 - x^2(x + 1)$? Why or why not? Sketch the curve $y^2 = x^2(x + 1)$.

NOTE: (a) is an example of a "nonsingular elliptic curve", while (b) is a "singular elliptic curve".

2. Let $f : M \to N$ denote a smooth map between manifolds and suppose $g : M \to M$ is a diffeomorphism which satisfies $f \circ g = f$. Given a point $x \in M$, prove that x is a regular point of f if and only if $g(x)$ is a regular point of f.

For the following problems, define the following terms: $M_{n\times n}(R)$ denotes the set of $n \times n$ matrices with real entries. A' denotes the transpose of a square matrix A.

3. Define $O(n, R) = \{A \in M_{n\times n}(R) : A'A = I_n\}$. Prove $O(n, R)$ is a smooth manifold, and find its dimension. Hint: Let $S_n(R)$ denote the set of $n \times n$ symmetric matrices. Define $f : M_{n\times n}(R) \to S_n(R)$ by $f(A) = A'A$. First, prove that I is a regular point of f. To do this, I would recommend calculating $df_I(V)$ for an arbitrary matrix V. Then show directly that df_I is onto. Now let A be an arbitrary element of $O(n, R)$. Show that $L_A : M_{n\times n}(R) \to M_{n\times n}(R)$ defined by $L_A(X) = AX$ is a diffeomorphism. Now use the result of 2.

NOTE: The result of 3. gives a space that is a manifold and a group at the same time (and in which the functions $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth). Such spaces are called *Lie groups* and are extremely important in geometry and topology.

4. As many of you know, a *covering map* is a continuous surjective map $f : X \to Y$, where X and Y are topological spaces, with the following property: For each $y \in Y$, there exists an open neighborhood V of y in Y such that

$$f^{-1}(V) = \bigcup_{\alpha\in A} U_\alpha,$$

where the sets U_α are pairwise disjoint open sets, and for each $\alpha \in A$, the restriction $f : U_\alpha \to V$ is a homeomorphism. Prove that if M and N are smooth manifolds of the same dimension, and M is compact and nonempty, and N is connected, and $f : M \to N$ is a smooth map for which every $x \in M$ is a regular point, then $f : M \to N$ is a covering map. Give a counterexample showing that you cannot drop the hypothesis that M is compact.