A Matrix Method to Solve a System of ***n*** Linear Equations in ***n*** unknowns:

1. Write the augmented matrix that represents the system.

2. Perform row operations to simplify the augmented matrix to one having zeros below the diagonal of the coefficient portion of the matrix.

   (An entry is on the diagonal of the coefficient portion of the matrix if it is located in *row i* and *column i* for some positive integer $i \leq n$.)

   - If the augmented matrix is equivalent to a matrix with zeros below the diagonal and all non-zero entries on the diagonal, then the corresponding system has a unique solution.

   - If the aumented matrix is equivalent to a matrix with zeros below the diagonal and at least one zero on the diagonal, then the corresponding system does not have a unique solution.

   In this case, examination of the rows which contain a zero on the diagonal entry will determine whether the corresponding system has no solution or an infinite number of solutions.

Examples:

The system of equations represented by the following augmented matrices have a **unique solution**:

$$\begin{bmatrix} 2 & 3 & 4 & 11 \\ 0 & -1 & 3 & 10 \\ 0 & 0 & -2 & -6 \end{bmatrix}, \begin{bmatrix} 2 & 3 & 4 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The system of equations represented by the following augmented matrix has **no solution**:

$$\begin{bmatrix} 2 & 3 & 4 & 11 \\ 0 & -1 & 3 & 10 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

(Note that the *third row* of the above matrix represents the equation: $0x + 0y + 0z = -6$ or $0 = -6$ which is not a true statement. Therefore the corresponding system has no solution.)

The system of equations represented by the following augmented matrix has an **infinite number of solutions**:

$$\begin{bmatrix} 2 & 3 & 4 & 11 \\ 0 & -1 & 3 & 10 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(Note that the *third row* of the above matrix represents the equation: $0x + 0y + 0z = 0$ or $0 = 0$ which is a true for any values x,y and z.)

Use matrices to solve the following system of equations:

$$x + 3y - z = -3$$
$$3x - y + 2z = 1$$
$$2x - y + z = -1$$

The augmented matrix corresponding to the above system of equations is:

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 3 & -1 & 2 & 1 \\ 2 & -1 & 1 & -1 \end{bmatrix}$$

Our strategy is to use elementary row operations to zero out the entries in: Row 2 and Column 1
Row 3 and Column 1
Row 3 and Column 2.

Hint 1:

One way to zero out the entry in row 2, column 1 is:

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 3 & -1 & 2 & 1 \\ 2 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_1=-3r_1} \begin{bmatrix} -3 & -9 & 3 & 9 \\ 3 & -1 & 2 & 1 \\ 2 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2=r_1+r_2} \begin{bmatrix} -3 & -9 & 3 & 9 \\ 0 & -10 & 5 & 10 \\ 2 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_1-\frac{1}{3}r_1} \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -10 & 5 & 10 \\ 2 & -1 & 1 & -1 \end{bmatrix}$$

A <u>more efficient</u> way is:

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 3 & -1 & 2 & 1 \\ 2 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_2=-3r_1+r_2} \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -10 & 5 & 10 \\ 2 & -1 & 1 & -1 \end{bmatrix}$$

"Scratch" for the above operation:

$$\begin{array}{rrrrr} -3r_1: & -3 & -9 & 3 & 9 \\ r_2: & 3 & -1 & 2 & 1 \\ \hline R_2: & 0 & -10 & 5 & 10 \end{array}$$

Hint 2:

When all entries of a row have a common factor, consider dividing each term in that row by the common factor. If you can reduce the magnitude of the entries in a row without introducing fractions, your subsequent calculations will involve smaller numbers.

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -10 & 5 & 10 \\ 2 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{\frac{1}{5}R_2} \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -2 & 1 & 2 \\ 2 & -1 & 1 & -1 \end{bmatrix}$$

Next, zero out the entry in row 3, column 1:

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -2 & 1 & 2 \\ 2 & -1 & 1 & -1 \end{bmatrix} \xrightarrow{R_3=-2r_1+r_3} \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -2 & 1 & 2 \\ 0 & -7 & 3 & 5 \end{bmatrix}$$

AFTER zeroing out the **necessary entries in column one**, **<u>USE ROW 2</u>** to zero out the **necessary entry in <u>column 2</u>**, which is, the entry in row 3, column 2.

$$\begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -2 & 1 & 2 \\ 0 & -7 & 3 & 5 \end{bmatrix} \xrightarrow{R_3=-7r_2+2r_3} \begin{bmatrix} 1 & 3 & -1 & -3 \\ 0 & -2 & 1 & 2 \\ 0 & 0 & -1 & -4 \end{bmatrix}$$

At this point, we know that the given system of equations has a **unique solution**.

Use back substitution to determine the solution:

Converting each row into its equivalent equation form,

from R_3: $-z = -4 \Rightarrow z = \frac{-4}{-1} \Rightarrow z = 4$

from R_2: $-2y + 1(4) = 2 \Rightarrow -2y = -2 \Rightarrow y = 1$

from R_1: $x + 3(1) - (4) = -3 \Rightarrow x - 1 = -3 \Rightarrow x = -2$.

So the solution is: **x = - 2, y = 1, z = 4**.