Section 10.4 - Determinants

<u>2 by 2 determinants</u>:

If D = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2 x 2 matrix, then the <u>determinant of D</u>

is det(D) = $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

<u>3 by 3 determinants</u>:

If D = $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3 by 3 matrix, then the

<u>determinant of D</u> is det (D) = $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$$

$$- (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

Calculating the determinant of a 3 by 3 matrix by **expanding a row or column:**

If $D = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a 3 by 3 matrix, then the

determinant of D may be calculated by expanding about row 1 in the matrix:

$$\det(D) = a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}).$$

Also, the determinant may be calculated by expanding about column 1 in the matrix:

$$\det(D) = a_{11}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} - a_{21}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{31}\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{21}(a_{12}a_{33} - a_{13}a_{32}) + a_{31}(a_{12}a_{23} - a_{13}a_{22})$$

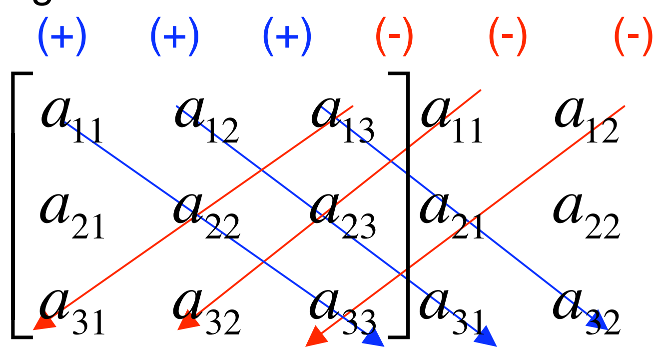
For expansions about other rows and columns see the text and/or course notes for section 10.4.

Shortcut for calculating the determinant of a 3 by 3 matrix (will not work for larger matrices):

To calculate the determinant of the 3 by 3 matrix:

$$D = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$ begin by copying the entries from the first two columns of the matrix to the right of the matrix.

Then multiply the entries on the diagonals with the appropriate sign as shown below :



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$$a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - (a_{13}a_{22}a_{31} + a_{11}a_{23}a_{32} + a_{12}a_{21}a_{33})$$

Example:

$$\begin{vmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} \begin{matrix} 2 & 1 \\ -1 & 2 \\ 1 & -2 \end{matrix} =$$

$2 \cdot 2 \cdot (-3) + 1 \cdot 4 \cdot 1 + (-1) \cdot (-1) \cdot (-2) - (-1) \cdot 2 \cdot 1 - 2 \cdot 4 \cdot (-2) - 1 \cdot (-1) \cdot (-3)$=-12+4-2+2+16-3= 5

10.4 - **Cramer's Rule**
for Two Equations containing Two Variables:

Consider the system of equations:

$$\begin{cases} ax + by = s \\ cx + dy = t \end{cases}$$

Define:

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \quad D_x = \begin{vmatrix} s & b \\ t & d \end{vmatrix}, \quad D_y = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$$

Then Cramer's Rule says that:

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}$$, provided $D \neq 0$.

10.4 - **Cramer's Rule for <u>Three Equations containing Three Variables:</u>**

The solution to the system of equations:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

is given by

$$x = \frac{D_x}{D}, \quad y = \frac{D_y}{D}, \quad z = \frac{D_z}{D} \quad \text{where}$$

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

$$D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix}, \quad D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix}, \quad D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}.$$

Determinants and the elementary row operations:

1. The value of the determinant changes sign when two rows are interchanged.

For example: $\begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} a & b & c \\ 1 & 2 & 3 \\ 0 & 0 & 1 \end{vmatrix}$

2. If a row is multiplied by a number k, then the determinant is multiplied by k.

For example: $\begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ 0 & 0 & 5 \end{vmatrix} = 5 \cdot \begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix}$

3. If a multiple of one row is added to another row, then the value of the determinant does not change.

For example: $\begin{vmatrix} 1 & 2 & 3 \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1-a & 2-b & 3-c \\ a & b & c \\ 0 & 0 & 1 \end{vmatrix}$