Elementary Topology

Note: This problem list was written primarily by Phil Bowers and John Bryant. It has been edited by a few others along the way.

Definition. A *topology* on a set X is a collection $\mathcal{T}$ of subsets of X that satisfies the following three properties:

(i) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$,

(ii) $\mathcal{T}$ is closed under finite intersections; that is, if $U_1, \dots, U_n \in \mathcal{T}$, then

$$\bigcap\{U_i : 1 \leq i \leq n\} \in \mathcal{T},$$

(iii) $\mathcal{T}$ is closed under arbitrary unions; that is, if $U_\gamma \in \mathcal{T}$ for all $\gamma \in \Gamma$, then

$$\bigcup\{U_\gamma : \gamma \in \Gamma\} \in \mathcal{T}.$$

The pair $(X, \mathcal{T})$ is called a *topological space*, or a *space*. Elements of $\mathcal{T}$ are called *open sets* in X (more precisely, open sets in $(X, \mathcal{T})$ or open sets in the topology $\mathcal{T}$).

Given a set X, $\mathcal{T}_I = \{\emptyset, X\}$ is called the *indiscrete topology* on X and $\mathcal{T}_D = \mathcal{P}(X)$ (or 2^X), the set of all subsets of X, is called the *discrete topology* on X.

Problem List

1. Let X be a set and let $\{A_\gamma : \gamma \in \Gamma\}$ be an indexed collection of subsets of X (that is, $A_\gamma \subset X$ for every $\gamma \in \Gamma$). Then

   (i) $X - \bigcap_{\gamma\in\Gamma} A_\gamma = \bigcup_{\gamma\in\Gamma}(X - A_\gamma)$.

   (ii) $X - \bigcup_{\gamma\in\Gamma} A_\gamma = \bigcap_{\gamma\in\Gamma}(X - A_\gamma)$

2. Let $\{A_1, A_2, A_3, \dots\}$ be a countable collection of subsets of a set X. If each A_i is countable then $\bigcup_{i=1}^\infty A_i$ is countable.

3. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. Suppose that for each $x \in A$, there is an open set U such that $x \in U \subset A$. Then A is open in X.

4. Let $\mathcal{C}$ be any family of subsets of a set X. Then there is a unique, smallest topology $\mathcal{T}(\mathcal{C})$ on X with $\mathcal{C} \subset \mathcal{T}(\mathcal{C})$.

5. *Cofinite topology*. Let X be any set and let $\mathcal{T} = \{A \subset X : X - A \text{ is finite or } A = \emptyset\}$. Then $\mathcal{T}$ is a topology on X.

6. *Cocountable topology*. Let X by any set and let $\mathcal{T} = \{A \subset X : X - A \text{ is countable or } A = \emptyset\}$. Then $\mathcal{T}$ is a topology on X.

7. *Euclidean Topology on the real line*. Let $\mathbb{R}$ be the real line and let $U \subset \mathbb{R}$ be open provided that for each $x \in U$, there is an $\varepsilon > 0$ such that $N_\varepsilon(x) = \{y \in \mathbb{R} : |x - y| < \varepsilon\} \subset U$. Show that this describes a topology on $\mathbb{R}$.

8. *Euclidean Topology on n-dimensional euclidean space.* Let $\mathbb{R}^n$ consist of all ordered n-tuples of real numbers. Given

$$x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \text{ let } \|x\| = \left[\sum_{i=1}^{n} x_i^2\right]^{1/2}.$$

A subset $U \subset \mathbb{R}^n$ is open provided that for each $x \in U$, there is an $\varepsilon > 0$ such that $N_\varepsilon(x) = \{y \in \mathbb{R}^n : \|x - y\| < \varepsilon\} \subset U$. Show that this describes a topology on $\mathbb{R}^n$.

9. Let $\mathcal{T}_X$ and $\mathcal{T}_Y$ be topologies on X and Y, respectively. Is $\mathcal{T} = \{A \times B : A \in \mathcal{T}_X, B \in \mathcal{T}_Y\}$ a topology on $X \times Y$?

10. $\mathcal{T}$ is the discrete topology on X iff (if, and only if) every point in X is an open set. [When no confusion arises, we make no distinction between x and $\{x\}$.]

Definition. Let $(X, \mathcal{T})$ be a topological space. A subcollection $\mathcal{B} \subset \mathcal{T}$ is a *basis* for $\mathcal{T}$ if each open set is the union of members of $\mathcal{B}$.

11. Let $(X, \mathcal{T})$ be a topological space. A family $\mathcal{B} \subset \mathcal{T}$ is basis for $\mathcal{T}$ iff, for each $U \in \mathcal{T}$ and $x \in U$, there is a $B \in \mathcal{B}$ with $x \in B \subset U$.

12. Let $\mathcal{B} \subset \mathcal{T}$ be a basis for $\mathcal{T}$. A set $U \subset X$ is open iff, for each $x \in U$, there is a $B \in \mathcal{B}$ with $x \in B \subset U$.

13. Let $\mathcal{B}$ be a family of subsets of a set X that forms a cover of X (i.e., $X = \cup\{B : B \in \mathcal{B}\}$) and suppose that for each pair B, $B' \in \mathcal{B}$ and each $x \in B \cap B'$, there exists a $B'' \in \mathcal{B}$ with $x \in B'' \subset B \cap B'$. Then $\mathcal{B}$ is a basis for a unique topology $\mathcal{T}(\mathcal{B})$ on X.

Definition. Two bases $\mathcal{B}$ and $\mathcal{B}'$ in X are *equivalent* if $\mathcal{T}(\mathcal{B}) = \mathcal{T}(\mathcal{B}')$.

14. Two bases $\mathcal{B}$, $\mathcal{B}'$ in X are equivalent iff both of the following hold:

    (i) for each $B \in \mathcal{B}$ and $x \in B$, there is a $B' \in \mathcal{B}'$ with $x \in B' \subset B$,

    (ii) for each $B' \in \mathcal{B}'$ and $x \in B'$, there is a $B \in \mathcal{B}$ with $x \in B \subset B'$.

15. Let $\mathcal{T}$ be the euclidean topology on $\mathbb{R}^2$. Let

$$\mathcal{B}_1 = \{N_\varepsilon(x) : x \in \mathbb{R}^2 \text{ and } \varepsilon > 0\}$$

and

$$\mathcal{B}_2 = \{N_\varepsilon(x) : x \in \mathbb{R}^2,\ \varepsilon > 0, \text{ and } \varepsilon \text{ is rational}\}.$$

Then both $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for $\mathcal{T}$.

16. Consider the following collections of subsets of $\mathbb{R}$:

$$\mathcal{B}_1 = \{(a, b) : a < b\}\ ;\ \ \mathcal{B}_2 = \{[a, b) : a < b\}.$$

Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are bases for topologies $\mathcal{T}(\mathcal{B}_1)$ and $\mathcal{T}(\mathcal{B}_2)$, respectively, on $\mathbb{R}$. Are $\mathcal{T}(\mathcal{B}_1)$ and $\mathcal{T}(\mathcal{B}_2)$ equal?

We call $\mathcal{T}(\mathcal{B}_1)$ the *standard topology* on $\mathbb{R}$. We call $\mathcal{T}(\mathcal{B}_2)$ the *lower limit topology* on $\mathbb{R}$, and we write $\mathbb{R}_\ell$ for $\mathbb{R}$ with the lower limit topology.

17. Is the euclidean topology on $\mathbb{R}$ the same as the standard topology?

Definition. Let $(X, \mathcal{T})$ be a topological space. A subcollection $\mathcal{C} \subset \mathcal{T}$ is a *subbasis* for $\mathcal{T}$ provided $\mathcal{T} = \mathcal{T}(\mathcal{C})$.

18. The collection of open rays in $\mathbb{R}$ is a subbasis for the standard topology on $\mathbb{R}$.

19. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. Let $\mathcal{B} = \{U \times V \subset X \times Y : U \in \mathcal{T}, V \in \mathcal{S}\}$. Then $\mathcal{B}$ is a basis for a (necessarily) unique topology $\mathcal{T}(\mathcal{B})$ on $X \times Y$.

Definition. $\mathcal{T}(\mathcal{B})$ is called the *product topology* on $X \times Y$.

Definition. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be defined by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$. These are called the *projection mappings*.

20. Let $(X, \mathcal{T})$ and $(Y, \mathcal{S})$ be topological spaces. Let $\mathcal{C} = \{\pi_1^{-1}(U) : U \in \mathcal{T}\} \cup \{\pi_2^{-1}(V) : V \in \mathcal{S}\}$. Then $\mathcal{T}(\mathcal{C})$ is the product topology on $X \times Y$; i.e., $\mathcal{C}$ is a subbasis for the product topology on $X \times Y$.

Definition. Let $(X, \mathcal{T})$ be a topological space and let Y be a subset of X. The *subspace (relative, induced) topology* on Y is $\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$. The space $(Y, \mathcal{T}_Y)$ is called a *subspace* of $(X, \mathcal{T})$. Sometimes, we suppress explicit mention of the topologies and say that Y is a subspace of X.

21. Let $(Y, \mathcal{T}_Y)$ be a subspace of $(X, \mathcal{T})$.

    (i) If $\mathcal{B}$ is a basis for $\mathcal{T}$, then $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$.

    (ii) If $\mathcal{C}$ is a subbasis for $\mathcal{T}$, then $\mathcal{C}_Y = \{C \cap Y : C \in \mathcal{C}\}$ is a subbasis for $\mathcal{T}_Y$.

22. Let $\mathbb{R}$ be the reals with the standard topology and $\mathbb{R}_\ell$ the reals with the lower limit topology.

    (i) Draw pictures in the plane that represent basic open sets in $\mathbb{R} \times \mathbb{R}$, in $\mathbb{R}_\ell \times \mathbb{R}$, and in $\mathbb{R}_\ell \times \mathbb{R}_\ell$.

    (ii) Let L be a straight line in the plane. Describe the topology that L inherits as a subspace of $\mathbb{R} \times \mathbb{R}$, of $\mathbb{R}_\ell \times \mathbb{R}$, and of $\mathbb{R}_\ell \times \mathbb{R}_\ell$. [Be careful; you might get different topologies depending on which line L you pick.]

Definition. Let $(X, \mathcal{T})$ be a topological space. A subset $A \subset X$ is *closed* provided $X - A$ is open; that is, provided $X - A \in \mathcal{T}$.

23. Let $(X, \mathcal{T})$ be a topological space.

    (i) $\emptyset$ and X are closed sets.

    (ii) The union of a finite collection of closed sets is closed.

    (iii) The intersection of an arbitrary collection of closed sets is closed.

Definition. Let $(X, \mathcal{T})$ be a topological space. A *neighborhood* (nbhd) of a point $x \in X$ is any open set containing x.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. A point $x \in X$ is a *limit point* of A provided, for each nbhd U of x, $U \cap (A - \{x\}) \neq \emptyset$. The set of all limit points of A in X is called the *derived set* of A and is denoted by A'. The *closure* of A in X, denoted by $\mathrm{Cl}(A)$ or $\mathrm{Cl}_X(A)$, is the set $A \cup A'$: $\mathrm{Cl}(A) = A \cup A'$.

24. Given $A \subset X$ and $x \in X$, $x \in \mathrm{Cl}(A)$ iff for every nbhd U of x, $U \cap A \neq \emptyset$.

25. Let $(X, \mathcal{T})$ be a topological space and suppose $A, B \subset X$.

    (i) If $A \subset B$, then $A' \subset B'$.

(ii) $(A \cap B)' \subset A' \cap B'$.

(iii) $(A \cup B)' = A' \cup B'$.

26. Let $(X, \mathcal{T})$ be a topological space and suppose $A, B \subset X$.

(i) $A \subset \mathrm{Cl}(A)$ and $\mathrm{Cl}(A)$ is a closed set.

(ii) A is closed iff $A = \mathrm{Cl}(A)$.

(iii) $\mathrm{Cl}(A)$ is the smallest closed set containing A.

(iv) If $A \subset B$, then $\mathrm{Cl}(A) \subset \mathrm{Cl}(B)$.

(v) $\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$.

(vi) $\mathrm{Cl}(A \cup B) = \mathrm{Cl}(A) \cup \mathrm{Cl}(B)$.

(vii) $\mathrm{Cl}(A \cap B) \subset \mathrm{Cl}(A) \cap \mathrm{Cl}(B)$.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. A point $x \in A$ is an *interior point* of A in X provided there is a nbhd N of x with $x \in N \subset A$. The *interior* of A, denoted by $\mathrm{Int}(A)$ or $\mathrm{Int}_X(A)$, is the set of all interior points of A in X.

27. Let $(X, \mathcal{T})$ be a topological space and suppose $A, B \subset X$.

(i) $\mathrm{Int}(A) \subset A$ and $\mathrm{Int}(A)$ is an open set.

(ii) A is open iff $A = \mathrm{Int}(A)$.

(iii) $\mathrm{Int}(A)$ is the largest open set contained in A.

(iv) If $A \subset B$, then $\mathrm{Int}(A) \subset \mathrm{Int}(B)$.

(v) $\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$.

(vi) $\mathrm{Int}(A \cup B) \supset \mathrm{Int}(A) \cup \mathrm{Int}(B)$.

(vii) $\mathrm{Int}(A \cap B) = \mathrm{Int}(A) \cap \mathrm{Int}(B)$.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $A \subset X$. A point $x \in X$ is a *boundary point* of A provided each nbhd of x meets both A and $X - A$ (i.e., for every nbhd N if x, $N \cap A \neq \emptyset$ and $N \cap (X - A) \neq \emptyset$). The set of all boundary points of A in X is called the *boundary* of A in X and is denoted by $\mathrm{Bd}(A)$ or $\mathrm{Bd}_X(A)$.

28. Let $(X, \mathcal{T})$ be a topological space and let $A, B \subset X$.

(i) $\mathrm{Bd}(A) = \mathrm{Cl}(A) - \mathrm{Int}(A)$.

(ii) $\mathrm{Bd}(A)$ is closed.

(iii) $\mathrm{Cl}(A) = \mathrm{Int}(A) \cup \mathrm{Bd}(A)$.

What can you say about the relationship between $\mathrm{Bd}(A \cap B)$ and $\mathrm{Bd}(A) \cap \mathrm{Bd}(B)$; between $\mathrm{Bd}(A \cup B)$ and $\mathrm{Bd}(A) \cup \mathrm{Bd}(B)$?

Terminology. Whenever it is possible to do so without creating confusion we shall henceforth refer to a "topological space X" or a "space X", meaning a set X with an underlying topology $\mathcal{T}$. A subset A of X is "open" ("closed") provided $A \in \mathcal{T}$ ($(X - A) \in \mathcal{T}$). Likewise, we may refer to a "basis" (or "subbasis") for X or a "basic open set" in X, meaning an underlying subset $\mathcal{B}$(or $\mathcal{C}$) of $\mathcal{T}$ that forms a basis (or subbasis) for $\mathcal{T}$ or one of its members.

Definition. Let X be a topological space. A subset $D \subset X$ is *dense* in X provided $\mathrm{Cl}(D) = X$.

29. The following are equivalent:

    (i) D is dense in X.

    (ii) If F is closed and $D \subset F$, then $F = X$.

    (iii) Each nonempty basic open set meets D.

    (iv) Each nonempty open set meets D.

    (v) $\mathrm{Int}(X - D) = \emptyset$.

30. Find a countable dense subset of $\mathbb{R}$ (with the standard topology), of $\mathbb{R}^n$.

31. Suppose a subset D of a space X meets every nonempty member of a subbasis. Is D necessarily dense in X?

32. Let Y be a subspace of a topological space X.

    (i) $A \subset Y$ is closed in Y iff $A = F \cap Y$ for some closed subset F of X.

    (ii) If $A \subset Y$, then $\mathrm{Cl}_Y(A) = Y \cap \mathrm{Cl}_X(A)$.

    (iii) If $A \subset Y$, then $Y \cap \mathrm{Int}_X(A) \subset \mathrm{Int}_Y(A)$.

    (iv) If $A \subset Y$, then $\mathrm{Bd}_Y(A) \subset Y \cap \mathrm{Bd}_X(A)$.

33. (i) If D is dense in X, is $D \cap Y$ dense in Y?

    (ii) Show that equality does not necessarily hold in 32. (iii) and (iv).

Terminology. Let $f: X \to Y$ be a function. There are induced functions $f: \mathcal{P}(X) \to \mathcal{P}(Y)$ and $f^{-1}: \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by $f(A) = \{f(a): a \in A\}$ for $A \subset X$, and $f^{-1}(B) = \{x \in X: f(x) \in B\}$ for $B \subset Y$.

34. Given a function $f: X \to Y$ and a family $\{B_\gamma\}_{\gamma \in \Gamma}$ of subsets of Y, then

    (i) $f^{-1}(\cap_{\gamma \in \Gamma} B_\gamma) = \cap_{\gamma \in \Gamma} f^{-1}(B_\gamma)$.

    (ii) $f^{-1}(\cup_{\gamma \in \Gamma} B_\gamma) = \cup_{\gamma \in \Gamma} f^{-1}(B_\gamma)$.

    (iii) $f^{-1}(Y - B) = X - f^{-1}(B)$ for $B \subset Y$.

35. Given a function $f: X \to Y$ and a family $\{A_\gamma\}_{\gamma \in \Gamma}$ of subsets of X, establish the appropriate analogs of 34. (i)-(iii).

Definition. Suppose that X and Y are topological spaces. A function $f: X \to Y$ is *continuous at a point* $x \in X$ provided for each nbhd V of $f(x)$ there is a nbhd U of x with $f(U) \subset V$. The function $f: X \to Y$ is *continuous* if it is continuous at each $x \in X$.

36. Given a function $f: X \to Y$, the following are equivalent:

    (i) f is continuous.

    (ii) $f^{-1}(V)$ is open for each open set V in Y.

    (iii) $f^{-1}(C)$ is closed for each closed set C in Y.

Definition. If $f: X \to Y$ and $A \subset X$, the *restriction of f to A* is the function $f|_A: A \to Y$ defined by $f|_A(a) = f(a)$ for all $a \in A$.

37. (i) If $f\colon X \to Y$ and $g\colon Y \to Z$ are continuous, then $g \circ f\colon X \to Z$ is continuous.

(ii) If $f\colon X \to Y$ is continuous and $A \subset X$ is a subspace, then $f|_A\colon A \to Y$ is continuous.

(iii) The projection mappings $\pi_1\colon X \times Y \to X$ and $\pi_2\colon X \times Y \to Y$ are continuous.

38. Suppose $f\colon X \to Y$ and suppose $\mathcal{B}$ (respectively, $\mathcal{C}$) is a basis (respectively, subbasis) for the topology on Y.

(i) f is continuous iff $f^{-1}(B)$ is open for each $B \in \mathcal{B}$.

(ii) f is continuous iff $f^{-1}(C)$ is open for each $C \in \mathcal{C}$.

39. $f\colon X \to \mathbb{R}$ is continuous iff for each real number b, $f^{-1}((-\infty, b))$ and $f^{-1}((b, \infty))$ are open.

A.

(i) Let X be a topological space, and define the *diagonal map* $\Delta : X \to X \times X$ by $\Delta(x) = (x, x)$. Prove that Δ is continuous.

(ii) If X, Y, Z, and S are sets and $f : X \to Y$ and $g : Z \to S$ are functions, define

$$f \times g : X \times Z \to Y \times S$$

by $(f \times g)(x, z) = (f(x), g(z))$. Prove that if X, Y, Z, and S are all topological spaces and f and g are continuous, then $f \times g$ is continuous.

B.

(i) Define $A : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $A(x, y) = x + y$. Prove that A is continuous.

(ii) Define $M : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $M(x, y) = xy$. Prove that M is continuous.

(iii) Define $I : \mathbb{R} - \{0\} \to \mathbb{R}$ by $I(x) = 1/x$. Prove that I is continuous.

40. Suppose $f, g\colon X \to \mathbb{R}$ are continuous.

(i) $|f|^a$ is continuous for each $a \geq 0$.

(ii) $af + bg$ is continuous for all real numbers a and b.

(iii) $f \cdot g$ is continuous.

(iv) $1/f$ is continuous on $\{x \in X\colon f(x) \neq 0\}$.

[All operations are pointwise.]

Definition. A function $f\colon X \to Y$ is *open* if $f(U)$ is open for each open set U; f is *closed* if $f(F)$ is closed for each closed set F.

41. Give examples that show that continuous, open, and closed functions are "independent" concepts.

42. Suppose $f\colon X \to Y$ is closed. For any subset $S \subset Y$ and open set U containing $f^{-1}(S)$, there is an open set V containing S with $f^{-1}(V) \subset U$.

43. $f\colon X \to Y$ is open iff $f(B)$ is open for each basic open set B.

44. $f\colon X \to Y$ is open iff $f(C)$ is open for each subbasic open set C.

Definition. A bijection $f\colon X \to Y$ is called a *homeomorphism* provided both f and $f^{-1}\colon Y \to X$ are continuous. X and Y are said to be *homeomorphic*, and we denote this relationship by $X \approx Y$.

45. Given a bijection $f\colon X \to Y$, the following are equivalent.

(i) f is homeomorphism.

(ii) f is continuous and open.

(iii) f is continuous and closed.

(iv) f induces a bijection from $\mathcal{T}_X$ to $\mathcal{T}_Y$.

46. If $f\colon X \to Y$ is a homeomorphism and $A \subset X$, then $f|_A\colon A \to f(A)$ is a homeomorphism.

Definition. $f\colon X \to Y$ is an *embedding* if $f\colon X \to f(X)$ is a homeomorphism.

47. Define $i_1\colon \mathbb{R} \to \mathbb{R}^2$ by $i_1(x) = (x, 0)$, $d\colon \mathbb{R} \to \mathbb{R}^2$ by $d(x) = (x, x)$, and $e\colon [0, 1) \to \mathbb{R}^2$ by $e(x) = (\cos 2\pi x, \sin 2\pi x)$. Which of these are embeddings?

Definition. For $y \in Y$, let $i_y\colon X \to X \times Y$ be defined by $i_y(x) = (x, y)$ and let $S_y = X \times \{y\} \subset X \times Y$. The function i_y is the *inclusion of X over y* and S_y is the *slice of X through y*.

48. The function i_y is an embedding with image S_y.

Definition. A collection $\mathcal{A}$ of subsets of a space X is *locally finite* if each point of X has a nbhd which meets only finitely many $A \in \mathcal{A}$. A collection $\mathcal{A}$ is a *cover* of X provided $X = \cup\mathcal{A}$.

49. Suppose $\mathcal{A}$ is a locally finite collection of closed subsets of a space X. Then $\cup\mathcal{A}$ is a closed subset of X.

50. Suppose $\mathcal{A}$ is a locally finite cover of X by closed sets and that, for each $A \in \mathcal{A}$, $f_A\colon A \to Y$ is continuous and $f_A = f_B$ in $A \cap B$ for each $A, B \in \mathcal{A}$. Then there is a (unique) continuous function $f\colon X \to Y$ with $f = f_A$ on A for each $A \in \mathcal{A}$.

Definition. Two subsets H and K of a space X are said to be *separated*, written $H|K$, provided $H \neq \emptyset$, $K \neq \emptyset$, $\mathrm{Cl}(H) \cap K = \emptyset$, and $H \cap \mathrm{Cl}(K) = \emptyset$.

Definition. A space X is *connected* if it is **not** the union of two separated subsets. A subset $A \subset X$ is connected if it is connected as a subspace of X.

51. Suppose that $H|K$ and $A \subset H \cup K$. If A is connected, then either $A \subset H$ or $A \subset K$.

52. A subset A of $\mathbb{R}$ is connected iff A has the following property: if $a, b \in A$, $c \in \mathbb{R}$, and $a < c < b$, then $c \in A$. Describe the connected subsets of $\mathbb{R}$.

53. The following are equivalent.

(i) X is connected.

(ii) The only subsets of X that are both open and closed are X and $\emptyset$.

(iii) X is not the union of two non-empty, disjoint open sets.

(iv) If $f\colon X \to \{0, 1\}$ is continuous, where $\{0, 1\}$ has the discrete topology, then f is not onto.

54. If X is connected and if $f\colon X \to Y$ is continuous, then $f(X)$ is a connected subset of Y.

55. Suppose $\mathcal{A}$ is a collection of connected subspaces of a space X such that $\cap\mathcal{A} \neq \emptyset$. Then $\cup\mathcal{A}$ is connected.

56. Suppose $\mathcal{A}$ is a collection of connected subspaces of a space X such that for all $A, B \in \mathcal{A}$, A and B are not separated. Then $\cup\mathcal{A}$ is connected.

57. Suppose that A is a connected subset of a space X. If $A \subset B \subset \text{Cl}(A)$, then B is connected. In particular, the closure of a connected set is connected.

58. Show that the following subsets of $\mathbb{R}^2$ are connected.

    (i) The union of the x-axis and the y-axis.

    (ii) The graph of $f : \mathbb{R} \to \mathbb{R}$, where f is an arbitrary continuous function.

    (iii) $\{(x, y) : 0 < x \leq 1 \text{ and } y = \sin(1/x)\} \cup (\{0\} \times [-1, 1])$. This diabolical set is often called the "topologist's sine curve."

59. If X and Y are connected, then so is $X \times Y$.

60. **Intermediate Value Theorem**. Suppose X is a connected space, $f\colon X \to \mathbb{R}$ is continuous, and $f(a) < r < f(b)$ for some points $a, b \in X$. Then there exists $x \in X$ such that $f(x) = r$.

61. X is connected iff every open covering $\mathcal{U}$ of X has the following property: for each pair of non-empty sets $U, V \in \mathcal{U}$ there are finitely many sets $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $U \cap U_1 \neq \emptyset$, $U_i \cap U_{i+1} \neq \emptyset$ $(i = 1, 2, \ldots, n-1)$, and $U_n \cap V \neq \emptyset$.

Definition. Given points x and y in a space X, a *path* from x to y in X is a continuous function $f\colon [a, b] \to X$ of some closed interval in $\mathbb{R}$ into X such that $f(a) = x$ and $f(b) = y$. A space X is *path connected* if every pair of points in X can be joined by a path in X.

62. If $n \geq 2$, then $\mathbb{R}^n - \{0\}$ is path connected.

63. If $n \geq 2$, then $\mathbb{R}$ is not homomorphic to $\mathbb{R}^n$.

64. Each path connected space is connected, but a connected space need not be path connected.

65. If X and Y are path connected, then so is $X \times Y$.

66. A connected open subset of $\mathbb{R}^n$ is path connected.

Separation Axioms

Suppose X is a topological space.

T_0: X is a *T_0-space* if for $x \neq y$ there is an open set U containing one of x or y, but not the other.

T_1: X is a *T_1-space* if for $x \neq y$ there is an open set U containing x, but not y.

T_2: X is a *T_2-space* (or a *Hausdorff space*) if for every $X \neq Y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

Definition. A space X is *regular* if for every $x \in X$ and every closed set $F \subset X$ not containing x, there are disjoint open sets U and V such that $x \in U$ and $F \subset V$.

Definition. A space X is *normal* if for every pair F, G of disjoint closed sets in X there are disjoint open sets U and V such that $F \subset U$ and $G \subset V$.

T_3: X is a *T_3-space* if X is T_1 and regular.

T_4: X is a *T_4-space* if X is T_1 and normal.

67. If X is a *T_1-space*, then points in X are closed subsets of X.

68. $T_4 \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$, but none of these implications can be reversed.

69. (i) X is regular iff given $x \in X$ and a nbhd U of x, there exists a nbhd V of x such that $x \in V \subset \mathrm{Cl}(V) \subset U$.

    (ii) X is normal iff given a closed set F in X and an open set $U \supset F$ there exists an open set V such that $F \subset V \subset \mathrm{Cl}(V) \subset U$.

70. A subspace of a Hausdorff space is Hausdorff; $X \times Y$ is Hausdorff iff each of X and Y is Hausdorff.

71. A subspace of a regular space is regular; $X \times Y$ is regular iff each of X and Y is regular.

72. A closed subspace of a normal space is normal; if $X \times Y$ is normal, then each of X and Y is normal.

Definition. A space X is *completely normal* if each of its subspaces is normal.

73. A space X is completely normal iff for every pair of separated subsets H and K of X there are disjoint open sets U and V such that $H \subset U$ and $K \subset V$.

Definition. A space X is *second countable (satisfies the second axiom of countability)* if it has a countable basis. We abbreviate this by "X is 2°".

74. Every subspace of a 2° space is a 2° space; $X \times Y$ is 2° iff each of X and Y is 2°.

75. If X is 2°, then every open covering of X has a countable subcovering; i.e., if $\mathcal{U}$ is a collection of open sets that covers X, then there is a subcollection $\mathcal{V} \subset \mathcal{U}$ such that V is countable and covers X.

Definition. A space X is *Lindelœf* if every open covering contains a countable subcovering.

Definition. A space X is *separable* if it contains a countable dense set.

76. If X is 2°, then X is separable.

77. $\mathbb{R}^n$ is a 2° space. $\mathbb{R}_\ell$ is separable, but not 2°.

78. A subspace of a separable space need not be separable. An open subspace of a separable space, however, is separable.

79. If X is 2°, then every subspace of X is separable.

80. $\mathbb{R}_\ell$ is Lindelœf but $\mathbb{R}_\ell \times \mathbb{R}_\ell$ is not Lindelœf.

Definition. A *metric* or *distance function* on a set X is a function $d: X \times X \to \mathbb{R}$ having the following properties:

(i) $d(x, y) \geq 0$ for all $x, y \in X$,

(ii) $d(x, y) = 0$ iff $x = y$,

(iii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iv) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$. (*Triangle inequality*)

Given $\varepsilon > 0$, $\{y \in X \colon d(x, y) < \varepsilon\}$ is called the *ε-ball about x* or the *ε-neighborhood about x* and is denoted by any of the following:

$$B_d(x, \varepsilon), \quad B(x, \varepsilon), \quad B_\varepsilon(x), \quad N(x, \varepsilon), \quad N_\varepsilon(x).$$

81. Suppose X is a set with a metric d. Let $\mathcal{B} = \{B_d(x, \varepsilon) \colon x \in X, \varepsilon > 0\}$. Then $\mathcal{B}$ is a basis for a topology $\mathcal{T}_d$ on X. We call $\mathcal{T}_d = \mathcal{T}(\mathcal{B})$ the *metric topology on X induced by d*, and we call (X, d) a *metric space* with metric d.

Definition. A topological space $(X, \mathcal{T})$ is called a *metrizable space* if its topology is induced by a metric on X; i.e., if there is a metric d on X such that $\mathcal{T} = \mathcal{T}(\mathcal{B})$, where $\mathcal{B} = \{B_d(x, \varepsilon) \colon x \in X, \varepsilon > 0\}$.

Definition. Two metrics d and d' on a set X are *equivalent*, denoted $d \sim d'$, provided $\mathcal{T}_d = \mathcal{T}_{d'}$.

82. Define a metric d' on $\mathbb{R}^2$ by $d'((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$. Then $d' \sim d$, where d is the *euclidean metric* on $\mathbb{R}^2$ defined by $d(x, y) = \|x - y\|$ (see # 8).

83. Suppose d and d' are metrics on X. $d \sim d'$ iff for each $x \in X$ and $\varepsilon > 0$, the following two conditions hold:

    (i) there exists $\delta > 0$ such that $d(x, y) < \delta \Rightarrow d'(x, y) < \varepsilon$,

    (ii) there exists $\delta' > 0$ such that $d'(x, y) < \delta' \Rightarrow d(x, y) < \varepsilon$.

84. Let d be a metric on X. Define $d^* \colon X \times X \to \mathbb{R}$ by $d^*(x, y) = \min\{d(x, y), 1\}$. Then d^* is a metric on X that is equivalent to d.

Definition. Let (X, d) be a metric space.

(i) Given $x \in X$ and $A \subset X$, $A \neq \emptyset$, $d(x, A) = \inf\{d(x, a) \colon a \in A\}$.

(ii) Given nonempty subsets A, B of X, $d(A, B) = \inf\{d(a, b) \colon a \in A \text{ and } b \in B\} = \inf\{d(a, B) \colon a \in A\}$.

(iii) Given $A \subset X$, $A \neq \emptyset$, the *diameter of A*, denoted $\operatorname{diam}_d A$, or $\operatorname{diam} A$, is $\operatorname{diam}_d A = \sup\{d(x, y) \colon x, y \in A\}$.

(iv) Given $A \subset X$, $A \neq 0$, the *restriction of d to A*, denoted d_A, is the metric $d_A = d|_{A \times A}$ on A.

85. Let d be a metric on X.

    (i) $d(x, A) = 0$ iff $x \in \operatorname{Cl}(A)$. Thus, $\operatorname{Cl}(A) = \{x \in X \colon d(x, A) = 0\}$.

    (ii) If A is a nonempty subset of X, then the function $f \colon X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is continuous.

86. Every subspace of a metric space is metrizable. In fact, if (X, d) is a metric space and $A \subset X$, then the restriction of d to A metrizes the subspace topology on A.

Definition. A *neighborhood bases at a point x* in a space X is a collection $\mathcal{B}_x$ of nbhds of x having the property that if U is any nbhd of x there exists $B \in \mathcal{B}_x$ such that $B \subset U$. A space X is *first countable (satisfies the first axiom of countability)*, denoted 1°, if there is a countable neighborhood basis at each point of X.

87. Every metric space is 1°.

88. Let d be a metric on X. Then the following are equivalent.

(i) X is 2°.

(ii) X is separable.

(iii) X is Lindelœf.

89. Every metric space is (completely) normal.

90. Suppose $f\colon X \to Y$. Then f is continuous at $x \in X$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \varepsilon$.

Definition. Let $\mathbb{N}$ denote the set of natural numbers. A *sequence* in a space X is a function $f\colon \mathbb{N} \to X$. If $f(n) = x_n$, we usually denote the sequence by $\{x_n\}_{n=1}^{\infty}$, or simply $\{x_n\}$. A sequence $\{x_n\}$ in a space X is said to *converge* to a point x in X, denoted $x_n \to x$ or $\lim x_n = x$, provided that for every nbhd U of x in X, there exists $\mathrm{N} \in \mathbb{N}$ such that if $n \geq \mathrm{N}$, then $x_n \in U$.

91. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X converges to $x \in X$ iff for every $\varepsilon > 0$, there exists $\mathrm{N} \in \mathbb{N}$ such that $n \geq \mathrm{N}$ implies $d(x_n, x) < \varepsilon$.

92. Let X be 1° and suppose $A \subset X$. Then $x \in \mathrm{Cl}(A)$ iff there is a sequence $\{a_n\}$ in A that converges to x.

93. Let X be 1° and suppose $f\colon X \to Y$, where Y is any space. Then f is continuous at $x \in X$ iff $f(x_n) \to f(x)$ for every sequence $x_n \to x$.

94. Is $\mathbb{R}_\ell$ metrizable? Is $\mathbb{R}_\ell \times \mathbb{R}_\ell$ metrizable?

Definition. A space X is *compact* if every open covering of X has a finite subcovering; that is, X is compact iff for every cover $\mathcal{U}$ of X by open sets, there exists $U_1, U_2, \ldots, U_n \in \mathcal{U}$ such that $\{U_1, U_2, \ldots, U_n\}$ covers X.

95. The following are equivalent.

(i) X is compact.

(ii) X satisfies the *finite intersection property* for closed sets: If $\mathcal{F}$ is a family of closed sets in X such that $\cap\mathcal{F} = \emptyset$, then there is a finite subset $\{F_1, F_2, \ldots, F_n\}$ of $\mathcal{F}$ such that $F_1 \cap F_2 \cap \ldots \cap F_n = \emptyset$.

96. If $f\colon X \to Y$ is continuous and X is compact, then $f(X)$ is compact.

97. A compact subset of a Hausdorff space X is closed in X.

98. A closed subspace of a compact space is compact.

99. A compact Hausdorff space is regular.

100. A compact Hausdorff space is normal.

101. $X \times Y$ is compact iff each of X and Y are compact.

102. Suppose X is a compact space, Y is Hausdorff, and $f\colon X \to Y$ is continuous. Then

(i) f is a closed map.

(ii) If f is a bijection, then f is a homeomorphism.

103. If X is compact and if $p: X \times Y \to Y$ is the projection, then p is a closed mapping.

104. Suppose $A \subset X$ and Y is compact. Let U be a nbhd of $A \times Y$ in $X \times Y$. Then there is a nbhd $V \supset A$ in X such that $V \times Y \subset U$.

Definition. A subset A of a metric space (X, d) is *bounded* if there is a point $x \in X$ and a number $M > 0$ such that $A \subset B(x, M)$.

105. If A is a compact subset of metric space (X, d), then A is closed and bounded.

106. Every closed interval in $\mathbb{R}$ is compact.

107. **Heine-Borel Theorem**. A subset A of $\mathbb{R}^n$ is compact iff A is closed and bounded.

108. **Max-min Theorem**. Suppose X is compact and $f: X \to \mathbb{R}$ is continuous. Then there are points $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for every $x \in X$.

109. Let (X, d) be a metric space. If $A \subset X$ is closed, $C \subset X$ is compact, and $A \cap C = \emptyset$, then $d(A, C) > 0$. Can the compactness hypothesis on C be dropped?

110. Let (X, d) be a compact metric space, and let $\mathcal{U}$ be an open cover of X. Then there exists a positive number λ, called a *Lebesgue number* for the cover $\mathcal{U}$, with the following property: each ball $B(x, \lambda)$ is contained in at least one element U of $\mathcal{U}$.

111. Let (X, d) be a compact metric space, let (Y, d') be a metric space, and suppose $f: X \to Y$ is continuous. Then for each $\varepsilon > 0$, there is a $\lambda > 0$ (depending only on ε) such that $f(B(x, \lambda)) \subset B(f(x), \varepsilon))$ for every $x \in X$ (f is *uniformly continuous*).

112. **Bolzano-Weierstrass Property**. If X is compact, then every infinite subset of X has a limit point in X.

113. The unit interval $[0, 1]$ is not compact as a subspace of $\mathbb{R}_\ell$.

114. Suppose X is a compact space and $F_1 \supset F_2 \supset \ldots$ is a descending sequence of nonempty, closed subsets of X. Then $\cap_{i=1}^{\infty} F_i \neq \emptyset$.

115. A metric space (X, d) is compact iff every continuous real valued function on X is bounded.

116. Let X be a Hausdorff space and suppose $F_1 \supset F_2 \supset \ldots$ is a descending sequence of compact, connected, nonempty subsets of X. Then $\cap_{i=1}^{\infty} F_i$ is a compact, connected, nonempty subset of X. (A compact, connected Hausdorff space is called a *continuum*.)

117. Let (X, d) be a compact metric space and let $f: X \to X$ be a continuous function.

    (i) f is a *contraction* if there is a non-negative number $\alpha < 1$ such that $d(f(x), f(y)) \leq \alpha d(x, y)$ for all points $x, y \in X$. Show that if f is a contraction then there is a unique point $a \in X$ such that $f(a) = a$.

    (ii) f is an *isometry* if $d(f(x), f(y)) = d(x, y)$ for all $x, y \in X$. Show that if f is an isometry then f is surjective.

    (iii) Give examples to show that (i) and (ii) are false if X is not compact.

118. Suppose Y is compact and $f\colon Y \to Y$ is continuous. Show that there is a nonempty subset A of Y such that $f(A) = A$.

119. Let $(X, \mathcal{T}_X)$ be a topological space, Y a set, and $\pi : X \to Y$ a surjection. Define a collection of subsets of Y by

$$\mathcal{T}_Y = \{U \subset Y : \pi^{-1}(U) \in \mathcal{T}_X\}.$$

(i) Prove that $\mathcal{T}_Y$ is a topology on Y. It is called the *quotient topology.*

(ii) Prove that π is continuous when Y is given the quotient topology.

(iii) Prove that $\mathcal{T}_Y$ is the largest topology we can put on Y that will make π continuous.

120. Let X and Z be a topological spaces, $\pi : X \to Y$ a surjection, and give Y the resulting quotient topology. If $f : Y \to Z$ is a function, then f is continuous if and only if $f \circ \pi : X \to Z$ is continuous.

Definition. Let X be a space, and suppose $\sim$ is an equivalence relation on X. For each $x \in X$, let $[x]$ denote the equivalence class of x; i.e. $[x] = \{y \in X : x \sim y\}$. Let $X/{\sim}$ denote the set of equivalence classes of X under $\sim$. Define the projection map $\pi : X \to X/{\sim}$ by $\pi(x) = [x]$. Since $\pi : X \to X/{\sim}$ is a surjection, we may define the *quotient topology* on $X/{\sim}$ as above. In fact, whenever we refer to $X/{\sim}$ as a topological space, it is assumed we mean the quotient topology, unless otherwise indicated.

121. Let X be a space with an equivalence relation $\sim$ and projection mapping $\pi : X \to X/{\sim}$. Let $f : X \to Z$ be a continuous function that is constant on each equivalence class under $\sim$. Then there is a unique map $g : (X/{\sim}) \to Z$ such that $f = g \circ \pi$, and g is continuous.

122. Suppose X is compact and Y is Hausdorff. Let $f : X \to Y$ be continuous and surjective. Define a relation $\sim$ on X by $x_0 \sim x_1$ iff $f(x_0) = f(x_1)$. Then $\sim$ is an equivalence relation, and $X/{\sim}$ is homeomorphic to Y.

123. Let X be a topological space, $\sim$ an equivalence relation on X, and $X/{\sim}$ the corresponding quotient space. Prove:

(i) If X is compact, then $X/{\sim}$ is compact.

(ii) If X is connected, then $X/{\sim}$ is connected.

124. Define an equivalence relation on $X = [0, 1] \times [0, 1]$ by $(s_0, t_0) \sim (s_1, t_1)$ iff $t_0 = t_1$ and $t_1 > 0$. Prove that $X/{\sim}$ is not Hausdorff.

Definition. Define the *n-dimensional unit ball* $\mathbf{B}^n \subset \mathbb{R}^n$ by $\mathbf{B}^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$. Define the *$n$-dimensional unit sphere* $\mathbf{S}^n \subset \mathbb{R}^{n+1}$ by $\mathbf{S}^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$.

125. Define an equivalence relation on $\mathbf{B}^n$ by $x \sim y$ iff $\|x\| = \|y\| = 1$. Prove that $\mathbf{B}^n/{\sim}$ is homeomorphic to $\mathbf{S}^n$.

Definition. Define an equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ by $x \sim y$ iff there exists a $\lambda \in \mathbb{R} - \{0\}$ such that $x = \lambda y$. Define *real projective n-space* by

$$\mathbb{R}\mathbf{P}^n = (\mathbb{R}^{n+1} - \{0\})/{\sim}.$$

126. Define an equivalence relation $\sim$ on $\mathbf{S}^n$ by $x \sim y$ iff $x = \pm y$. Prove that $\mathbf{S}^n/{\sim}$ is homeomorphic to $\mathbb{R}\mathbf{P}^n$.

127. Define an equivalence relation $\sim$ on $\mathbf{B}^n$ by $x \sim y$ iff $\|x\| = \|y\| = 1$ and $x = \pm y$. Prove that $\mathbf{B}^n/\sim$ is homeomorphic to $\mathbb{R}\mathbf{P}^n$.

128. Prove that $\mathbb{R}\mathbf{P}^n$ is compact, connected, and Hausdorff.

129. Prove that each point in $\mathbb{R}\mathbf{P}^n$ has a neighborhood that is homeomorphic to an open set in $\mathbb{R}^n$.

Definition. Let $\mathbf{C}$ denote the set of complex numbers. Define an equivalence relation on $\mathbf{C}^{n+1} - \{0\}$ by $x \sim y$ iff there exists $\lambda \in \mathbf{C} - \{0\}$ such that $x = \lambda y$. Define *complex projective n-space* by

$$\mathbf{CP}^n = (\mathbf{C}^{n+1} - \{0\})/\sim.$$

130. Considering $\mathbf{S}^{2n+1}$ as a subset of $\mathbf{C}^{n+1}$, define an equivalence relation $\sim$ on $\mathbf{S}^{2n+1}$ such that $\mathbf{S}^{2n+1}/\sim$ is homeomorphic to $\mathbf{CP}^n$.

131. Prove that $\mathbf{CP}^n$ is compact, connected and Hausdorff, and that every point in $\mathbf{CP}^n$ has a neighborhood that is homeomorphic to $\mathbf{C}^n$.

132. Prove that $\mathbf{CP}^1$ is homeomorphic to $\mathbf{S}^2$.

133. Using 2 of the last 3 problems, define a surjective map $H : \mathbf{S}^3 \to \mathbf{S}^2$. This beautiful map is called the Hopf map.

134. $\mathbf{S}^1 = \{(x, y) \in \mathbb{R}^2 \,|x^2 + y^2 = 1\}$ is called the *unit circle* or the 1-*sphere*. Show that there is no 1:1 continuous function of $\mathbf{S}^1$ into $\mathbb{R}$ and that there is no surjection of $\mathbf{S}^1$ onto $\mathbb{R}$.

Definition. A subset A of a space X is *relatively compact* if its closure $\mathrm{Cl}(A)$ in X is compact. A space X is *locally compact* if it is Hausdorff and each point has a relatively compact neighborhood.

135. The following are equivalent:

(i) X is locally compact.

(ii) For each $x \in X$ and neighborhood U of x, there is a relatively compact open set V with $x \in V \subset \mathrm{Cl}(V) \subset U$.

(iii) For each compact set C and open set $U \supset C$, there is a relatively compact open set V with $C \subset V \subset \mathrm{Cl}(V) \subset U$.

(iv) X has a basis consisting of relatively compact open sets.

Definition. A *compactification* of a space X is a pair $(\tilde{X}, h)$ consisting of a compact space $\tilde{X}$ and a homeomorphism h of X onto a *dense* subset of $\tilde{X}$.

136. *One-point compactification.*

(i) Any locally compact space X can be embedded in a compact space $\tilde{X}$ so that $\tilde{X} - X$ is a single point.

(ii) (Uniqueness). Any two spaces $\tilde{X}$ and $\tilde{Y}$ having property (i) are homeomorphic.

137. The one-point compactification of $\mathbb{R}^1$ is $\mathbf{S}^1$. More generally, let

$$\mathbf{S}^n = \{x \in \mathbb{R}^{n+1}\colon \|x\| = \left(\sum_{i=1}^{n+1} x_i^2\right)^{1/2} = 1\},$$

the *n-sphere*. Then $\mathbf{S}^n$ is the one-point compactification of $\mathbb{R}^n$.

138. Let X be locally compact. Then the one-point compactification $\tilde{X}$ of X is metrizable if and only if X is 2°.

139. *Baire Property*. Let Y be locally compact and for each $i \in \mathbb{N}$, let D_i be a dense open subset of Y. Then $\cap_{i \in \mathbb{N}} D_i$ is dense in Y.

Definition. A space Y is a *Baire space* if the intersection of each countable family of open dense sets in Y is dense. Thus, every locally compact space is a Baire space.

140. Let Y be a Baire space. If $\{A_n | n \in \mathbb{N}\}$ is a countable closed covering of Y, then at least one A_n must contain an open set. That is, $\mathrm{Int}_Y(A_n) \neq \emptyset$ for some $n \in \mathbb{N}$.

141. $\mathbb{R}^1$ is a Baire space. $\mathbb{R}_\ell$ is a Baire space.

142. The set of rational numbers in $\mathbb{R}^1$ is not a Baire space. The set of irrationals is a Baire space.

Definition. Let (X, d) be a metric space. A sequence $\{x_n\}$ in X is called *d-Cauchy* if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, whenever $n, m \geq N$.

143. Let (X, d) be a metric space.

   (i) Every convergent sequence in X is d-Cauchy.

   (ii) Every subsequence of a d-Cauchy sequence is d-Cauchy.

   (iii) If a d-Cauchy sequence in X has a limit point, then it converges to that point.

   (iv) If a d-Cauchy sequence does not converge, then it has no convergent subsequence.

Definition. Let X be a metrizable space. A metric d for X is called *complete* if every d-Cauchy sequence in X converges. In this case (X, d) is called a *complete metric space*.

Definition. A metrizable space X is called *topologically complete* if a complete metric for X exists. To indicate that d is a complete metric for X, we say that X is *d-complete*.

144. Let (X, d) be a metric space, and assume that d has the property: there exists $\varepsilon > 0$ such that for all $x \in X$, $B_d(x, \varepsilon)$ is relatively compact. Then d is a complete metric for X.

145. Give an example of a metric space (X, d) that is topologically complete, but not complete.

146. Every locally compact metric space is topologically complete. Furthermore, if X is compact metric, then every metric for X is a complete metric.

147. If X and Y are homeomorphic spaces and X is topologically complete, then so is Y.

148. If X is topologically complete, then every closed subspace A is topologically complete. Furthermore, if X is d-complete, then A is d_A-complete, where d_A is the restriction of d to A.

149. If (X, d) is a metric space (not necessarily complete) and $A \subset X$ is d_A-complete, then A is closed in X.

150. A countable product $\prod_{i=1}^{\infty} Y_i$ is topologically complete if and only if each factor Y_i is topologically complete.

151. **Baire's Theorem for Complete Spaces**. Any topologically complete space is a Baire space.

152. Let X be d-complete and let $f: X \to X$ be *d-contractive* (i.e., there exists α, $0 \le \alpha < 1$, such that $d(f(x), f(y)) \le \alpha d(x, y)$ for all $x, y \in X$). Then f is continuous and has exactly one fixed point.

153. Let X be a metrizable space and d a given metric on X. Then X can be isometrically embedded as a dense subset of a complete space (X^*, d^*).

154. X^* and d^* in 139 are unique up to isometry: if X is embedded isometrically in a complete space (X°, d°), then (X^*, d^*) and (X°, d°) are isometric.

Definition. A subset F of a space X is called an *F_σ-subset* of X if $F = \cup_{i \in \mathbb{N}} F_i$, for some countable collection $\{F_i\}$ of closed subsets of X. A subset G of X is a *G_δ-subset* of X if $G = \cap_{i \in \mathbb{N}} G_i$ for some countable collection $\{G_i\}$ of open subsets of X. Thus, countable unions of closed sets are F_σ's, and countable intersections of open sets are G_δ's.

155. The irrationals in $\mathbb{R}$ is a G_δ-subset of $\mathbb{R}$.

156. Let X be an arbitrary metric space, and suppose $A \subset X$. Let Y be complete and let $f: A \to Y$ be a continuous map. Then there is a G_δ-subset $G \supset A$ of X and a continuous function $F: G \to Y$ such that $F|_A = f$.

157. **Lavrentieff's Theorem**. If X and Y are complete metric spaces and h is a homeomorphism of $A \subset X$ onto $B \subset Y$, then h can be extended to a homeomorphism h^* of A^* onto B^*, where A^* and B^* are G_δ-subsets of X and Y, respectively, and $A \subset A^* \subset \mathrm{Cl}_X A$ and $B \subset B^* \subset \mathrm{Cl}_Y B$.

158. Let Y be complete and let $A \subset Y$ be a topologically complete subset. Then A is a G_δ-subset of Y.

159. **Mazurkiewicz's Theorem**. Let Y be a complete space. Then $A \subset Y$ is topologically complete if and only if A is a G_δ-subset of Y.

160. The set of irrational numbers in $\mathbb{R}$ is topologically complete.

161. The set of rational numbers in $\mathbb{R}$ is not topologically complete.

162. Show that if A is any G_δ-subset of $\mathbb{R}$, then there is a function $f: \mathbb{R} \to \mathbb{R}$ that is continuous at all points of A and discontinuous at all other points of $\mathbb{R}$.

163. Let X be a metric space such that $X = A \cup B$, where A and B are topologically complete. Then X is topologically complete. [Hint: Prove that the union of two G_δ-subsets is a G_δ-subset and use 139.]

The Axiom of Choice and the Tychonoff Theorem

Definition. Let $\{A_\gamma : \gamma \in \Gamma\}$ be a collection of sets. A *choice function* for $\{A_\gamma : \gamma \in \Gamma\}$ is a function $f: \Gamma \to \cup A_\gamma$ such that $f(\gamma) \in A_\gamma$ for all $\gamma \in \Gamma$. The *cartesian product* of $\{A_\gamma : \gamma \in \Gamma\}$, denoted by $\prod_{\gamma \in \Gamma} A_\gamma$, (or, just $\prod A_\gamma$, if no confusion arises) is the set of all choice functions for $\{A_\gamma : \gamma \in \Gamma\}$.

Axiom of Choice. If $\{A_\gamma : \gamma \in \Gamma\}$ is a nonempty collection of nonempty sets, then $\prod A_\gamma \neq \emptyset$.

Recall that a relation R on a set X is a subset of $X \times X$. We will usually write xRy instead of $(x, y) \in R$ when x is related to y. A *partial order* on a set X is a reflexive, antisymmetric, and transitive relation. Thus if $\le$ is a partial order on X, then

(i) $x \leq x$ for all $x \in X$ (*reflexive*),

(ii) $x \leq y$ and $y \leq x \Rightarrow x = y$ (*antisymmetric*),

(iii) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (*transitive*).

If, for every $x, y \in X$, either $x \leq y$ or $y \leq x$, then we say that $\leq$ is a *total order* on X. A totally ordered set is called a *chain*. Let X be a set with partial order $\leq$. A *maximal* element of X is an element $a \in X$ such that for all $x \in X$, if $a \leq x$, then $a = x$; i.e., $a \in X$ is a maximal element if X contains no element strictly greater than a. An element $a \in X$ is the *greatest* element of X if $x \leq a$ for all $x \in X$. Necessarily, a greatest element of X, if it exists, is unique. There may be many maximal elements of X.

Let X be a partially ordered set with order $\leq$ and let $E \subset X$. An element $a \in X$ is an *upper bound* of E in case $x \leq a$ for all $x \in E$.

Zorn's Lemma. If X is a partially ordered set such that every chain in X has an upper bound, then X contains a maximal element.

[A *chain in* X is a subset $E \subset X$ that is totally ordered by $\leq$. Note that we do not require the upper bound to be in E.]

Theorem A. *The Axiom of Choice is equivalent to Zorn's Lemma.*

We shall prove that the Axiom of Choice implies Zorn's Lemma.

If $\{X_\gamma : \gamma \in \Gamma\}$ is a collection of topological spaces, then the collection of sets of the form $\prod U_\gamma \subset \prod X_\gamma$, where U_γ is open X_γ for all $\gamma \in \Gamma$ and $U_\gamma = X_\gamma$ for all but finitely many $\gamma \in \Gamma$, forms a basis for a topology on $\prod X_\gamma$, called the *product topology*.

Tychonoff Theorem. *If* $\{X_\gamma : \gamma \in \Gamma\}$ *is a collection of compact spaces, then* $\prod X_\gamma$ *is compact (in the product topology).*

Theorem B. *The Tychonoff Theorem is equivalent to the Axiom of Choice.*

We will prove that the Axiom of Choice implies the Tychonoff Theorem. More specifically, we shall show that Zorn's Lemma implies the Tychonoff Theorem.