HOMEWORK 4 **DIFFERENTIAL TOPOLOGY** **MTG 5932-02**

due Monday, October 6

In this problem set, I will always let $D^n = \{(x_1, \ldots, x_n) \in R^n : x_1^2 + \ldots + x_n^2 \leq 1\}$ denote the closed n-dimensional disc.

1. Define a function $\lambda : R \to R$ by

$$\lambda(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ e^{-1/t} & \text{for } t > 0 \end{cases}$$

Prove that λ is C^∞ at 0. (Clearly it is at every other point.)

2. Prove that if $X \subset R^k$ is a manifold with boundary and $Y \subset R^l$ is a manifold without boundary, then $X \times Y \subset R^{k+l}$ is a manifold whose boundary is equal to $(\partial X) \times Y$.

3. As an important example of 2., consider the "solid torus" $S^1 \times D^2$. What is its boundary? Find a nice diffeomorphism of $S^1 \times D^2$ with a subset of R^3, and draw me a picture of this subset. By the way, in $\partial(S^1 \times D^2)$, a curve of the form $\{*\} \times \partial D^2$ is called a "meridian," while a curve of the form $S^1 \times \{*\}$, with $* \in \partial D^2$, is called a longitude. Draw these curves in the picture you just drew of a solid torus.

Consider the three-dimensional sphere $S^3 = \{(x, y, z, t) \in R^4 : x^2 + y^2 + z^2 + t^2 = 1\}$. Define $D_+ = \{(x, y, z, t) \in S^3 : t \geq 0\}$ and $D_- = \{(x, y, z, t) \in S^3 : t \leq 0\}$. It's easy to see that D_+ and D_- are each diffeomorphic to the closed three dimensional disc D^3, and that they intersect along their common boundary which is diffeomorphic to S^2.

4. We now give another nice way to decompose S^3 into two pieces. Redefine S^3 as $\{(z, w) \in C^2 : |z|^2 + |w|^2 = 1\}$. Note that if we identify C^2 with R^4, this is the same definition as above. Define $T_+ = \{(z, w) \in S^3 : |z|^2 \geq 1/2\}$ and $T_- = \{(z, w) \in S^3 : |z|^2 \leq 1/2\}$. Prove that T_+ and T_- are each diffeomorphic to $S^1 \times D^2$, and that they intersect along their common boundary. Consider a meridian and a longitude in ∂T_+. To which curves in ∂T_- are they being identified?

5. There is a beautiful map $g : S^3 \to S^2$ called the "Hopf map". It can be defined as follows. First, let $K = S^3 \cap (\{0\} \times C)$ and let $L = S^3 \cap (C \times \{0\})$. Define $f_1 : S^3 - K \to C$ by $f_1(z, w) = w/z$, and define $f_2 : S^3 - L \to C$ by $f_2(z, w) = \bar{z}/\bar{w}$.
Now, define $g : S^3 \to S^2$ by

$$g(z, w) = \begin{cases} h_+^{-1} \circ f_1, & \text{for } (z, w) \in S^3 - K \\ h_-^{-1} \circ f_2, & \text{for } (z, w) \in S^3 - L \end{cases}$$

where h_+ and h_- are the stereographic projections introduced in class (and in Milnor). Verify for yourself that this is well-defined and that g is smooth (you don't have to write this down). Prove every point in S^2 is a regular value of g. Calculate the 1-manifolds $g^{-1}(y)$, for $y = (1, 0, 0), (0, 1, 0)$, and $(0, 0, -1)$. Note that these manifolds all lie in T_+ (or maybe T_- if I've made a mistake!). Draw these three 1-manifolds in the solid torus you drew in 3.