HOMEWORK **Group Representation Theory** **MTG 5932-01**

Note: The notation (**D**) next to a problem means it's already been presented, so you can no longer choose that problem for a presentation.

1.(**D**) Define a function $\rho : Z \to \mathrm{GL}(C^2)$, where Z denotes the integers and C denotes the complex numbers, by

$$\rho(n) = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$$

(a) Verify that ρ is a representation.

(b) Show that C^2 contains a non-trivial subspace which is invariant under ρ, i.e., a nontrivial subrepresentation.

(c) Show that the subspace found in (b) does not have any complement in C^2 which is invariant under ρ. This example shows that Theorem 1 in Section 1.3 of Serre need not hold if G is infinite.

2.(**D**) Let Z_3 denote the cyclic group with three elements, and let $\rho : Z_3 \to \mathrm{GL}(V)$ denote its regular representation. Recall that in class we identified a 1-dimensional subrepresentation W of V. (In fact, we did this for the regular representation of an arbitrary finite group.) Find a stable complement W_0 of W. Is the restriction of ρ to W_0 irreducible? Continue this process (if necessary!) to find the decomposition of ρ into a direct sum of irreducibles. Give an orthonormal basis of each of these irreducibles, and give an explicit matrix form of each irreducible in terms of these bases. Hint: it might help to think about eigenvalues and eigenvectors of ρ_s for a nontrivial element $s \in Z_3$.

3.(**D**) Prove that every irreducible representation of Z_n is 1-dimensional. How many different (i.e., non-isomorphic) irreducible representations does Z_n have?

4. In problem 3, you found all the different irreducible representations of Z_n. Verify directly that the characters of these representations comprise an orthonormal basis for the set W of all functions $Z_n \to C$, where the Hermitian inner product on W is given by

$$(\alpha|\beta) = \frac{1}{|G|} \sum_{s \in G} \alpha(s)\overline{\beta(s)}.$$

The following problems from Serre: (A * next to a problem means that because the problem is longer than usual, two people may collaborate on it, and both get credit.)

p. 12: 2.1(**D**), 2.2(**D**), 2.3(**D**), 2.4(**D**)
p. 17: 2.6*
p. 18: 2.7(**D**)
p. 26: 3.1(**D**), 3.2*(**D**)
p. 31: 3.4(**D**), 3.5, 3.6(**D**)
p. 49: 6.3*
p. 50: 6.4, 6.5
p. 53: 6.7, 6.8(**D**), 6.9, 6.10
p. 57: 7.2, 7.3
p. 63: 8.1
p. 65: 8.3, 8.4, 8.5
p. 66: 8.7, 8.8, 8.9