Preliminary to Math Induction - An Infinite Sequence of Propositions:

In Section 11.1, formulas are used to define an infinite sequence of numbers.

For instance the sequence $\{1, 2, 4, 8, 16, \cdots\}$ is generated by the formula $\{2^{n-1}\}_{n=1}^{\infty}$

and $\{1, 3, 5, 7, 9, \cdots\}$ is generated by the formula $\{2n-1\}_{n=1}^{\infty}$.

In Section 11.4, math induction is used to prove all the propositions (or statements) in an infinite sequence of propositions are true. The sequence of propostions can by defined by a formula analogous to the formulas used to generate sequences of numbers.

Notice the pattern in the following sequence of equations (propositions):

$1 = 1^2$

$1 + 3 = 2^2$

$1 + 3 + 5 = 3^2$

$1 + 3 + 5 + 7 = 4^2$

$1 + 3 + 5 + 7 + 9 = 5^2$, etc.

The n^{th} equation in this sequence of equations is defined by:

$1 + 3 + \cdots + (2n-1) = n^2$

(Notice that in the nth equation: the number of terms in the sum on the left hand side is n, the last term in the sum on the left is given by the formula 2n-1 and the sum of the n terms equals n^2.)

Another sequence of propositions is:

$1 = 1*2/2$

$1 + 2 = 2*3/2$

$1 + 2 + 3 = 3*4/2$

$1 + 2 + 3 + 4 = 4*5/2$

$1 + 2 + 3 + 4 + 5 = 5*6/2$, etc.

This sequence of equations is defined by the formula: $1 + 2 + 3 + \cdots + n = n(n+1)/2$.

Notation: P_n will be used to denote the n^{th} proposition in a sequence of propositions.

If P_n is defined by the formula $1 + 3 + \cdots + (2n-1) = n^2$, then the sequence of propositions which it generates is:

P_1 : $1 = 1^2$

P_2 : $1 + 3 = 2^2$

P_3 : $1 + 3 + 5 = 3^2$

P_4 : $1 + 3 + 5 + 7 = 4^2$

P_5 : $1 + 3 + 5 + 7 + 9 = 5^2$, etc.

In proof by induction, we must show that IF a PARTICULAR proposition in a sequence of propositions is TRUE, then the NEXT proposition in the sequence must ALSO be TRUE.

(Recall the recursive definition of a sequence. If we are able to calculate a particular term in a recursive sequence, then we could use that value to calculate the next term in the sequence. Just as a recursive definition has two parts - the part which defines the initial term(s) in the sequence and the part which gives the rule used to generate subsequent terms in the sequence, so a proof by math induction has two parts - the Basis Step (which establishes the truth of the sequence of propositions for an initial value(s)) and the Inductive Step (which establishes that IF a particular proposition in a sequence of propositions is TRUE, then the NEXT proposition in the sequence must ALSO be TRUE.))

The notation we will use for the PARTICULAR proposition is P_k and for the NEXT proposition is P_{k+1}.

If P_n is defined by the formula $1 + 3 + \cdots + (2n-1) = n^2$, then

P_k : $1 + 3 + \cdots + (2k-1) = k^2$

P_{k+1} : $1 + 3 + \cdots + [2(k+1)-1] = (k+1)^2$, or since $2(k+1) - 1 = 2k+2 - 1 = 2k+1$,

an equvalient statement would be:

P_{k+1} : $1 + 3 + \cdots + (2k+1) = (k+1)^2$

Proof by Math Induction:

To prove an infinite sequence of propositions (or statements) $\{P_n\}_{n=1}^{\infty}$ is

true for every positive integer n,

1) Show that P_1 is true. ***(Basis Step)***

2) Show that IF P_k is true for a particular k,

THEN P_{k+1} (the NEXT proposition in the sequence of propositions)

MUST ALSO be true.

That is, show $P_k \Rightarrow P_{k+1}$ ***(Inductive Step)***

.

The following are examples of complete proofs using math induction, followed by quesitons similar to those found on the eGrade test bank concerning the construction of the proposition P_{k+1} in the Inductive Step of Proof by Induction.

Sample Proofs by Math Induction:

1) **Example - proving a sum formula is true for all positive integers n:**

Prove that $1*2+2*3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$, whenever n is a positive integer.

Proof: Define P_n to be the statement: $1*2+2*3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$.

We wish to show that P_n is true for all positive integers n.

Basis Step: Substituting n=1 into the definition of P_n yields the statement:

$1*2=\frac{1(1+1)(1+2)}{3}$. Since 1*2=2 and $\frac{1(1+1)(1+2)}{3}=\frac{1*2*3}{3}=2$,

we see that P_1 is true and the basis step is established.

Inductive Step: We must show that $P_k \longrightarrow P_{k+1}$ for every integer k ≥ 1.

Assume that P_k is true for some integer k ≥ 1. That is, assume

$1*2+2*3+\cdots+k(k+1)=\frac{k(k+1)(k+2)}{3}$ (the Inductive Hypothesis).

We must show this implies $1*2+2*3+\cdots+k(k+1)+(k+1)(k+2)=\frac{(k+1)(k+2)(k+3)}{3}$

Starting with the left hand side of the equation which we need to establish:

$$1*2+2*3+\cdots+k(k+1)+(k+1)(k+2) = \left[1*2+2*3+\cdots+k(k+1)\right]+(k+1)(k+2) \quad \text{assoc. prop.}$$

$$= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{by } \underline{\text{INDUCTIVE HYPOTHESIS}}$$

$$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3} \quad \text{muliplying the second term by } \frac{3}{3}$$

$$= \frac{(k+1)\left[k(k+2)+3(k+2)\right]}{3} \quad \text{factoring (k+1) from each term}$$

$$= \frac{(k+1)(k+2)[k+3]}{3} \quad \text{factoring (k+2) from each term inside the parenteses}$$

Thus P_{k+1} holds and we have shown that $P_k \longrightarrow P_{k+1}$ for every integer k ≥ 1.

By the Principle of Math Induction, P_n is true for every integer n ≥ 1.

Question:

To Prove by induction that $1*2+2*3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$ is true for all positive integers n, we assume that $1*2+2*3+\cdots+k(k+1)=\frac{k(k+1)(k+2)}{3}$ is true for some positive integer k, and show that $1*2+2*3+\cdots+k(k+1)+A=\frac{(k+1)(k+2)(k+3)}{3}$ where A is

a) k(k+1) + 1 b) k(k+1) + 2 c) (k+1)(k+1) d) (k+1)(k+2) e) none of these

(Hint: In the inductive step, we must prove that if P_k is true, it must follow that P_{k+1} is true. What is P_{k+1}?)

Question:

To Prove by induction that $1*2+2*3+\cdots+n(n+1)=\frac{n(n+1)(n+2)}{3}$ is true for all positive integers n, we assume that $1*2+2*3+\cdots+k(k+1)=\frac{k(k+1)(k+2)}{3}$ is true for some positive integer k, and show that

$$1*2+2*3+\cdots+k(k+1)+(k+1)[(k+1)+1]=A$$ where A is

a) $\frac{k(k+1)(k+2)+1}{3}$ b) $\frac{k(k+1)(k+2)}{3}+1$ c) $\frac{(k+1)(k+2)(k+3)}{3}$

d) $\frac{k(k+1)(k+2)+3}{3}$ e) none of these

(Hint: In the inductive step, we must prove that if P_k is true, it must follow that P_{k+1} is true. What is P_{k+1}?)

2) **Example - proving a divisibility statement is true for all positive integers n:**
To Prove: 3 divides $n^3 + 2n$ when n > 0.

Proof: **Setup:** Define P_n to be the statement: 3 divides $n^3 + 2n, \ \forall n \in P$.

We wish to show that P_n is true for all integers n in P.

Basis Step: We must show P_1 is true, that is, we must show that 3 divides $(1)^3 + 2 \cdot 1$ or equivalently that $1^3 + 2$ = 3m for some integer m. Since $1^3 + 2$ = 3 = 3·1, P_1 is true. This establishes the basis step.

Inductive Step: We must show that $P_k \longrightarrow P_{k+1}$ for every k≥1. Assume that P_k is true, that is, $k^3 + 2k = 3 \cdot m$ for some integer k≥ 1 and integer some m. We must show this implies that P_{k+1} is true. That is, we must show that $(k+1)^3 + 2(k+1) = 3 \cdot l$ for some integer *l.* But $(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$ distributing

$= (k^3 + 2k) + 3k^2 + 3k + 3$ commutative & assoc. prop.

$= 3 \cdot m + 3k^2 + 3k + 3$ by INDUCTIVE HYPOTHESIS

$= 3\left(m + k^2 + k + 1\right)$ factoring

Then P_{k+1} holds with $l = m + k^2 + k + 1.$
We have thus shown that $P_k \longrightarrow P_{k+1}$.
By the Principle of Math Induction, P(n) is true for every positive integer n.

Question: To Prove by induction that $n^3 + 2n$ is divisible by 3 is true for all positive integers n, we assume $k^3 + 2k$ is divisible by 3 is true for some positive integer k, and show that A is divisible by 3, where A is

a) $k^3 + 2k + 1$ b) $(k+1)^3 + 2(k+1)$ c) $k^3 + 2k + 2$ d) none of these

(Hint: In the inductive step, we must prove that if P_k is true, it must follow that P_{k+1} is true. What is P_{k+1}?)

Question: To Prove by induction that $n^3 + 2n$ is divisible by 3 is true for all positive integers n, we assume $k^3 + 2k$ is divisible by 3 is true for some positive integer k, and show that $k^3 + 2k$+A is divisible by 3, where A is

a) $3k^2 + 3k + 3$ b) $(k+1)^3 + 2(k+1)$ c) $k^3 + 2k + 2$ d) none of these

(Hint: In the inductive step, we must prove that if P(k) is true, it must follow that P_{k+1} is true. What is P_{k+1}? What do we have to add to the expression guarenteed to be divisible by 3 in P_k to obtain the expression which we need to show is divisible by 3 for P_{k+1} to be true?)

---

3) **Example - proving a divisibility statement is true for all positive integers n:**

To Prove: $n^2 - 1$ is divisible by 8 whenever n is an odd positive integer.

Proof: **Setup:** Define P_n to be the statement: 8 divides $(2n-1)^2 - 1, \ \forall n \in P$.

We wish to show that P_n is true for all integers n in P.

Basis Step: We must show P_1 is true, that is, we must show that 8 divides $(2 \cdot 1 - 1)^2 - 1$ or equivalently that $1^2 - 1$ = 8*m for some integer m. Since $1^2 - 1$ = 0 = 8*0, P(1) is true. This establishes the basis step.

Inductive Step: We must show that $P_k \rightarrow P_{k+1}$ for every k≥1.

Assume that P_k is true, that is, $(2k-1)^2 - 1$ for some integer m. We must show this implies that P_{k+1} is true. That is, we must show that $\left(2(k+1)-1\right)^2 - 1$ for some integer *l*. But

$\left(2(k+1)-1\right)^2 - 1$	$= (2k+2-1)^2 - 1$	distributing
	$= (2k-1+2)^2 - 1$	commutative prop. of addition
	$= \left[(2k-1)+2\right]^2 - 1$	associtative prop. of addition
	$= (2k-1)^2 + 2(2k-1) + (2k-1)2 + 4 - 1$	distributing
	$= (2k-1)^2 - 1 + 4(2k-1) + 4$	algebraic simplification
	$= 8 * m + 4(2k-1) + 4$	BY THE INDUCTIVE HYPOTHESIS
	$= 8 * m + 8k$ *=8(m+k)*	factoring

Then P_{k+1} holds with *l=m+k.*

We have thus shown that $P_k \rightarrow P_{k+1}$.

By the Principle of Math Induction, P(n) is true for every positive integer n.

Question: To Prove by induction that $(2n-1)^2 - 1, \ \forall n \in P$ is divisible by 8 is true for all positive integers n, we assume $(2k)^2 - 1$ is divisible by 8 is true for some positive integer k, and show that A is divisible by 8, where A is

a) $(2k-1)^2 - 1 + 1$ b) $(2k)^2 - 1$ c) $\left(2(k+1)-1\right)^2 - 1$ d) none of these

(Hint: In the inductive step, we must prove that if P_k is true, it must follow that P_{k+1} is true. What is P_{k+1}?)

Answers to **Questions**: d, c, b, a, c.

II. The following two "proofs" are examples of faulty induction proofs. What is wrong with these proofs?

a. Claim: All horses are the same color.

"Proof." Let P_n be the propostition that all the horses in a set of n horses are the same color.

Clearly, P_1 is true.

Now assume that P_k is true, so that all the horses in any set of k horses are the same color. Consider any k + 1 horses: number these as horses 1,2,3,...,k,k+1. Now the first k of these horses all must have the same color, and the last k of these must also have the same color. Since the set of the first k horses and the set of the last k horses overlap, all k + 1 must be the same color. This shows that P_{k+1} is true and finishes the proof by induction.

(Hint: The inductive step fails when k=1.)

---

b. Claim: $n^2 + 5n + 1$ is even for all positive integers n.

"Proof." Let P_n be the proposition $n^2 + 5n + 1$ is even, $n \in \mathbf{P}$.

Assume $k^2 + 5k + 1$ is even for some positive integer k. Then there exists an integer j such that $k^2 + 5k + 1 = 2j$.

Then $(k + 1)^2 + 5(k+1) + 1 = k^2 + 2k + 1 + 5k + 5 + 1$ distributing

$= k^2 + 5k + 1 + 2k + 6$ algebraic simplification

$= 2j + 2k + 6$ BY INDUCTIVE HYPOTHESIS

$= 2(j + k + 3)$ factoring

Thus P_{k+1} is true and by the Principle of Math Induction, P_n is true for every $n \in \mathbf{P}$.

(Hint: A proof by induction requires two steps - a basis step and an inductive step. One step in missing in this proof. Thus the proof is not valid. Try to find any value of n for which P_n is true. Since the inductive step in this proof is valid, if you could find one initial value k for which P_k were true, then P_n would be true for all subsequent values n, that is, all integers $n \geq k$.)