KMS STATES AND COMPLEX MULTIPLICATION

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1. Introduction

The following problem in operator algebra has been open for several years.

Problem 1.1. *For some number field K (other than $\mathbb{Q}$) exhibit an explicit quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$ with the following properties:*

1. *The partition function $Z(\beta)$ is the Dedekind zeta function of K.*
2. *The system has a phase transition with spontaneous symmetry breaking at the pole $\beta = 1$ of the zeta function.*
3. *There is a unique equilibrium state above critical temperature $\beta = 1$.*
4. *The quotient C_K/D_K of the idèle class group C_K of K by the connected component D_K of the identity acts as symmetries of the system $(\mathcal{A}, \sigma_t)$.*
5. *There is a subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ with the property that the values of extremal ground states on elements of $\mathcal{A}_0$ are algebraic numbers and generate the maximal abelian extension K^{ab}.*
6. *The Galois action on these values is realized by the induced action of C_K/D_K on the ground states, via the class field theory isomorphism*

$$\theta : C_K/D_K \to \mathrm{Gal}(K^{ab}/K). \tag{1.1}$$

The problem originates from the work of Bost–Connes, [3], [4], where a system with all the properties listed above was constructed for $K = \mathbb{Q}$. Important developments in the direction of generalizing the Bost–Connes system to other number fields were obtained by Harari and Leichtnam [11], Cohen [6], Arledge, Laca and Raeburn [1], Laca and van Frankenhuijsen [12]. These results all assume restrictions on the class number of K. It was widely believed that a system satisfying all the properties of Problem 1.1 would exist (supposedly for any number field and certainly at least in the case where K is an imaginary quadratic field). However, a complete construction (without special assumptions on the class number) had not been obtained so far for any case other than $\mathbb{Q}$.

The purpose of the present paper is to give a complete solution to Problem 1.1, for K an imaginary quadratic field, without any restriction on the class number of K. In an accompanying paper [8], we explain the geometry underlying and motivating the construction presented in this paper, and we make a detailed comparison between the properties of the system described here, the original Bost–Connes system [4] and the GL_2-system of [7].

2. Quantum statistical mechanics for imaginary quadratic fields

In this section we construct a quantum statistical mechanical system $(\mathcal{A}_K, \sigma_t)$ associated to an imaginary quadratic field K.

We begin by recalling some basic notions and notation that we will use through the paper.

For any ring R, we write R^* for the group of invertible elements, while $R^\times$ denotes the set of nonzero elements of R, which is a semigroup if R is an integral domain.

We write $\mathcal{O}$ for the ring of algebraic integers of the imaginary quadratic field $K = \mathbb{Q}(\sqrt{-d})$, where d is a positive integer.

We also use the notation

$$\hat{\mathcal{O}} := \mathcal{O} \otimes \hat{\mathbb{Z}} \quad \mathbb{A}_{K,f} = \mathbb{A}_f \otimes_{\mathbb{Q}} K \quad \text{and} \quad \mathbb{I}_K = \mathbb{A}_{K,f}^* = \mathrm{GL}_1(\mathbb{A}_{K,f}), \tag{2.1}$$

where $\hat{\mathbb{Z}}$ is the profinite completion of $\mathbb{Z}$ and $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ is the ring of finite adeles of $\mathbb{Q}$. Notice that K^* embeds diagonally into $\mathbb{I}_K$.
We can write the ring of integers in the form $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ and the imaginary quadratic field as $K = \mathbb{Q}(\tau)$, where we fix the embedding $K \hookrightarrow \mathbb{C}$ so that $\tau \in \mathbb{H}$. We can regard $\mathbb{C}$ as a K-vector space and in particular an $\mathcal{O}$-module.

2.1. K-lattices and commensurability.

We introduce the main geometric notions underlying our system.

Definition 2.1. *For K an imaginary quadratic field, a 1-dimensional K-lattice (Λ, ϕ) is a finitely generated $\mathcal{O}$-submodule $\Lambda \subset \mathbb{C}$, such that $\Lambda \otimes_{\mathcal{O}} K \cong K$, together with a morphism of $\mathcal{O}$-modules*

$$\phi : K/\mathcal{O} \to K\Lambda/\Lambda. \tag{2.2}$$

A 1-dimensional K-lattice is invertible *if ϕ is an isomorphism of $\mathcal{O}$-modules.*

Notice that in the definition we assume that the $\mathcal{O}$-module structure is compatible with the embeddings of both $\mathcal{O}$ and Λ in $\mathbb{C}$.

Lemma 2.2. *As an $\mathcal{O}$-module, Λ is projective.*

Proof. As an $\mathcal{O}$-module Λ is isomorphic to a finitely generated $\mathcal{O}$-submodule of K, hence to an ideal in $\mathcal{O}$. Every ideal in a Dedekind domain $\mathcal{O}$ is finitely generated projective over $\mathcal{O}$. ☐

Definition 2.3. *Two 1-dimensional K-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable if $K\Lambda_1 = K\Lambda_2$ and $\phi_1 = \phi_2$ modulo $\Lambda_1 + \Lambda_2$.*

One checks easily, as in Proposition 1.11 of [7], that commensurability is indeed an equivalence relation.

Lemma 2.4. *Up to scaling by some $\lambda \in \mathbb{C}^*$, any K-lattice Λ is equivalent to a K-lattice $\Lambda' = \lambda\Lambda \subset K \subset \mathbb{C}$. The lattice Λ' is unique modulo K^*.*

Proof. The K-vector space $K\Lambda$ is 1-dimensional. If ξ is a generator, then $\xi^{-1}\Lambda \subset K$. The remaining ambiguity is only by scaling by elements in K^*. ☐

Proposition 2.5. *For invertible 1-dimensional K-lattices, the element of $K_0(\mathcal{O})$ associated to the $\mathcal{O}$-module Λ is an invariant of the commensurability class up to scaling.*

Proof. Two invertible 1-dimensional K-lattices that are commensurable are in fact equal. The same holds for lattices up to scaling. Thus, the corresponding $\mathcal{O}$-module class is well defined. ☐

There is a canonical isomorphism $K_0(\mathcal{O}) \cong \mathbb{Z} + \mathrm{Cl}(\mathcal{O})$ (*cf.* Corollary 1.11, [14]), where the $\mathbb{Z}$ part is given by the rank, which is equal to one in our case, hence the invariant of Proposition 2.5 is the class in the class group $\mathrm{Cl}(\mathcal{O})$.

In contrast to Proposition 2.5, every 1-dimensional K-lattice is commensurable to a K-lattice whose $\mathcal{O}$-module structure is trivial. This follows, since every ideal in $\mathcal{O}$ is commensurable to $\mathcal{O}$.

Proposition 2.6. *The data (Λ, ϕ) of a 1-dimensional K-lattice are equivalent to data (ρ, s) of an element $\rho \in \hat{\mathcal{O}}$ and $s \in \mathbb{A}_K^*/K^*$, modulo the $\hat{\mathcal{O}}^*$-action given by $(\rho, s) \mapsto (x^{-1}\rho, xs)$, $x \in \hat{\mathcal{O}}^*$. Thus, the space of 1-dimensional K-lattices is given by*

$$\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*). \tag{2.3}$$

Proof. The $\mathcal{O}$-module Λ can be described in the form $\Lambda_s = s_\infty^{-1}(s_f\hat{\mathcal{O}} \cap K)$, where $s = (s_f, s_\infty) \in \mathbb{A}_K^*$. This satisfies $\Lambda_{ks} = \Lambda_s$ for all $k \in (\hat{\mathcal{O}}^* \times 1)\, K^* \subset \mathbb{A}_K^*$. Indeed, up to scaling, Λ can be identified with an ideal in $\mathcal{O}$. These can be written in the form $s_f\hat{\mathcal{O}} \cap K$ (*cf.* [17] §5.2). If $\Lambda_s = \Lambda_{s'}$, then $s'_\infty s_\infty^{-1} \in K^*$

and one is reduced to the condition $s_f\hat{\mathcal{O}} \cap K = s'_f\hat{\mathcal{O}} \cap K$, which implies $s'_f s_f^{-1} \in \hat{\mathcal{O}}^*$. The data ϕ of the 1-dimensional K-lattice can be described by the composite map $\phi = s_\infty^{-1}(s_f \circ \rho)$

$$\begin{tikzcd} K/\mathcal{O} \arrow[r, "\rho"] \arrow[d, "\cong"] & K/\mathcal{O} \arrow[r] \arrow[d, "\cong"] & K/(s_f\hat{\mathcal{O}} \cap K) \arrow[d, "\cong"] \\ \mathbb{A}_{K,f}/\hat{\mathcal{O}} \arrow[r, "\rho"] & \mathbb{A}_{K,f}/\hat{\mathcal{O}} \arrow[r, "s_f"] & \mathbb{A}_{K,f}/s_f\hat{\mathcal{O}} \end{tikzcd} \tag{2.4}$$

where ρ is an element in $\hat{\mathcal{O}}$. By construction the map $(\rho, s) \mapsto (\Lambda_s, s_\infty^{-1}(s_f \circ \rho))$ passes to the quotient $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*)$ and the above shows that it gives a bijection with the space of 1-dimensional K-lattices.

□

Notice that, even though Λ and $\mathcal{O}$ are not isomorphic as $\mathcal{O}$-modules, on the quotients we have $K/\Lambda \simeq K/\mathcal{O}$ as $\mathcal{O}$-modules, with the isomorphism realized by s_f in the diagram (2.4).

Proposition 2.7. *Let $\mathbb{A}^{\cdot}_K = \mathbb{A}_{K,f} \times \mathbb{C}^*$ be the subset of adèles of K with nontrivial archimedean component. The map $\Theta(\rho, s) = \rho\, s$,*

$$\Theta : \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*) \to \mathbb{A}^{\cdot}_K/K^*, \tag{2.5}$$

preserves commensurability and induces an identification of the set of commensurability classes of 1-dimensional K-lattices (not up to scale) with the space $\mathbb{A}^{\cdot}_K/K^*$.

Proof. The map is well defined, because ρs is invariant under the action $(\rho, s) \mapsto (x^{-1}\rho, xs)$ of $x \in \hat{\mathcal{O}}^*$. It is clearly surjective. It remains to show that two K-lattices have the same image if and only if they are in the same commensurability class. First we show that we can reduce to the case of principal K-lattices, without changing the value of the map Θ. Given a K-lattice (Λ, ϕ), we write $\Lambda = \lambda J$, where $J \subset \mathcal{O}$ is an ideal, hence $\Lambda = \lambda(s_f\hat{\mathcal{O}} \cap K)$, where $\lambda = s_\infty^{-1} \in \mathbb{C}^*$ and $s_f \in \hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}$. Then (Λ, ϕ) is commensurate to the principal K-lattice $(\lambda\mathcal{O}, \phi)$. If (ρ, s) is the pair associated to (Λ, ϕ), with $s = (s_f, s_\infty)$ as above, then the corresponding pair (ρ', s') for $(\lambda\mathcal{O}, \phi)$ is given by $\rho' = s_f\,\rho$ and $s' = (1, s_\infty)$. Thus, we have $\Theta(\Lambda, \phi) = \Theta(\lambda\mathcal{O}, \phi)$. We can then reduce to proving the statement in the case of principal K-lattices $(s_\infty^{-1}\mathcal{O}, s_\infty^{-1}\rho)$. In this case, the equality $s_\infty\rho = ks'_\infty\rho'$, for $k \in K^*$, means that we have $s_\infty = ks'_\infty$ and $\rho = k\rho'$. In turn, this is the relation of commensurability for principal K-lattices.

□

Thus, we obtain, for 1-dimensional K-lattices, the following Lemma as an immediate corollary,

Lemma 2.8. *The map defined as $\Upsilon : (\Lambda, \phi) \mapsto \rho \in \hat{\mathcal{O}}/K^*$ for principal K-lattices extends to an identification, given by $\Upsilon : (\Lambda, \phi) \mapsto s_f\,\rho \in \mathbb{A}_{K,f}/K^*$, of the set of commensurability classes of 1-dimensional K-lattices up to scaling with the quotient*

$$\hat{\mathcal{O}}/K^* = \mathbb{A}_{K,f}/K^*. \tag{2.6}$$

2.2. **Algebra of coordinates.**

We now describe the noncommutative algebra of coordinates of the space of commensurability classes of 1-dimensional K-lattices up to scaling.

To this purpose, we first consider the groupoid $\tilde{\mathcal{R}}_K$ of the equivalence relation of commensurability on 1-dimensional K-lattices (not up to scaling). By construction, this groupoid is a subgroupoid of the groupoid $\tilde{\mathcal{R}}$ of commensurability classes of 2-dimensional $\mathbb{Q}$-lattices. Its structure as a locally compact étale groupoid is inherited from this embedding.

The groupoid $\tilde{\mathcal{R}}_K$ corresponds to the quotient $\mathbb{A}^{\cdot}_K/K^*$. Its C^*-algebra is given up to Morita equivalence by the crossed product

$$C_0(\mathbb{A}^{\cdot}_K) \rtimes K^*. \tag{2.7}$$

The case of commensurability classes of 1-dimensional K-lattices up to scaling is more delicate. Its noncommutative algebra of coordinates is given by the algebra $\mathcal{A}_K = C^*(G_K)$ obtained by taking the quotient by scaling $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$ of the groupoid of the equivalence relation of commensurability.

Proposition 2.9. *The quotient $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$ is a groupoid.*

Proof. The simplest way to check this is to write $\tilde{\mathcal{R}}_K$ as the union of the two groupoids $\tilde{\mathcal{R}}_K = G_0 \cup G_1$ corresponding respectively to pairs of commensurable K-lattices (L, L') with $L = (\Lambda, 0), L' = (\Lambda', 0)$ and (L, L') with $L = (\Lambda, \phi \neq 0), L' = (\Lambda', \phi' \neq 0)$. The scaling action of $\mathbb{C}^*$ on G_0 is the identity on $\mathcal{O}^*$ and the corresponding action of $\mathbb{C}^*/\mathcal{O}^*$ is free on the units of G_0. Thus the quotient $G_0/\mathbb{C}^*$ is a groupoid. Similarly the action of $\mathbb{C}^*$ on G_1 is free on the units of G_1 and the quotient $G_1/\mathbb{C}^*$ is a groupoid. $\square$

The quotient topology turns G_K into a locally compact étale groupoid. The algebra of coordinates $\mathcal{A}_K = C^*(G_K)$ is the convolution algebra of weight zero functions on the groupoid $\tilde{\mathcal{R}}_K$ of the equivalence relation of commensurability on K-lattices. Elements in the algebra are functions of pairs $(\Lambda, \phi), (\Lambda', \phi')$ of commensurable 1-dimensional K-lattices satisfying, for all $\lambda \in \mathbb{C}^*$,

$$f(\lambda(\Lambda, \phi), \lambda(\Lambda', \phi')) = f((\Lambda, \phi), (\Lambda', \phi')).$$

Lemma 2.10. *The algebra $\mathcal{A}_K$ is unital.*

Proof. The set $G_K^{(0)}$ of units of G_K is the quotient of $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_K/K^*)$ by the action of $\mathbb{C}^*$. This gives the compact space

$$X = G_K^{(0)} = \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}^*_{K,f}/K^*). \tag{2.8}$$

Notice that $\mathbb{A}^*_{K,f}/(K^* \times \hat{\mathcal{O}}^*)$ is $\mathrm{Cl}(\mathcal{O})$. Since the set of units is compact the convolution algebra is unital.

$\square$

There is a homomorphism $\mathfrak{n}$ from the groupoid $\tilde{\mathcal{R}}_K$ to $\mathbb{R}^*_+$ given by the covolume of a commensurable pair of K-lattices. More precisely given such a pair $(L, L') = ((\Lambda, \phi), (\Lambda', \phi'))$ we let

$$|L/L'| = \mathrm{covolume}(\Lambda')/\mathrm{covolume}(\Lambda) \tag{2.9}$$

This is invariant under scaling both lattices, so it is defined on $G_K = \tilde{\mathcal{R}}_K/\mathbb{C}^*$. Up to scale, we can identify the lattices in a commensurable pair with ideals in $\mathcal{O}$. The covolume is then given by the ratio of the norms. This defines a time evolution on the algebra $\mathcal{A}_K$ by

$$\sigma_t(f)(L, L') = |L/L'|^{it} f(L, L'). \tag{2.10}$$

We construct representations for the algebra $\mathcal{A}_K$. For an étale groupoid G_K, every unit $y \in G_K^{(0)}$ defines a representation π_y by left convolution of the algebra of G_K in the Hilbert space $\mathcal{H}_y = \ell^2((G_K)_y)$, where $(G_K)_y$ is the set of elements with source y. The representations corresponding to points that have a nontrivial automorphism group will no longer be irreducible. As in the GL_2-case, this defines the norm on $\mathcal{A}_K$ as

$$\|f\| = \sup_y \|\pi_y(f)\|. \tag{2.11}$$

Lemma 2.11. (1) *Given an invertible K-lattice (Λ, ϕ), the map*

$$(\Lambda', \phi') \mapsto J = \{x \in \mathcal{O} | x\Lambda' \subset \Lambda\} \tag{2.12}$$

gives a bijection of the set of K-lattices commensurable to (Λ, ϕ) with the set of ideals in $\mathcal{O}$.

(2) *Invertible K-lattices define positive energy representations.*
(3) *The partition function is the Dedekind zeta function $\zeta_K(\beta)$ of K.*

Proof. (1) As in Theorem 1.26 of [7], we use the fact (Lemma 1.27 of [7]) that, if Λ is an invertible 2-dimensional $\mathbb{Q}$-lattice and Λ' is commensurable to Λ, then $\Lambda \subset \Lambda'$. The map above is well defined, since $J \subset \mathcal{O}$ is an ideal. Moreover, $J\Lambda' = \Lambda$, since $\mathcal{O}$ is a Dedekind domain. The map is injective, since J determines Λ' as the $\mathcal{O}$-module $\{x \in \mathbb{C} | xJ \subset \Lambda\}$ and the corresponding ϕ and ϕ' agree. This also shows that the map is surjective. We then use the notation

$$J^{-1}(\Lambda, \phi) = (\Lambda', \phi). \tag{2.13}$$

(2) For an invertible K-lattice, the above gives an identification of $(G_K)_y$ with the set $\mathcal{J}$ of ideals $J \subset \mathcal{O}$. The covolume is then given by the norm. The corresponding Hamiltonian is of the form

$$H\, \epsilon_J = \log \mathfrak{n}(J)\, \epsilon_J, \tag{2.14}$$

with non-negative eigenvalues.
(3) The partition function of the system $(\mathcal{A}_K, \sigma_t)$ is then given by the Dedekind zeta function

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \sum_{J \text{ ideal in } \mathcal{O}} \mathfrak{n}(J)^{-\beta} = \zeta_K(\beta). \tag{2.15}$$

□

We give a more explicit description of the action $L \mapsto J^{-1}L$ on K-lattices, for J an ideal of $\mathcal{O}$. This will be useful later.

Proposition 2.12. *Let $L = (\Lambda, \phi)$ be a K-lattice and $J \subset \mathcal{O}$ an ideal. If L is represented by a pair (ρ, s), then $J^{-1}L$ is represented by the commensurable pair $(s_J\rho, s_J^{-1}s)$, where s_J is a finite idèle such that $J = s_J\hat{\mathcal{O}} \cap K$.*

Proof. By [17] §5.2, there is a finite idèle s_J, such that we can write the ideal J in the form $J = s_J\hat{\mathcal{O}} \cap K$. The pair $(s_J\rho, s_J^{-1}s)$ defines an element in $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$. In fact, first notice that s_J is in fact in $\hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*$, hence the product $s_J\rho \in \hat{\mathcal{O}}$. It is well defined modulo $\hat{\mathcal{O}}^*$, and by direct inspection one sees that the class it defines in $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$ is that of $J^{-1}L$. By Proposition 2.7, in order to check that the K-lattices (ρ, s) and $(s_J\rho, s_J^{-1}s)$ lie in the same commensurability class, it is sufficient to see that $\Theta(\rho, s) = \rho s = \Theta(s_J\rho, s_J^{-1}s)$.

□

2.3. **Symmetries of the system.**

Recall the following general facts about symmetries of a quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$, with a unital C^*-algebra $\mathcal{A}$ and a 1-parameter group of automorphisms σ_t, $(t \in \mathbb{R})$.

Definition 2.13. *An element $g \in \mathrm{Aut}(\mathcal{A})$ acts as an automorphism of $(\mathcal{A}, \sigma_t)$ if $g\sigma_t = \sigma_t g$, for all $t \in \mathbb{R}$. A unitary $u \in \mathcal{A}$ such that $\sigma_t(u) = u$, for all $t \in \mathbb{R}$, acts as an inner automorphism of $(\mathcal{A}, \sigma_t)$, by*

$$(\mathrm{Ad}u)\,(a) := u\,a\,u^*\,, \quad \forall a \in \mathcal{A}\,. \tag{2.16}$$

A $*$-homomorphism $\rho : \mathcal{A} \to \mathcal{A}$ acts as an endomorphism of $(\mathcal{A}, \sigma_t)$ if $\rho\sigma_t = \sigma_t\rho$, for all $t \in \mathbb{R}$. An isometry $u \in \mathcal{A}$, $u^*\,u = 1$, satisfying $\sigma_t(u) = \lambda^{it}\,u$, for all $t \in \mathbb{R}$ and for some $\lambda \in \mathbb{R}_+^*$, defines an inner endomorphism of $(\mathcal{A}, \sigma_t)$, again of the form (2.16).

In the case of the system $(\mathcal{A}_K, \sigma_t)$ defined above, we have the following symmetries.

Proposition 2.14. *The semigroup $\hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*$ acts on the algebra $\mathcal{A}_K$ by endomorphisms. The subgroup $\hat{\mathcal{O}}^*$ acts on $\mathcal{A}_K$ by automorphisms. The subsemigroup $\mathcal{O}^\times$ acts by inner endomorphisms.*

Proof. Given an ideal $J \subset \mathcal{O}$, consider the set of K-lattices (Λ, ϕ) such that ϕ is well defined modulo $J\Lambda$. Namely, the map $\phi : K/\mathcal{O} \longrightarrow K\Lambda/\Lambda$ factorises as $K/\mathcal{O} \longrightarrow K\Lambda/J\Lambda \to K\Lambda/\Lambda$. We say, in this case, that the K-lattice (Λ, ϕ) is divisible by J. The above condition gives a closed and open subset of the set of K-lattices up to scaling. We denote by $e_J \in \mathcal{A}_K$ the corresponding idempotent. Let $s \in \hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}$. Let $J = s\hat{\mathcal{O}} \cap K$. Given a commensurable pair (Λ, ϕ) and (Λ', ϕ'), and an element $f \in \mathcal{A}_K$, we define

$$\theta_s(f)((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} f((\Lambda, s^{-1}\phi), (\Lambda', s^{-1}\phi')) & \text{both } K\text{-lattices are divisible by } J \\ 0 & \text{otherwise.} \end{cases} \tag{2.17}$$

Formula (2.17) defines an endomorphism of $\mathcal{A}_K$ with range the algebra reduced by e_J. It is, in fact, an isomorphism with the reduced algebra. Clearly, for $s \in \hat{\mathcal{O}}^*$, the above defines an automorphism. For $s \in \mathcal{O}^\times$, the endomorphism (2.17) is inner. In fact, for $s \in \mathcal{O}^\times$, let $\mu_s \in \mathcal{A}_K$ be given by

$$\mu_s((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} 1 & \Lambda = s^{-1}\Lambda' \text{ and } \phi' = \phi \\ 0 & \text{otherwise.} \end{cases} \tag{2.18}$$

Then the range of μ_s is the projection e_J, where J is the principal ideal generated by s. Then we have

$$\theta_s(f) = \mu_s \, f \, \mu_s^*, \quad \forall s \in \mathcal{O}^\times.$$

The action of symmetries $\hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}$ is compatible with the time evolution,

$$\theta_s \, \sigma_t = \sigma_t \, \theta_s, \qquad \forall s \in \hat{\mathcal{O}} \cap \mathbb{A}^*_{K,f}, \, \forall t \in \mathbb{R}.$$

The isometries μ_s are eigenvectors of the time evolution, namely

$$\sigma_t(\mu_s) = \mathfrak{n}(s)^{it} \, \mu_s.$$

□

2.4. **KMS states and symmetries.**

We recall the definition of KMS states for a quantum statistical mechanical system $(\mathcal{A}, \sigma_t)$, (*cf.* [5], [10]) and their induced symmetries.

Definition 2.15. *Suppose given a triple $(\mathcal{A}, \sigma_t, \varphi)$, with φ a state on the algebra $\mathcal{A}$. The Kubo-Martin-Schwinger (KMS) condition at inverse temperature β is defined as follows.*

(1) *Assume $0 \le \beta < \infty$. The state φ is a KMS$_\beta$ state if, for all $x, y \in \mathcal{A}$, there exists a holomorphic function $F_{x,y}(z)$ on the strip $0 < \mathrm{Im}(z) < \beta$, which extends as a continuous function to the boundary of the strip, with the property that*

$$F_{x,y}(t) = \varphi(x\sigma_t(y)) \quad \text{and} \quad F_{x,y}(t + i\beta) = \varphi(\sigma_t(y)x), \qquad \forall t \in \mathbb{R}. \tag{2.19}$$

(2) *Assume $\beta = \infty$. The state φ is a KMS$_\infty$ state if it is a weak limit of KMS$_\beta$ states. Namely, for all $a \in \mathcal{A}$,*

$$\varphi_\infty(a) = \lim_{\beta \to \infty} \varphi_\beta(a). \tag{2.20}$$

For any given $\beta < \infty$, the set of KMS$_\beta$ states is a compact convex Choquet simplex [5, II §5] whose set of extreme points $\mathcal{E}_\beta$ consists of the factor states. One can express any KMS$_\beta$ state uniquely in terms of extremal states, because of the uniqueness of the barycentric decomposition of a Choquet simplex. With the definition above, the set of KMS$_\infty$ states is a weakly compact convex set, so that we can still consider the set $\mathcal{E}_\infty$ of its extremal points.

Symmetries $\mathrm{Aut}(A, \sigma_t)$ and $\mathrm{End}(\mathcal{A}_t, \sigma_t)$ of the system, as in Definition 2.13, induce symmetries of the KMS states as follows (*cf.* [7]).

Lemma 2.16. *Suppose given $(\mathcal{A}_t, \sigma_t)$, with a unital C^*-algebra $\mathcal{A}$ and a 1-parameter family of automorphisms σ_t. Assume that, for sufficiently large β, the map*

$$W_\beta(\varphi)(a) = \frac{\mathrm{Tr}(\pi_\varphi(a)\, e^{-\beta H})}{\mathrm{Tr}(\, e^{-\beta H})}\,, \quad \forall a \in \mathcal{A} \tag{2.21}$$

is a bijection $W_\beta : \mathcal{E}_\infty \to \mathcal{E}_\beta$.

(1) *The induced action*
$$\varphi \mapsto g^*(\varphi) := \varphi \circ g, \tag{2.22}$$
for $g \in \mathrm{Aut}(A, \sigma_t)$ and $\varphi \in \mathcal{E}_\beta$, descends to an action of the quotient group $\mathrm{Aut}(A, \sigma_t)/\mathcal{U}$.

(2) *For $\varphi \in \mathcal{E}_\beta$, consider endomorphisms $\rho \in \mathrm{End}(\mathcal{A}_t, \sigma_t)$ such that $\varphi(\rho(1)) \neq 0$. Then the induced action on KMS$_\beta$ states is given by*
$$\varphi \mapsto \rho^*(\varphi) := \frac{\varphi \circ \rho}{\varphi(\rho(1))} \tag{2.23}$$
when $\beta < \infty$ and by
$$\rho^*(\varphi)(a) := \lim_{\beta\to\infty} \rho^*(W_\beta(\varphi))(a) \qquad \forall a \in \mathcal{A}, \tag{2.24}$$
when $\varphi \in \mathcal{E}_\infty$.

(3) *The action on KMS states of $\rho \in \mathrm{End}(\mathcal{A}_t, \sigma_t)$ descends to an action modulo the inner action (2.16) of isometries that are eigenvectors of the time evolution, as in Definition 2.13.*

Proof. The argument is the same as in [7]. One verifies using the KMS condition that the inner action (2.16) is trivial on KMS states, both in the case of automorphisms and of endomorphisms. □

In the case of the system $(\mathcal{A}_K, \sigma_t)$ for an imaginary quadratic field K, we have the following symmetries of KMS states.

Proposition 2.17. *The quotient C_K/D_K of the idèle class group C_K of K by the connected component of identity D_K acts as symmetries of the KMS states of the system $(\mathcal{A}_K, \sigma_t)$. The action of the subgroup $\hat{\mathcal{O}}^*/\mathcal{O}^*$ is by automorphisms.*

Proof. Recall that $\mathbb{A}_{K,f} = \hat{\mathcal{O}}.K^*$. Thus, we can pass to the corresponding group of symmetries, modulo inner, which is given by the group $\mathbb{A}^*_{K,f}/K^*$, which is isomorphic to C_K/D_K (*cf.* [17] §5, [2] §9). We have an exact sequence of groups

$$1 \to \hat{\mathcal{O}}^*/\mathcal{O}^* \to \mathbb{A}^*_{K,f}/K^* \to \mathrm{Cl}(\mathcal{O}) \to 1,$$

where $\mathrm{Cl}(\mathcal{O})$ is the class group of the ring $\mathcal{O}$, with $\#\mathrm{Cl}(\mathcal{O}) = h_K$, the class number of K. □

This shows that, in the very special case of class number $h_K = 1$, symmetries are given only through an action by automorphisms, as in the original case of the Bost–Connes system. In the case where $h_K \neq 1$, the nontrivial elements of $\mathrm{Cl}(\mathcal{O})$ have representatives in $\mathbb{A}^*_{K,f}/K^*$ that act by endomorphisms.

In order to compute the value of KMS states on the projection e_J associated to an ideal J of the ring $\mathcal{O}$ of integers (*i.e.* the characteristic function of the set of K-lattices divisible by J) we introduce an isometry $\mu_J \in \mathcal{A}_K$ such that its range is e_J. This isometry is simply given with our notations by

$$\mu_J((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} 1 & \Lambda = J^{-1}\Lambda' \text{ and } \phi' = \phi \\ 0 & \text{otherwise.} \end{cases} \tag{2.25}$$

which is similar to equation (2.18) and reduces to that one when J is principal (generated by s). Thus, this would seem to imply that it is not only the subsemigroup $\mathcal{O}^\times$ that acts by inner endomorphisms, but in fact a bigger one, using the isometries $\mu_J \in \mathcal{A}_K$. To see what happens, one needs to compare the endomorphism $f \to \mu_J f \mu_J^*$ with the endomorphism θ_s. In the first case one gets

$$\mu_J f \mu_J^*((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} f((J\,\Lambda, \phi), (J\,\Lambda', \phi')) & \text{both } K\text{-lattices are divisible by } J \\ 0 & \text{otherwise,} \end{cases} \tag{2.26}$$

while in the second case one gets the formula (2.17) *i.e.*

$$\theta_s(f)((\Lambda,\phi),(\Lambda',\phi')) = \begin{cases} f((\Lambda, s^{-1}\phi),(\Lambda', s^{-1}\phi')) & \text{both } K\text{-lattices are divisible by } J \\ 0 & \text{otherwise.} \end{cases} \tag{2.27}$$

The key point here is that the scaling is only allowed by elements of K^* and the scaling relation between lattices $(s\,\Lambda,\phi)$ and $(\Lambda, s^{-1}\phi)$ holds only for $s \in K^*$, but not for ideles. Thus, even though the μ_J always exist (for any ideal), they implement the endomorphism θ_s only in the case where J is principal.

2.5. **The arithmetic subalgebra.**

In order to show that the system $(\mathcal{A}_K, \sigma_t)$ solves Problem 1.1, we need to identify a suitable arithmetic subalgebra $\mathcal{A}_{K,0}$ of $\mathcal{A}_K$.

The algebra $\mathcal{A}_{K,0}$ is obtained by embedding the system $(\mathcal{A}_K, \sigma_t)$ as a sub-system of the GL_2-system of [7] and restricting the arithmetic algebra of the GL_2-system to the subgroupoid of commensurability classes of 1-dimensional K-lattices up to scale.

We recall the notion of 2-dimensional $\mathbb{Q}$-lattices and commensurability.

Definition 2.18. *A 2-dimensional $\mathbb{Q}$-lattice is a pair (Λ,ϕ), with Λ a lattice in $\mathbb{C}$, and*

$$\phi : \mathbb{Q}^2/\mathbb{Z}^2 \longrightarrow \mathbb{Q}\Lambda/\Lambda \tag{2.28}$$

a homomorphism of abelian groups. A $\mathbb{Q}$-lattice is invertible if the map (2.28) *is an isomorphism. Two $\mathbb{Q}$-lattices (Λ_1,ϕ_1) and (Λ_2,ϕ_2) are commensurable if the lattices are commensurable (i.e. $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$) and the maps agree modulo the sum of the lattices,*

$$\phi_1 \equiv \phi_2 \mod \Lambda_1 + \Lambda_2.$$

Let $\mathcal{O} = \mathbb{Z} + \mathbb{Z}\tau$ be the ring of integers of an imaginary quadratic field $K = \mathbb{Q}(\tau)$. The choice of such a $\tau \in \mathbb{H}$ determines an embedding

$$q_\tau : K \hookrightarrow M_2(\mathbb{Q}). \tag{2.29}$$

The image of its restriction $q_\tau : K^* \hookrightarrow \mathrm{GL}_2^+(\mathbb{Q})$ is characterized by the property that (*cf.* [17] Proposition 4.6)

$$q_\tau(K^*) = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : \ g(\tau) = \tau\}. \tag{2.30}$$

For $g = q_\tau(x)$ with $x \in K^*$, we have

$$\det(g) = \mathfrak{n}(x), \tag{2.31}$$

where $\mathfrak{n} : K^* \to \mathbb{Q}^*$ is the norm map.

The relation between 1-dimensional K-lattices and 2-dimensional $\mathbb{Q}$-lattices is explained in the following Lemma.

Lemma 2.19. *A 1-dimensional K-lattice is, in particular, a 2-dimensional $\mathbb{Q}$-lattice. Two 1-dimensional K-lattices are commensurable iff the underlying $\mathbb{Q}$-lattices are commensurable.*

Proof. First notice that $K\Lambda = \mathbb{Q}\Lambda$, since $\mathbb{Q}\mathcal{O} = K$. This, together with $\Lambda \otimes_{\mathcal{O}} K \cong K$, shows that the $\mathbb{Q}$-vector space $\mathbb{Q}\Lambda$ is 2-dimensional. Since $\mathbb{R}\Lambda = \mathbb{C}$, and Λ is finitely generated as an abelian group, this shows that Λ is a lattice. The basis $\{1,\tau\}$ provides an identification of $K/\mathcal{O}$ with $\mathbb{Q}^2/\mathbb{Z}^2$, so that we can view ϕ as a homomorphism of abelian groups $\phi : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}\Lambda/\Lambda$. The pair (Λ,ϕ) thus gives a two dimensional $\mathbb{Q}$-lattice.

The second statement holds, since for 1-dimensional K-lattices we have $K\Lambda = \mathbb{Q}\Lambda$. ∎

The GL_2-system is constructed in [7] by first considering the groupoid $\mathcal{R}_2$ of the equivalence relation of commensurability on the set of 2-dimensional $\mathbb{Q}$-lattices. This is a locally compact étale groupoid. One then takes the quotient with respect to the scaling action of $\mathbb{C}^*$. Unlike what happens in Proposition 2.9, the quotient $\mathcal{R}_2/\mathbb{C}^*$ is *not* a groupoid. However, one can still define a convolution algebra associated to $\mathcal{R}_2/\mathbb{C}^*$ by restricting the convolution product of the algebra of $\mathcal{R}_2$ to weight zero functions with $\mathbb{C}^*$-compact support. Elements of the resulting algebra $\mathcal{A}_2$ are functions of pairs of commensurable 2-dimensional $\mathbb{Q}$-lattices, invariant under the scaling action,

$$f(\lambda(\Lambda,\phi),\lambda(\Lambda',\phi')) = f((\Lambda,\phi),(\Lambda',\phi')), \qquad \forall \lambda \in \mathbb{C}^*.$$

The following result is a direct consequence of Lemma 2.19.

Lemma 2.20. *The groupoid $\tilde{\mathcal{R}}_K$ of the equivalence relation of commensurability of 1-dimensional K-lattices is a subgroupoid of the groupoid $\mathcal{R}_2$ of commensurability of 2-dimensional $\mathbb{Q}$-lattices. Its structure as a locally compact étale groupoid is inherited from this embedding.*

More explicitly, the convolution algebra $\mathcal{A}_2$ is the Hecke algebra of functions on

$$\mathcal{U} = \{(g,\alpha,z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H},\ g\alpha \in M_2(\hat{\mathbb{Z}})\} \tag{2.32}$$

invariant under the $\Gamma \times \Gamma$ action

$$(g,\alpha,z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2\alpha, \gamma_2(z)), \tag{2.33}$$

with convolution

$$(f_1 * f_2)(g,\alpha,z) = \sum_{s\in\Gamma\backslash\mathrm{GL}_2^+(\mathbb{Q}),\, s\alpha\in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\alpha, s(z))\, f_2(s,\alpha,z) \tag{2.34}$$

and adjoint $f^*(g,\alpha,z) = \overline{f(g^{-1}, g\alpha, g(z))}$.

The C^*-algebra completion of $\mathcal{A}_2$ is taken with respect to the sup of the norms in the representations π_y by left convolution on the Hilbert space $\mathcal{H}_y = \ell^2(\mathcal{R}_y)$, where $\mathcal{R}_y$ is the set of elements with source $y \in \mathcal{R}^{(0)}$. (We refer the reader to [7] for details on the definition and properties of this algebra.)

The time evolution of the GL_2-system is given by the covolume of a commensurable pair of 2-dimensional $\mathbb{Q}$-lattices, that is,

$$\sigma_t(f)(g,\alpha,\tau) = \det(g)^{it} f(g,\alpha,\tau), \tag{2.35}$$

where, for the pair of commensurable $\mathbb{Q}$-lattices associated to the data (g,α,τ), one has

$$\det(g) = \mathrm{covolume}(\Lambda')/\mathrm{covolume}(\Lambda). \tag{2.36}$$

Lemma 2.21. *The time evolution* (2.10) *of* $(\mathcal{A}_K,\sigma_t)$ *is the restriction to* $\mathcal{A}_K$ *of the time evolution* (2.35) *of the* GL_2*-system.*

Proof. By restriction from the GL_2-system, there is a homomorphism $\mathfrak{n}$ from the groupoid $\tilde{\mathcal{R}}_K$ to $\mathbb{R}_+^*$ given by the covolume of a commensurable pair of K-lattices. Thus, by (2.31), the time evolution of the system $(\mathcal{A}_K,\sigma_t)$ is the restriction to $\mathcal{A}_K$ of the time evolution of the GL_2-system. □

The modular field F is the field of modular functions over $\mathbb{Q}^{ab}$ (*cf. e.g.* [13]). This is the union of the fields F_N of modular functions of level N rational over the cyclotomic field $\mathbb{Q}(\zeta_N)$, that is, such that the q-expansion at a cusp has coefficients in the cyclotomic field $\mathbb{Q}(\zeta_N)$.

The arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ of the GL_2-system is defined as follows (*cf.* [7]).

Definition 2.22. *Let $\mathcal{A}_{2,\mathbb{Q}}$ be the subalgebra of continuous functions on the quotient $\mathcal{R}_2/\mathbb{C}^*$, with the convolution product* (2.34)*, which satisfy the following properties.*

- *The support of f in $\Gamma\backslash\mathrm{GL}_2^+(\mathbb{Q})$ is finite.*
- *The function f is of finite level.*
- *The functions $f_{(g,m)}$ satisfy $f_{(g,m)} \in F$, for all (g,m).*

- *The function f satisfies the* cyclotomic condition*:*
$$f_{(g,\alpha(u)m)} = \mathrm{cycl}(u)\, f_{(g,m)},$$
for all $g \in \mathrm{GL}_2^+(\mathbb{Q})$ diagonal and all $u \in \hat{\mathbb{Z}}^*$, with
$$\alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

Here $\mathrm{cycl} : \hat{\mathbb{Z}}^* \to \mathrm{Aut}(F)$ *is the action of* $\hat{\mathbb{Z}}^* \simeq \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ *on the coefficients of the q-expansion.*

The definition of this algebra is very natural for the GL_2-system. The conditions listed above amount to imposing the simplest possible "algebraicity condition" on the unique continuous modulus (the q-parameter) that functions in the algebra depend on. The cyclotomic condition is a consistency condition on the roots of unity that appear in the coefficients of the q-series, which eliminates trivial cases that would not behave well with respect to a Galois action on the values of states.

Notice that the condition that the functions f have finite support in the left coset space of the action of Γ on $\mathrm{GL}_2(\mathbb{Q})$ is compatible with the right Γ-invariance, because the inclusion $\mathrm{SL}_2(\mathbb{Z}) \subset \mathrm{GL}_2(\mathbb{Q})$ is quasi-normal, so that the image under the projection onto the left coset of a right Γ-orbit is finite.

We define the arithmetic subalgebra of the system $(\mathcal{A}_K, \sigma_t)$ as follows.

Definition 2.23. *The algebra $\mathcal{A}_{K,0}$ is the K-algebra obtained by*

$$\mathcal{A}_{K,0} = \mathcal{A}_{2,\mathbb{Q}}|_{G_K} \otimes_{\mathbb{Q}} K. \tag{2.37}$$

Here $\mathcal{A}_{2,\mathbb{Q}}|_{G_K}$ denotes the $\mathbb{Q}$-algebra obtained by restricting elements of the algebra $\mathcal{A}_{2,\mathbb{Q}}$ of Definition 2.22 to the $\mathbb{C}^*$-quotient G_K of the subgroupoid $\tilde{\mathcal{R}}_K \subset \tilde{\mathcal{R}}_2$.

Since $\mathcal{A}_K$ is unital, one sees easily that $\mathcal{A}_{K,0}$ is a subalgebra of $\mathcal{A}_K$, even though $\mathcal{A}_{2,\mathbb{Q}}$ is only an algebra of unbounded multipliers of $\mathcal{A}_2$ (*cf.* [7]).

3. KMS states and class field theory for imaginary quadratic fields

In this section we prove that the system $(\mathcal{A}_K, \sigma_t)$ gives a complete solution to Problem 1.1 for the imaginary quadratic field K.

Theorem 3.1. *Consider the system $(\mathcal{A}_K, \sigma_t)$ described in the previous section. The extremal KMS states of this system satisfy:*

- *In the range $0 < \beta \leq 1$ there is a unique KMS state.*
- *For $\beta > 1$, extremal KMS$_\beta$ states are parameterized by invertible K-lattices,*
$$\mathcal{E}_\beta \simeq \mathbb{A}^*_{K,f}/K^* \tag{3.1}$$
with a free and transitive action of $C_K/D_K \cong \mathbb{A}^*_{K,f}/K^*$ as symmetries.
- *In this range, the extremal KMS$_\beta$ state associated to an invertible K-lattice $L = (\Lambda, \phi)$ is of the form*
$$\varphi_{\beta,L}(f) = \zeta_K(\beta)^{-1} \sum_{J \text{ ideal in } \mathcal{O}} f(J^{-1}L, J^{-1}L)\, \mathfrak{n}(J)^{-\beta}, \tag{3.2}$$
where $\zeta_K(\beta)$ is the Dedekind zeta function, and $J^{-1}L$ defined as in (2.13)*.*
- *The set of extremal KMS$_\infty$ states (as weak limits of KMS$_\beta$ states) is still given by* (3.1)*.*
- *The extremal KMS$_\infty$ states $\varphi_{\infty,L}$ of the CM system, evaluated on the arithmetic subalgebra $\mathcal{A}_{K,0}$, take values in K^{ab}, with $\varphi_{\infty,L}(\mathcal{A}_{K,0}) = K^{ab}$.*
- *The class field theory isomorphism* (1.1) *intertwines the action of $\mathbb{A}^*_{K,f}/K^*$ by symmetries of the system $(\mathcal{A}_K, \sigma_t)$ and the action of* $\mathrm{Gal}(K^{ab}/K)$ *on the image of $\mathcal{A}_{K,0}$ under the extremal KMS$_\infty$ states. Namely, for all $\varphi_{\infty,L} \in \mathcal{E}_\infty$ and for all $f \in \mathcal{A}_{K,0}$,*
$$\alpha(\varphi_{\infty,L}(f)) = (\varphi_{\infty,L} \circ \theta^{-1}(\alpha))(f), \quad \forall \alpha \in \mathrm{Gal}(K^{ab}/K). \tag{3.3}$$

The proof of Theorem 3.1 is given in the following subsections.

3.1. **KMS states at low temperature.**

The partition function $Z_K(\beta)$ of (2.15) converges for $\beta > 1$. We have also seen in the previous section that invertible K-lattices $L = (\Lambda, \phi)$ determine positive energy representations of $\mathcal{A}_K$ on the Hilbert space $\mathcal{H} = \ell^2(\mathcal{J})$ where $\mathcal{J}$ is the set of ideals of $\mathcal{O}$. Thus, the formula

$$\varphi_{\beta,L}(f) = \frac{\mathrm{Tr}\,(\pi_L(f)\,\exp(-\beta H))}{\mathrm{Tr}(\exp(-\beta H))} \tag{3.4}$$

defines an extremal KMS_β state, with the Hamiltonian H of (2.14). These states are of the form (3.2). It is not hard to see that distinct elements in $\mathbb{A}_{K,f}/K^*$ define distinct states $\varphi_{\beta,L}$.

This shows that we have an injection of $\mathbb{A}_{K,f}/K^* \subset \mathcal{E}_\beta$. We need to show that, conversely, every extremal KMS_β state is of the form (3.2).

In order to prove the second and third statements of Theorem 3.1 we shall proceed in two steps. The first shows (Proposition 3.4 below) that KMS_β states are given by measures on the space X of K-lattices (up to scaling), as in (3.5) below. The second shows that when $\beta > 1$ this measure is carried by the commensurability classes of invertible K-lattices.

Let p denotes the projection from K-lattices to their class $p(L) \in X$ modulo scaling, which we can write as the projection

$$p : \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_K^*/K^*) \to \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} (\mathbb{A}_{K,f}^*/K^*).$$

We obtain the following result.

Lemma 3.2. *Let $\gamma = (L, L') \in \tilde{\mathcal{R}}_K$ with $p(L) = p(L') \in X$. Then either $L = L'$ or $\phi = \phi' = 0$.*

Proof. We first describe the elements $\gamma \in G_K$ such that $s(\gamma) = r(\gamma)$ *i.e.* $\gamma = (L, L') \in \tilde{\mathcal{R}}_K$ such that the classes of L and L' modulo the scaling action of $\mathbb{C}^*$ are the same elements of the set $X = G_K^{(0)}$ of (2.8). Modulo scaling, we can assume that the lattice $\Lambda \subset K$. Thus, since L and L' are commensurable, it follows that $\Lambda' \subset K$. Then, by hypothesis, there exists $\lambda \in \mathbb{C}^*$ such that $\lambda L = L'$. One has $\lambda \in K^*$ and $\phi' = \lambda\phi$. By the commensurability of the pair one also has $\phi' = \phi$ modulo $\Lambda + \Lambda'$. Writing $\lambda = \frac{a}{b}$, with $a, b \in \mathcal{O}$, we get $a\phi = b\phi$ and $\lambda = 1$ unless $\phi = 0$. $\square$

We let F be the finite closed subset of X given by the set of K-lattices up to scaling such that $\phi = 0$. Its cardinality is the class number of K. The groupoid G_K is the union $G_K = G_0 \cup G_1$ of the reduced groupoids by $F \subset X$ and its complement.

Lemma 3.3. *Let $\gamma \in G_1 \backslash G_1^{(0)}$. There exists a neighborhood V of γ in G_K such that*

$$r(V) \cap s(V) = \emptyset$$

where r and s are the range and source maps of G_K.

Proof. Let γ be the class modulo scaling of the commensurable pair (L, L'). By Lemma 3.2 one has $p(L) \neq p(L') \in X$. Let $c(\Lambda)$ and $c(\Lambda')$ denote the classes of Λ and Λ' in $K_0(\mathcal{O})$ (*cf.* Proposition 2.5). If these classes are different, $c(\Lambda) \neq c(\Lambda')$, then one can simply take a neighborhood V in such a way that all elements $\gamma_1 = (L_1, L_1') \in V$ are in the corresponding classes: $c(\Lambda_1) = c(\Lambda)$, $c(\Lambda_1') = c(\Lambda')$, for $L_1 = (\Lambda_1, \phi_1)$ and $L_1' = (\Lambda_1', \phi_1')$. This ensures that range and source are disjoint sets, $r(V) \cap s(V) = \emptyset$. Otherwise, there exists $\lambda \in K^*$ such that $\Lambda' = \lambda\Lambda$. Since $\Lambda' \neq \Lambda$, one has $\lambda \notin \mathcal{O}^*$. One has $\phi' = \phi \neq 0$. Thus, one is reduced to showing that, given $\rho \in \hat{\mathcal{O}}$, $\rho \neq 0$, and $\lambda \in K^*$, $\lambda \notin \mathcal{O}^*$, there exists a neighborhood W of ρ in $\hat{\mathcal{O}}$ such that $\lambda W \cap \mathcal{O}^* W = \emptyset$. This follows, using a place v such that $\rho_v \neq 0$. In fact, one has $\lambda\rho_v \notin \mathcal{O}^*\rho_v$ and the same holds in a suitable neighborhood. $\square$

We can now prove the following.

Proposition 3.4. *Let $\beta > 0$ and φ a KMS_β state on $(\mathcal{A}_K, \sigma_t)$. Then there exists a probability measure μ on X such that*

$$\varphi(f) = \int_X f(L, L)\, d\mu(L)\,, \quad \forall f \in \mathcal{A}_K\,. \tag{3.5}$$

Proof. It is enough to show that $\varphi(f) = 0$ provided f is a continuous function with compact support on G_K with support disjoint from $G_K^{(0)}$. Let $h_n \in C(X)$, $0 \leq h_n \leq 1$ with support disjoint from F and converging pointwise to 1 in the complement of F. Let $u_n \in \mathcal{A}_K$ be supported by the diagonal and given by h_n there.

The formula

$$\Phi(f)(\Lambda, \Lambda') := f((\Lambda, 0), (\Lambda', 0))) \quad \forall f \in \mathcal{A}_K \tag{3.6}$$

defines a homomorphism of $(\mathcal{A}_K, \sigma_t)$ to the C^* dynamical system $(C^*(G_0), \sigma_t)$ obtained by specialization to pairs of K-lattices with $\phi = 0$ as in [7].

Since there are unitary eigenvectors for σ_t for non trivial eigenvalues in the system $(C^*(G_0), \sigma_t)$ it has no non-zero KMS$_\beta$ positive functional. This shows that the pushforward of φ by Φ vanishes and by Proposition 1.5 of [7] that, with the notation introduced above,

$$\varphi(f) = \lim_n \varphi(f * u_n).$$

Thus, since $(f * u_n)(\gamma) = f(\gamma)\, h_n(s(\gamma))$, we can assume that $f(\gamma) = 0$ unless $s(\gamma) \in C$, where $C \subset X$ is a compact subset disjoint from F. Let $L \in C$ and V as in lemma 3.3 and let $h \in C_c(V)$. Then, upon applying the KMS$_\beta$ condition to the pair a, b with $a = f$ and b supported by the diagonal and equal to h there. One gets $\varphi(b * f) = \varphi(f * b)$. One has $(b * f)(\gamma) = h(r(\gamma))\, f(\gamma)$. Applying this to $f * b$ instead of f and using $h(r(\gamma))\, h(s(\gamma)) = 0\,, \quad \forall \gamma \in V$, we get $\varphi(f * b^2) = 0$ and $\varphi(f) = 0$, using a partition of unity. ∎

Lemma 3.5. *Let φ be a KMS$_\beta$ state on $(\mathcal{A}_K, \sigma_t)$. Then, for any ideal $J \subset \mathcal{O}$ one has*

$$\varphi(e_J) = \mathfrak{n}(J)^{-\beta}.$$

Proof. For each ideal J we let $\mu_J \in \mathcal{A}_K$ be given as above by (2.25)

$$\mu_J((\Lambda, \phi), (\Lambda', \phi')) = \begin{cases} 1 & \Lambda = J^{-1}\Lambda' \text{ and } \phi' = \phi \\ 0 & \text{otherwise.} \end{cases}$$

One has $\sigma_t(\mu_J) = \mathfrak{n}(J)^{it}\, \mu_J \forall t \in \mathbb{R}$ while $\mu_J^*\, \mu_J = 1$ and $\mu_J\, \mu_J^* = e_J$ thus the answer follows from the KMS condition. ∎

Given Proposition 2.6 above, we make the following definition.

Definition 3.6. *A K-lattice is* quasi-invertible *if the ρ in Proposition 2.6 is in $\hat{\mathcal{O}} \cap \mathbb{A}_{K,f}^*$.*

Then we have the following result.

Lemma 3.7.
1. *A K-lattice (Λ, ϕ) that is divisible by only finitely many ideals is either quasi-invertible, or there is a finite place v such that the component ϕ_v of ϕ satisfies $\phi_v = 0$.*
2. *A quasi-invertible K-lattice is commensurable to a unique invertible K-lattice.*

Proof. Let (ρ, s) be associated to the K-lattice (Λ, ϕ) as in Proposition 2.6. If $\rho \notin \mathbb{A}_{K,f}^*$, then either there exists a place v such that $\rho_v = 0$, or $\rho_v \neq 0$ for all v and there exists infinitely many places w such that $\rho_w^{-1} \notin \mathcal{O}_w$, where $\mathcal{O}_w$ is the local ring at w. This shows that the K-lattice is divisible by infinitely many ideals. For the second statement, if we have $\rho \in \mathbb{A}_{K,f}^*$, we can write it as a product $\rho = s'_f \rho'$ where $\rho' = 1$ and $s'_f = \rho$. The K-lattice obtained this way is commensurable to the given one by Proposition 2.7 and is invertible.

∎

Let us now complete the proof of the second and third statements of Theorem 3.1. Let φ be a KMS$_\beta$ state. Proposition 3.4 shows that there is a probability measure μ on X such that

$$\varphi(f) = \int_X f(L, L)\, d\mu(L)\,, \quad \forall f \in \mathcal{A}_K.$$

With $L=(\Lambda,\phi)\in X$, Lemma 3.5 shows that the probability $\varphi(e_J)$ that an ideal J divides L is $\mathfrak{n}(J)^{-\beta}$. Since the series $\sum \mathfrak{n}(J)^{-\beta}$ converges for $\beta>1$, it follows (*cf.* [16] Thm. 1.41) that, for almost all $L\in X$, L is only divisible by a finite number of ideals. Notice that the KMS condition implies that the measure defined above gives measure zero to the set of K-lattices (Λ,ϕ) such that the component $\phi_v=0$ for some finite place v.

By the first part of Lemma 3.7, the measure μ gives measure one to quasi-invertible K-lattices,

$$\mu\left((\hat{\mathcal{O}}\cap\mathbb{A}^*_{K,f})\times_{\hat{\mathcal{O}}^*}(\mathbb{A}^*_{K,f}/K^*)\right)=1.$$

Notice that these K-lattices form a Borel subset which is not closed. Then, by the second part of Lemma 3.7, the KMS_β condition shows that the measure μ is entirely determined by its restriction to invertible K-lattices, so that, for some probability measure ν,

$$\varphi=\int\varphi_{\beta,L}\;d\nu(L).$$

It follows that the Choquet simplex of extremal KMS_β states is the space of probability measures on the compact space $\hat{\mathcal{O}}\times_{\hat{\mathcal{O}}^*}(\mathbb{A}^*_{K,f}/K^*)$ of invertible K-lattices modulo scaling and its extreme points are the $\varphi_{\beta,L}$. $\square$

The action of the symmetry group $\mathbb{I}_K/K^*$ on $\mathcal{E}_\beta$ is then free and transitive. In fact, recall that (Lemma 1.28 [7]) the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on 2-dimensional $\mathbb{Q}$-lattices has as its only fixed points the $\mathbb{Q}$-lattices (Λ,ϕ) with $\phi=0$.

The action is given explicitly, for $L=(\Lambda,\phi)$ an invertible K-lattice and for $s\in\hat{\mathcal{O}}^*/\mathcal{O}^*\subset\mathbb{I}_K/K^*$, by

$$(\varphi_{\beta,L}\circ\theta_s)(f)=Z_K(\beta)^{-1}\sum_{J\in\mathcal{J}}f((J^{-1}\Lambda,s^{-1}\phi),(J^{-1}\Lambda,s^{-1}\phi))\;\mathfrak{n}(J)^{-\beta}=\varphi_{\beta,(\Lambda,s^{-1}\phi)}(f),\tag{3.7}$$

for $f\in\mathcal{A}_K$. More generally, for $s\in\hat{\mathcal{O}}\cap\mathbb{A}^*_{K,f}$ let $J_s=s\hat{\mathcal{O}}\cap K$, one then has,

$$\begin{aligned}(\varphi_{\beta,L}\circ\theta_s)(f)=&\quad Z_K(\beta)^{-1}\textstyle\sum_{J\in\mathcal{J}}\theta_s(f)((J^{-1}\Lambda,\phi),(J^{-1}\Lambda,\phi))\;\mathfrak{n}(J)^{-\beta}\\ =&\quad Z_K(\beta)^{-1}\textstyle\sum_{J\subset J_s}f((J^{-1}\Lambda,s^{-1}\phi),(J^{-1}\Lambda,s^{-1}\phi))\;\mathfrak{n}(J)^{-\beta}\\ =&\quad Z_K(\beta)^{-1}\textstyle\sum_{J\in\mathcal{J}}f(J^{-1}L_s,J^{-1}L_s)\;\mathfrak{n}(J\,J_s)^{-\beta}=\mathfrak{n}(J_s)^{-\beta}\,\varphi_{\beta,L_s}(f),\end{aligned}\tag{3.8}$$

for $f\in\mathcal{A}_K$, and with L_s the invertible K-lattice $(J_s^{-1}\Lambda,s^{-1}\phi)$. To prove the last equality one uses the basic property of Dedekind rings that any ideal $J\subset J_s$ can be written as a product of ideals $J=J'\,J_s$.

3.2. **KMS states at zero temperature and Galois action.**

The weak limits as $\beta\to\infty$ of states in $\mathcal{E}_\beta$ define states in $\mathcal{E}_\infty$ of the form

$$\varphi_{\infty,L}(f)=f(L,L).\tag{3.9}$$

The action of the symmetry group $\mathbb{A}_{K,f}/K^*$ on extremal KMS states at zero temperature is given, as in Lemma 2.16 by (2.24). In fact, for an invertible K-lattice L, evaluating $\varphi_{\infty,L}$ on $\theta_s(f)$ would not give a nontrivial action, while (2.24) gives the action

$$\Theta_s(\varphi_{\infty,L})(f)=\lim_{\beta\to\infty}\left(W_\beta(\varphi_{\infty,L})\circ\theta_s\right)(f),\tag{3.10}$$

with W_β as in (2.21). This gives

$$\Theta_s(\varphi_{\infty,L})=\varphi_{\infty,L_s},\tag{3.11}$$

with L_s as in (3.8).

Thus the action of the symmetry group $\mathbb{I}_K/K^*$ is given by

$$L=(\Lambda,\phi)\mapsto L_s=(J_s^{-1}\Lambda,s^{-1}\phi)\,,\quad\forall s\in\mathbb{I}_K/K^*\,.\tag{3.12}$$

When we evaluate states $\varphi_{\infty,L}$ on elements $f \in \mathcal{A}_{K,0}$ of the arithmetic subalgebra we obtain

$$\varphi_{\infty,L}(f) = f(L,L) = g(L)\,, \tag{3.13}$$

where the function g is the lattice function of weight 0 obtained as the restriction of f to the diagonal. By construction of $\mathcal{A}_{K,0}$, one obtains in this way all the evaluations $f \mapsto f(z)$ of elements of the modular field F on the finitely many modules $z \in \mathbb{H}$ of the classes of K-lattices.

Consider the subring B of F consisting of those modular functions $f \in F$ that are defined at τ. The theory of complex multiplication (*cf.* [17], §5) shows that the subfield $F_\tau \subset \mathbb{C}$ generated by the values $f(\tau)$, for $f \in B$, is the maximal abelian extension of K (we have fixed an embedding $K \subset \mathbb{C}$),

$$F_\tau = K^{ab}. \tag{3.14}$$

Moreover, the action of $\alpha \in \mathrm{Gal}(K^{ab}/K)$ on the values $f(z)$ is given by

$$\alpha f(z) = f^{\,\sigma q_\tau \theta^{-1}(\alpha)}(z). \tag{3.15}$$

In this formula the notation $f \mapsto f^\gamma$ denotes the action of an element $\gamma \in \mathrm{Aut}(F)$ on the elements $f \in F$. The map θ is the class field theory isomorphism (1.1),

$$\theta : \mathbb{A}^*_{K,f}/K^* \to \mathrm{Gal}(K^{ab}/K).$$

The map $q_\tau : \mathbb{A}^*_{K,f} \hookrightarrow \mathrm{GL}_2(\mathbb{A}_f)$ is the embedding determined by the choice of the basis $\{1,\tau\}$, as in (2.29). The map σ is as in the diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & K^* \arrow[r, "\iota"] & \mathrm{GL}_1(\mathbb{A}_{K,f}) \arrow[r, "\theta"] \arrow[d, "q_\tau"] & \mathrm{Gal}(K^{ab}/K) \arrow[r] & 1 \\ 1 \arrow[r] & \mathbb{Q}^* \arrow[r] & \mathrm{GL}_2(\mathbb{A}_f) \arrow[r, "\sigma"] & \mathrm{Aut}(F) \arrow[r] & 1. \end{tikzcd} \tag{3.16}$$

Thus, when we act by an element $\alpha \in \mathrm{Gal}(K^{ab}/K)$ on the values on $\mathcal{A}_{K,0}$ of an extremal KMS$_\infty$ state we have

$$\alpha\,\varphi_{\infty,L}(f) = \varphi_{\infty,L_s}(f) \tag{3.17}$$

where $s = \theta^{-1}(\alpha) \in \mathbb{I}_K/K^*$.

This result makes essential use of an important result of Shimura (*cf.* [17] §6.6) that characterizes the automorphism group of the modular field by the exact sequence

$$1 \to \mathbb{Q}^* \to \mathrm{GL}_2(\mathbb{A}_f) \to \mathrm{Aut}(F) \to 1.$$

We also use another deep result of the theory of modular curves, namely Shimura reciprocity (*cf.* [17] §6.8), which gives (3.15).

Thus, the intertwining of the *geometric action* of $\mathbb{A}^*_{K,f}/K^*$ as symmetries and the *Galois action* on the values of extremal states on elements of the arithmetic subalgebra relies essentially on the classical complex multiplication theory. For more general number fields, while it is possible to obtain systems with the right geometric action (see *e.g.* the construction of [9]) and the right partition function, the Galois aspect is not yet understood and might deserve further study even in the CM case. The distinction between the geometric action of symmetries and a Galois action on values of states becomes essential in possible generalizations of the GL_2 system either in the context of Shimura varieties or in the direction of the Langlands program, where one is working in the non-abelian context. Already in the GL_2 case of [7] the intertwining of geometric and Galois action is very subtle, with new phenomena that appear for non-generic extremal states, and also in that case one has to rely essentially on the classical theory of Shimura.

3.3. **Uniqueness of high temperature KMS state.**

The proof follows along the line of [15]. We first discuss uniqueness. By Proposition 3.4, one obtains a measure μ on the space X of K-lattices up to scale. As in Lemma 3.5, this measure fulfills the quasi-invariance condition

$$\int_X \mu_J\, f\, \mu_J^*\; d\mu = \mathfrak{n}(J)^{-\beta} \int_X f\, d\mu, \tag{3.18}$$

for all ideals J, where μ_J is as in (2.25). To prove uniqueness of such a measure, for $\beta \in (0,1]$, one proceeds in the same way as in [15], reducing the whole argument to an explicit formula for the orthogonal projection P from $L^2(X, d\mu)$ to the subspace of functions invariant under the semigroup action

$$L = (\Lambda, \phi) \mapsto J^{-1}L, \tag{3.19}$$

which preserves commensurability. As in [15], one can obtain such formula as a weak limit of the orthogonal projections P_A associated to finite sets A of non-archimedean places.

Let A be a finite set of non-archimedean places. Let $\mathcal{J}_A$ be the subsemigroup of the semigroup $\mathcal{J}$ of ideals, generated by the prime ideals in A. Any element $J \in \mathcal{J}_A$ can be uniquely written as a product

$$J = \prod_{v\in A} J_v^{n_v}, \tag{3.20}$$

where J_v is the prime ideal associated to the place $v \in A$.

Lemma 3.8. *Let $L = (\Lambda, \phi)$ be a K-lattice such that $\phi_v \neq 0$ for all $v \in A$. Let $J \in \mathcal{J}_A$, $J = \prod_{v\in A} J_v^{n_v}$ be the smallest ideal dividing L. Let $(\rho, s) \in \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$ be the pair associated to L. Then, for each $v \in A$, the valuation of ρ_v is equal to n_v.*

Proof. Let (ρ, s) be as above, and m_v be the valuation of ρ_v. Then it is enough to show that an ideal J divides L if and only if J is of the form (3.20), with $n_v \leq m_v$. The map ϕ is the composite of multiplication by ρ and an isomorphism, as in the diagram (2.4), hence the divisibility is determined by the valuations of ρ_v. □

Definition 3.9. *With A as above we shall say that a K-lattice $L = (\Lambda, \phi)$ is A-invertible iff the valuation of ρ_v is equal to zero far all $v \in A$.*

We now define basic test functions associated to a Hecke Grössencharakter. Given such a character χ, the restriction of χ to $\hat{\mathcal{O}}^*$ only depends on the projection on $\hat{\mathcal{O}}^*_{B_\chi} = \prod_{v\in B_\chi} \hat{\mathcal{O}}_v^*$, for B_χ a finite set of non-archimedean places. Let B be a finite set of non-archimedean places $B \supset B_\chi$. We consider the function $f = f_{B,\chi}$ on $\hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$, which is obtained as follows. For $(\rho, s) \in \hat{\mathcal{O}} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$, we let $f = 0$ unless $\rho_v \in \hat{\mathcal{O}}_v^*$ for all $v \in B$, while $f(\rho, s) = \chi(\rho' s)$, for any $\rho' \in \hat{\mathcal{O}}^*$ such that $\rho'_v = \rho_v$ for all $v \in B$. This is well defined, because the ambiguity in the choice of ρ' does not affect the value of χ, since $B_\chi \subset B$. The function f obtained this way is continuous.

Let $H(B)$ be the subspace of $L^2(X, d\mu)$ of functions that only depend on s and on the projection of ρ on $\hat{\mathcal{O}}_B$. Let us consider the map that assigns to a K-lattice L the smallest ideal $J \in \mathcal{J}_B$ dividing L, extended by zero if some $\phi_v = 0$. By Lemma 3.8, the value of this map only depends on the projection of ρ on $\hat{\mathcal{O}}_B$. Thus, we can consider corresponding projections $E_{B,J}$ in $H(B)$, for J as above. By construction the projections $E_{B,J}$ give a partition of unity on the Hilbert space $H(B)$. Note that $E_{B,0} = 0$, since the measure μ gives measure zero to the set of K-lattices with $\phi_v = 0$ for some v.

Let $V_J f(L) = f(J^{-1}L)$ implementing the semigroup action (3.19). For $J \in \mathcal{J}_B$, the operator $\mathfrak{n}(J)^{-\beta/2} V_J^*$ maps isometrically the range of $E_{B,\mathcal{O}}$ to the range of $E_{B,J}$.

Lemma 3.10. *The functions $V_J^* f_{B,\chi}$ span a dense subspace of $H(B)$.*

Proof. It is sufficient to prove that the $f_{B,\chi}$ form a dense subspace of the range of $E_{B,\mathcal{O}}$. The image of $\hat{\mathcal{O}}^*$ in $\hat{\mathcal{O}}_B^* \times \mathbb{A}_K^*/K^*$ by the map $u \mapsto (u, u^{-1})$ is a closed normal subgroup. We let $\hat{\mathcal{O}}_B^* \times_{\hat{\mathcal{O}}^*} \mathbb{A}_K^*/K^*$ be the quotient. This is a locally compact group. The quotient G_B by the connected component of identity D_K in $\mathbb{A}_K^*/K^*$ is a compact group. Then $C(G_B)$ is identified with a dense subspace of the range of $E_{B,\mathcal{O}}$. The characters of G_B are all the Grössencharakter χ that vanish on the connected component of identity and such that $B_\chi \subset B$. Thus, by Fourier transform, we obtain the density result. □

Let A be a finite set of non-archimedean places, and $\mathcal{J}_A$ as above. Let H_A be the subspace of functions constant on $\mathcal{J}_A$-orbits, and let P_A be the corresponding orthogonal projection. The P_A converge weakly to P.

Proposition 3.11. *Let $A \supset B$ be finite sets of non-archimedean places. Let L be an A-invertible K-lattice, and $f \in H(B)$, the restriction of $P_A f$ to the $\mathcal{J}_A$-orbit of L is constant and given by the formula*

$$P_A f|_{\mathcal{J}_A L} = \zeta_{K,A}(\beta)^{-1} \sum_{J \in \mathcal{J}_A} \mathfrak{n}(J)^{-\beta} f(J^{-1}L), \tag{3.21}$$

where $\zeta_{K,A}(\beta) = \sum_{J \in \mathcal{J}_A} \mathfrak{n}(J)^{-\beta}$.

Proof. By construction, the right hand side of the formula (3.21) defines an element f_A in $H_A \cap H(A)$. One checks, using the quasi-invariance condition (3.18) on the measure μ, that $\langle f_A, g\rangle = \langle f, g\rangle$ for all $g \in H_A$, as in [15]. □

Let L be an invertible K-lattice and χ a Grössencharakter vanishing on the connected component of identity D_K. We define $\chi(L)$ as $\chi(\rho s)$, for any representative (ρ, s) of L. This continues to make sense when L is an A-invertible K-lattice and $A \supset B_\chi$ taking $\chi(\rho' s)$ where $\rho' \in \hat{\mathcal{O}}^*$ and $\rho'_v = \rho_v$ for all $v \in A$.

Finally we recall that to a Grössencharakter χ vanishing on the connected component of identity D_K one associates a Dirichlet character $\tilde{\chi}$ defined for ideals J in $\mathcal{J}_{B_\chi^c}$, where B_χ^c is the complement of B_χ. More precisely, given $J \in \mathcal{J}_{B_\chi^c}$, let s_J be an idèle such that $J = s_J \hat{\mathcal{O}} \cap K$ and $(s_J)_v = 1$ for all places $v \in B_\chi$. One then define $\tilde{\chi}(J)$ to be the value $\chi(s_J)$. This is independent of the choice of such s_J.

Proposition 3.12. *Let $A \supset B \supset B_\chi$ and L an A-invertible K-lattice. The projection P_A of* (3.21) *applied to the function $f_{B,\chi}$ gives*

$$P_A f_{B,\chi}|_{\mathcal{J}_A L} = \frac{\chi(L)}{\zeta_{K,A}(\beta)} \sum_{J \in \mathcal{J}_{A \setminus B}} \mathfrak{n}(J)^{-\beta}\, \tilde{\chi}(J)^{-1}. \tag{3.22}$$

Proof. Among ideals in $\mathcal{J}_A$, those that have nontrivial components on B do not contribute to the sum (3.21) computing $P_A f_{B,\chi}|_{\mathcal{J}_A L}$. It remains to show that $f_{B,\chi}(J^{-1}L) = \chi(L)\tilde{\chi}(J)^{-1}$, for $J \in \mathcal{J}_{A\setminus B}$. Let $J = s_J\, \hat{\mathcal{O}} \cap K$ and $(s_J)_v = 1$ for all places $v \in B$. Let L be given by (ρ, s), we have $J^{-1}L$ given by $(\rho\, s_J, s\, s_J^{-1})$ using Proposition 2.12. Thus, for any choice of $\rho' \in \hat{\mathcal{O}}^*$ with $\rho'_v = (\rho\, s_J)_v$ for all $v \in B$ one has $f_{B,\chi}(J^{-1}L) = \chi(\rho' s\, s_J^{-1}) = \chi(\rho' s)\tilde{\chi}(J)^{-1}$. Note that $(s_J)_v = 1$ for all places $v \in B$ thus the choice of ρ' is governed by $\rho'_v = \rho_v$ for all $v \in B$. Since L is A-invertible and $A \supset B \supset B_\chi$ we get $\chi(\rho' s) = \chi(L)$ for a suitable choice of ρ'. □

It then follows as in [15] that $P_A f_{B,\chi}$ tend weakly to zero for χ nontrivial. The same argument gives an explicit formula for the measure, obtained as a limit of the $P_A f_{B,1}$, for the trivial character. In particular, the restriction of the measure to G_B is proportional to the Haar measure. Positivity is ensured by the fact that we are taking a projective limit of positive measures. This completes the proof of existence and uniqueness of the KMS_β state for $\beta \in (0, 1]$.

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