HOMEWORK 5 **DIFFERENTIAL TOPOLOGY** **MTG 5932-02**

due Wednesday, October 15, 2008

1. Show that if X is any topological space, and if two continuous maps $f, g : X \to S^n$ have the property that for all $x \in X$, $\|f(x) - g(x)\| < 2$, then f is homotopic to g. In addition, if X is a manifold and f and g are smooth, then they are smoothly homotopic.

2. If M is a manifold and $\dim M < n$, show that every smooth map $f : M \to S^n$ is smoothly homotopic to a constant map.

3. Define the antipodal map $A : S^n \to S^n$ by $A(x) = -x$. Prove that if $f : S^n \to S^n$ is a smooth map with no fixed points, then f is smoothly homotopic to A. From this, deduce that if $f : S^n \to S^n$ has no fixed points, then $\deg_2(f) = 1$.

4. If X and Y are connected manifolds without boundary of the same dimension, X is compact and Y is not compact, and $f : X \to Y$ is smooth, prove that $\deg_2(f) = 0$.

5. Suppose that V and W are finite dimensional oriented vector spaces, and let $L : V \to W$ be a surjective linear transformation. Let $K = \ker(L)$. We wish to use the given information to determine an orientation of K. Here's how: Let $(e_1, \ldots, e_k)$ be an ordered basis for K. To determine whether this ordered basis represents the correct orientation, do the following test: First, extend it to a positively oriented basis $(e_1, \ldots, e_k, f_1, \ldots, f_l)$ of V (why is this possible)? Then consider the ordered basis $(L(f_1), \ldots, L(f_l))$ for W. Define $(e_1, \ldots, e_k)$ to be a positively oriented basis for K if and only if $(L(f_1), \ldots, L(f_l))$ is a positively oriented basis for W. Show that this procedure determines a well-defined orientation for K. To do this, you should show (1) that the answer you get is independent of the extension you made and (2) if you do this test on two different bases of K which are positively related (i.e., their change of basis matrix has positive determinant), you will get the same answer.