Renormalization, Galois symmetries, and motives

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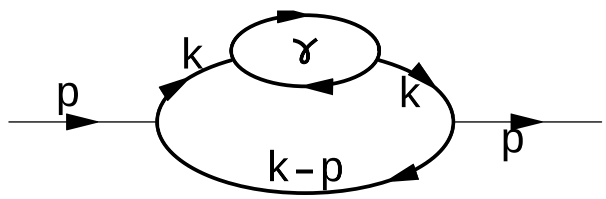
Quantum Fields and Motives
(an unlikely match)

- *Feynman diagrams: graphs and integrals*

$$(2\pi)^{-2D}\int \frac{1}{k^4}\frac{1}{(k-p)^2}\frac{1}{(k+\ell)^2}\frac{1}{\ell^2}\, d^Dk\, d^D\ell$$



Divergences $\Rightarrow$ Renormalization

- *Algebraic varieties and motives*

$\mathcal{V}_{\mathbb{K}}$ smooth proj alg varieties over $\mathbb{K}$ $\Rightarrow$ category of pure motives $\mathcal{M}_{\mathbb{K}}$

$$\text{Hom}((X,p,m),(Y,q,n)) = q\text{Corr}^{m-n}_{/\sim}(X,Y)\,p$$

$p^2 = p$, $q^2 = q$, $\mathbb{Q}(m) =$ Tate motives

Universal cohomology theory for algebraic varieties

What do they have in common?

Supporting evidence

- Multiple zeta values from Feynman integral calculations (Broadhurst–Kreimer)
- Parametric Feynman integrals as periods (Bloch–Esnault–Kreimer)
- Graph hypersurfaces and their motives (Belkale–Brosnan)
- Hopf algebras of renormalization (Connes–Kreimer)
- Flat equisingular connections and Galois symmetries (Connes-M.)
- Feynman integrals and Hodge structures (Bloch–Kreimer; M.)

Perturbative QFT in a nutshell

$\mathcal{T}$ = scalar field theory in spacetime dimension D

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

with Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Effective action and perturbative expansion (1PI graphs)

$$S_{eff}(\phi) = S_0(\phi) + \sum_{\Gamma} \frac{\Gamma(\phi)}{\#\mathrm{Aut}(\Gamma)}$$

$$\Gamma(\phi) = \frac{1}{N!}\int_{\sum_i p_i = 0} \widehat{\phi}(p_1)\cdots\widehat{\phi}(p_N) U_\mu^z(\Gamma(p_1,\ldots,p_N)) dp_1 \cdots dp_N$$

$$U(\Gamma(p_1,\ldots,p_N)) = \int I_\Gamma(k_1,\ldots,k_\ell,p_1,\ldots,p_N) d^D k_1 \cdots d^D k_\ell$$

$\ell = b_1(\Gamma)$ loops

Dimensional Regularization: $U_\mu^z(\Gamma(p_1,\ldots,p_N))$

$$= \int \mu^{z\ell} d^{D-z} k_1 \cdots d^{D-z} k_\ell I_\Gamma(k_1,\ldots,k_\ell,p_1,\ldots,p_N)$$

(Laurent series in $z \in \Delta^* \subset \mathbb{C}^*$)

Two aspects:

• Individual Feynman graphs
(Feynman rules, parametric representations)

Construction of $I_\Gamma(k_1, \ldots, k_\ell, p_1, \ldots, p_N)$:

- Internal lines $\Rightarrow$ propagator = quadratic form q_i

$$\frac{1}{q_1 \cdots q_n}, \quad q_i(k_i) = k_i^2 + m^2$$

- Vertices: conservation (valences = monomials in $\mathcal{L}$)

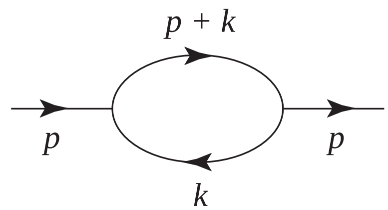
$$\sum_{e_i \in E(\Gamma): s(e_i) = v} k_i = 0$$

- Integration over k_i, internal edges
- Parameterizations of the integral
- Graph hypersurfaces and periods

• All Feynman graphs
(subdivergences and renormalization)

- Regularization and BPHZ renormalization
- Connes–Kreimer Hopf algebra
- Birkhoff factorization
- Beta function
- Equisingular connections
- Ward identities/gauge theories

Regularization (Dim Reg)



$$\int \frac{1}{k^2+m^2}\,\frac{1}{((p+k)^2+m^2)}\,d^Dk$$

ϕ^3-theory $D=4$ divergent

Schwinger parameters

$$\frac{1}{k^2+m^2}\,\frac{1}{(p+k)^2+m^2} = \int_{s>0,\,t>0} e^{-s(k^2+m^2)-t((p+k)^2+m^2)}\,ds\,dt$$

diagonalize quadratic form in exp

$$-Q(k) = -\lambda\,((k+xp)^2 + ((x-x^2)p^2+m^2))$$

with $s=(1-x)\lambda$ and $t=x\,\lambda \Rightarrow$ Gaussian $q=k+xp$

$$\int e^{-\lambda\,q^2}\,d^Dq = \pi^{D/2}\,\lambda^{-D/2}$$

$$\int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2+\lambda\,m^2)} \int e^{-\lambda\,q^2}\,d^Dq\;\lambda\,d\lambda\;dx$$

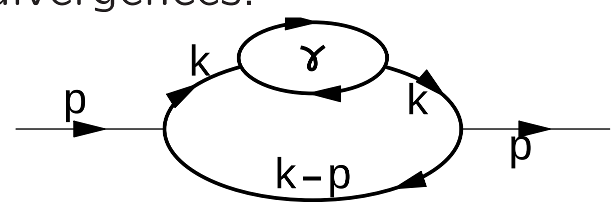
$$= \pi^{D/2} \int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2+\lambda\,m^2)}\,\lambda^{-D/2}\,\lambda\,d\lambda\,dx$$

$$= \pi^{D/2}\,\Gamma(2-D/2) \int_0^1 ((x-x^2)p^2+m^2)^{D/2-2}\,dx$$

Adjust bare constants to cancel polar part

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2(1-\delta Z)+\left(\frac{m^2-\delta m^2}{2}\right)\phi^2-\frac{g+\delta g}{6}\phi^3$$

But with subdivergences:



$$(2\pi)^{-2D}\int \frac{1}{k^4}\,\frac{1}{(k-p)^2}\,\frac{1}{(k+\ell)^2}\,\frac{1}{\ell^2}\,d^Dk\,d^D\ell$$

$$(4\pi)^{-D}\,\frac{\Gamma(2-\frac{D}{2})\Gamma(\frac{D}{2}-1)^3\Gamma(5-D)\Gamma(D-4)}{\Gamma(D-2)\Gamma(4-\frac{D}{2})\Gamma(\frac{3D}{2}-5)}\,(p^2)^{D-5}$$

$(p^2/\mu^2)^{-z} = \sum \frac{(-z)^n}{n!}\,\log^n(p^2/\mu^2)$

$$(4\pi)^{-6}\,\frac{1}{18}\,p^2\,(\log(p^2/\mu^2)+\text{constant})$$

non-polynomial term coefficient of $\frac{1}{z}$

Renormalization (BPHZ)

Preparation:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

$\Rightarrow$ coeff of pole is local

Counterterm:

$$C(\Gamma) = -T(\bar{R}(\Gamma))$$

Projection onto polar part of Laurent series

Renormalized value:

$$R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma)$$

$$= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

Connes–Kreimer Hopf algebra

$\mathcal{H} = \mathcal{H}(\mathcal{T})$ (depend on theory $\mathcal{L}(\phi)$)

Free commutative algebra in generators
Γ 1PI Feynman graphs

Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom):
Ward identities as Hopf ideals

Connes–Kreimer theory

• $\mathcal{H}$ dual to affine group scheme G (diffeographisms)

• $G(\mathbb{C})$ pro-unipotent Lie group $\Rightarrow$

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)$$

Birkhoff factorization of loops exists

• Recursive formula for Birkhoff = BPHZ

• loop = $\phi \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}(\{z\}))$
(germs of meromorphic functions)

• Feynman integral $U(\Gamma) = \phi(\Gamma)$
counterterms $C(\Gamma) = \phi_-(\Gamma)$
renormalized value $R(\Gamma) = \phi_+(\Gamma)|_{z=0}$

Bottom-up method: Feynman parameters

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n$$

$n = \#E_{int}(\Gamma)$, $N = \#E_{ext}(\Gamma)$

$$\epsilon_{e,v} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases}$$

- Schwinger parameters $q_1^{-k_1} \cdots q_n^{-k_n} =$

$$\frac{1}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)} \, s_1^{k_1 - 1} \cdots s_n^{k_n - 1} \, ds_1 \cdots ds_n.$$

- Feynman trick

$$\frac{1}{q_1 \cdots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \cdots dt_n$$

then change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$

$$\eta_{ik} = \begin{cases} +1 & \text{edge } e_i \in \text{loop } l_k, \text{ same orientation} \\ -1 & \text{edge } e_i \in \text{loop } l_k, \text{ reverse orientation} \\ 0 & \text{otherwise.} \end{cases}$$

• Parametric form

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n-D\ell/2}}$$

$\sigma_n = \{t \in \mathbb{R}_+^n | \sum_i t_i = 1\}$, vol form ω_n

• Graph polynomials

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e$$

$$(M_\Gamma)_{kr}(t) = \sum_{i=0}^{n} t_i \eta_{ik} \eta_{ir}$$

$$V_\Gamma(t,p) = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)} \quad \text{and} \quad P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

cut-sets C (compl of spanning tree plus one edge)

$s_C = (\sum_{v \in V(\Gamma_1)} P_v)^2$ with $P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e$

for $\sum_{e \in E_{ext}(\Gamma)} p_e = 0$ $\quad \deg \Psi_\Gamma = b_1(\Gamma) = \deg P_\Gamma - 1$

• Dim Reg

$$U_\mu(\Gamma)(z) = \mu^{-z\ell} \frac{\Gamma(n - \frac{(D+z)\ell}{2})}{(4\pi)^{\frac{\ell(D+z)}{2}}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{\frac{(D+z)}{2}} V_\Gamma(t,p)^{n-\frac{(D+z)\ell}{2}}}$$

• Divergent case $\Rightarrow$ Renormalization (BPHZ)

• Convergent: period

$$X_\Gamma = \{t = (t_1 : \cdots : t_n) \in \mathbb{P}^{n-1} | \Psi_\Gamma(t) = 0\}$$

On complement $\mathbb{P}^{n-1} \smallsetminus X_\Gamma$

$$\int_{\sigma_n} \frac{P_\Gamma(p,t)^{-n+D\ell/2}}{\Psi_\Gamma(t)^{-n+(\ell+1)D/2}} \omega_n$$

alg diff form on semi-alg set (period)

$$\Sigma_n = \{t = (t_1 : \cdots : t_n) \in \mathbb{P}^{n-1} | \prod_i t_i = 0\}$$

(coordinate simplex) then $\sigma_n \in H_{n-1}(\mathbb{P}^{n-1}, \Sigma_n, \mathbb{Z})$

Motive: $H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n)$

How complex?

• Classes $[X_\Gamma]$ generate $K_0(\mathcal{V})$ Grothendieck ring of varieties (Belkale-Brosnan)

• but is the part of the motive involved in the period simpler? a mixed Tate motive?

(Note: Tate part of $K_0(\mathcal{M})$ is just $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}]$)

A simple example: Banana graphs

Class in the Grothendieck group

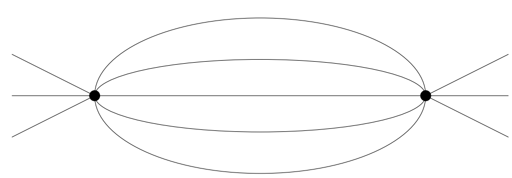
$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2}$$

where $\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathcal{V})$ Lefschetz motive

(also characteristic classes: Aluffi-M.)

$$\int_{\sigma_n} \frac{(t_1 \cdots t_n)^{(\frac{D}{2}-1)(n-1)-1}\,\omega_n}{\Psi_\Gamma(t)^{(\frac{D}{2}-1)n}}$$

$$\Psi_\Gamma(t) = t_1 \cdots t_n(\frac{1}{t_1} + \cdots + \frac{1}{t_n})$$



More information on X_Γ from other invariants e.g. Characteristic classes of singular varieties

More complicated examples: wheels with n-spokes (Bloch–Esnault–Kreimer)

Observations on the bottom-up method

- $\mathbb{P}^{n-1} \smallsetminus X_\Gamma$ can be very complicated motivically (by Belkale-Brosnan)
- but for ℓ loops, when X_Γ *not a cone*:

$$\mathbb{P}^{n-1} \smallsetminus X_\Gamma \hookrightarrow \mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$$

$\mathcal{D}_\ell =$ determinant variety
$\mathbb{P}^{n-1}$ mapped linearly

$$\Upsilon : \mathbb{P}^{n-1} \to \mathbb{P}^{\ell^2-1}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ki} \eta_{ir}$$

- The motive of $\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$ can be computed
- mixed Tate
- $P_\Gamma(t,p)$ extends (non-uniquely) to $\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$
- $\Psi_\Gamma(t)$ becomes $\det(x)$
- σ_n and Σ_n mapped linearly to $\mathbb{P}^{\ell^2-1}$
- difficulty: explicit control of $\Sigma \cap \mathcal{D}_\ell$ for

$$H^*(\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell, \Sigma \smallsetminus (\Sigma \cap \mathcal{D}_\ell))$$

(from Aluffi-M. work in progress)

Top-down method: Galois theory

(Connes–M.)

Compare renormalization and motives by comparing Tannakian categories

<u>Tannakian formalism</u>

$\mathcal{C}$ neutral Tannakian category $\Rightarrow$

$$\mathcal{C} \simeq \mathrm{Rep}_G$$

G affine group scheme

- abelian category (homological algebra)

- Hom k-vector spaces
- products and coproducts
- kernels and cokernels (decomp of morphisms)

- rigid tensor category

- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, $1 \in Obj(\mathcal{C})$
- $\vee : \mathcal{C} \to \mathcal{C}^{op}$
- $\epsilon : X \otimes X^\vee \to 1$, $\delta : 1 \to X^\vee \otimes X$
- $(\epsilon \otimes 1)(1 \otimes \delta) = 1_X$, $(1 \otimes \epsilon)(\delta \otimes 1) = 1_{X^\vee}$

- Tannakian

- fiber functor $\omega : \mathcal{C} \to Vect_K$ (faithful exact tensor)
- K=k neutral

Main results (Connes–M.)

• Counterterms as iterated integrals
('t Hooft–Gross relations)
• Solutions of irregular singular differential equations (flat equisingular connections)
• Flat equisingular vector bundles form a neutral Tannakian category $\mathcal{E}$
• Free graded Lie algebra $\mathcal{L} = \mathcal{F}(e_{-n}; n \in \mathbb{N})$

$$\mathcal{E} \simeq Rep_{\mathbb{U}^*}, \quad \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$$

$\mathbb{U} = \mathrm{Hom}(\mathcal{H}_{\mathbb{U}}, -)$, with $\mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^{\vee}$
• Motivic Galois group (Deligne–Goncharov)

$$\mathbb{U}^* \simeq \mathrm{Gal}(\mathcal{M}_S)$$

$\mathcal{M}_S$ mixed Tate motives on $S = \mathrm{Spec}(\mathbb{Z}[i][1/2])$

Counterterms as iterated integrals

$$\gamma_\mu(z) = \gamma_-(z)^{-1}\gamma_{\mu,+}(z)$$

Time ordered exponential

$$\gamma_-(z) = Te^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)dt} = 1 + \sum_{n=1}^\infty \frac{d_n(\beta)}{z^n}$$

$$d_n(\beta) = \int_{s_1\geq s_2\geq\cdots\geq s_n\geq 0} \theta_{-s_1}(\beta)\cdots\theta_{-s_n}(\beta)ds_1\cdots ds_n$$

with $\beta \in \mathrm{Lie}(G)$ beta function

(infinitesimal generator of renormalization group)

$$\theta_u(X) = u^n X, \quad u \in \mathbb{G}_m, \quad X \in \mathcal{H}, \deg(X) = n$$

1-param action generated by grading $Y(X) = nX$

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z)), \qquad \frac{\partial}{\partial\mu}\gamma_-(z) = 0$$

Birkhoff factorization:

$$\gamma_{\mu,+}(z) = Te^{-\frac{1}{z}\int_0^{-z\log\mu}\theta_{-t}(\beta)dt}\ \theta_{z\log\mu}(\gamma_{reg}(z))$$

so $\gamma_\mu(z)$ specified by β and $\gamma_{reg}(z)$

up to equivalence (same γ_-) just β

Iterated integrals to differential systems

$g(t) = Te^{\int_a^b \alpha(t)dt}$ solution of $dg(t) = g(t)\alpha(t)dt$

Differential field $(K = \mathbb{C}(\{z\}), \delta)$
affine group scheme G

$$G(K) \ni f \mapsto D(f) = f^{-1}\delta(f) \in \mathrm{Lie}G(K)$$

Differential equations: $D(f) = \omega$

flat $\mathrm{Lie}G(\mathbb{C})$-valued ω (singular at $z = 0 \in \Delta^*$)

Existence of solutions: trivial monodromy on Δ^*

$$M(\omega)(\ell) = Te^{\int_0^1 \ell^*\omega} = 1, \quad \ell \in \pi_1(\Delta^*)$$

Gauge equivalence $D(fh) = Dh + h^{-1}Df\,h$

(regular $h \in \mathbb{C}\{z\}$)

$$\omega' = Dh + h^{-1}\omega h \quad \Leftrightarrow \quad f_-^{\omega} = f_-^{\omega'}$$

for $D(f^\omega) = \omega$ and $D(f^{\omega'}) = \omega'$ solutions

Flat equisingular connections

fibration $\mathbb{G}_b \to B \to \Delta$, principal bundle $P = B \times G$

$$u(b,g) = (u(b), u^Y(g)) \quad \forall u \in \mathbb{G}_m$$

Flat connection ϖ on P^* *equisingular*

- $u \in \mathbb{G}_m$ action

$$\varpi(z, u(v)) = u^Y(\varpi(z,u))$$

- $D\gamma = \varpi$ solution $\Rightarrow$ restrictions

$$\sigma_1^*(\gamma) \sim \sigma_2^*(\gamma)$$

σ_1, σ_2 sections of B with $\sigma_1(0) = \sigma_2(0)$ and $f_1 \sim f_2$ iff $f_1^{-1} f_2 \in G(\mathcal{O})$

Restrictions to different sections same type of singularity

Up to gauge equivalence

Flat equisingular vector bundles
category $\mathcal{E}$

- $Obj(\mathcal{E})$ pairs $\Theta = (V, [\nabla])$

- $V =$ fin dim $\mathbb{Z}$-graded vector space

- $E = B \times V$ filtered $W^{-n}(V) = \oplus_{m \geq n} V_m$

- $\mathbb{G}_m$ action from grading

- ∇ compatible with filtration, trivial on $Gr^W_{-n}(V)$

- W-equivalent: $T \circ \nabla_1 = \nabla_2 \circ T$ for a $T \in \mathrm{Aut}(E)$ compatible with filtration, trivial on $Gr^W_{-n}(V)$

- ∇ equisingular: $\mathbb{G}_m$-invariant and restrictions to sections with same $\sigma(0)$ are W-equivalent

- $\mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$ linear maps $T : V \to V'$

- compatible with grading
- on $E \oplus E'$ connections

$$\begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix} \overset{W-equiv}{\simeq} \begin{pmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{pmatrix}$$

No longer depends on particular G (particular physical theory)

The Riemann–Hilbert correspondence

$$\mathcal{E} \simeq Rep_{\mathbb{U}^*}, \quad \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$$

$\mathbb{U}(A) = \mathrm{Hom}(\mathcal{H}_{\mathbb{U}}, A)$, Hopf algebra

$$\mathcal{L} = \mathcal{F}(e_{-1}, e_{-2}, e_{-3}, \cdots), \qquad \mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$$

free graded Lie algebra

Renormalization group $\mathbf{rg} : \mathbb{G}_a \to \mathbb{U}$ generator

$$e = \sum_{n=1}^{\infty} e_{-n}$$

Universal singular frame (source of counterterms)

$$\gamma_{\mathbb{U}}(z, v) = T e^{-\frac{1}{z} \int_0^v u^Y(e) \frac{du}{u}}$$

For $\Theta = (V, [\nabla])$ in $\mathcal{E}$ exists unique $\rho \in Rep_{\mathbb{U}^*}$

$$D\rho(\gamma_{\mathbb{U}}) \overset{W-equiv}{\simeq} \nabla$$

Fiber functor

$$\omega_n(\Theta) = \mathrm{Hom}(\mathbb{Q}(n), Gr^W_{-n}(\Theta))$$

$\mathbb{Q}(n) =$ 1-dim in degree n, trivial connection

Deligne–Goncharov: $\mathbb{U}^*$ is the motivic Galois group of a category of mixed Tate motives $\mathcal{M}_S \simeq Rep_{\mathbb{U}^*}$

Remark:

• Bottom-up approach with periods works well in *convergent (or log-divergent) case*

• Top-down approach with Tannakian categories works well in *divergent case*

• Do they meet?

Dim Reg of parametric integrals (divergent case): seen as oscillatory integrals and mixed Hodge structures (M. 2008)

Parametric Feynman integrals

$$\int_\sigma \frac{P_\Gamma(t,p)^{-n+D\ell/2}}{\Psi_\Gamma(t)^{-n+(\ell+1)D/2}}\omega_n$$

$$= \int_{\partial\sigma} \pi^*(\eta) + \int_\sigma df \wedge \frac{\pi^*(\eta)}{f}$$

forms on hypersurface complements $f = \Psi_\Gamma$

$$\pi^*(\eta) = \frac{\Delta(\omega)}{f^m}$$

$\Delta(\omega)$ = contraction with Euler vect field $E = \sum t_i \partial_i$
$\pi : \mathbb{A}^n \smallsetminus \{0\} \to \mathbb{P}^{n-1}$ $\qquad \Delta(\omega) = P_\Gamma^{-n+D\ell/2}\Omega_n$

$$\Delta(\omega_n) = \Omega_n = \sum_i (-1)^{i+1} t_i dt_1 \wedge \hat{dt_i} \wedge \cdots \wedge dt_n$$

$$m \operatorname{deg}(f) \int_\Sigma \frac{\omega}{f^m} = \int_{\partial\Sigma} \frac{\Delta(\omega)}{f^m} + \int_\Sigma df \wedge \frac{\Delta(\omega)}{f^{m+1}}$$

ω = closed k-form homogeneous of deg $m \operatorname{deg}(f)$

Nonisolated singularities

(example: Banana graphs: codim 2)

Slicing the Feynman integral

$$\Pi_\xi = \{t \in \mathbb{A}^n \,|\, \langle \xi_i, t\rangle = 0,\, i = 1, \dots, n-k\}$$

$$\Omega_n = \langle \xi_1, dt\rangle \wedge \cdots \wedge \langle \xi_{n-k}, dt\rangle \wedge \Omega_\xi$$

$$U(\Gamma)_\xi \sim \int \mathcal{F}_{\Sigma,k}(h_\Gamma)(\xi)\, \langle \xi, dt\rangle$$

$h_\Gamma = V_\Gamma^{-k+D\ell/2}/\Psi_\Gamma^{D/2}$

$$\mathcal{F}_{\Sigma,k}(h_\Gamma)(\xi) = \int_\sigma F_\Gamma(t,p) \prod_{i=1}^{n-k} \delta(\langle \xi_i, t\rangle)\, \Omega_\xi(t)$$

$$= \int_{\sigma \cap \Pi_\xi} h_\Gamma(t,p) \omega_\xi(t)$$

for $\langle \xi, dt\rangle \wedge \omega_\xi = \omega_n$ and $\langle \xi, dt\rangle = \langle \xi_1, dt\rangle \wedge \cdots \wedge \langle \xi_{n-k}, dt\rangle$

$\Rightarrow$ forms $h\Delta(\omega_\xi)/f^m$ span subspace of $H^r(F_\xi)$ of Milnor fiber, with $r = \dim \Pi_\xi - 1$

DimReg, Mellin transforms, Gelfand-Leray

$$F_{\Gamma,\xi}(z) = \int \Psi_\Gamma^z \chi_\xi P_\Gamma^\ell \Omega_\xi$$

from DimReg of (sliced) parametric Feynman integrals

$$J_{\Gamma,\xi}(s) = \int_{X_s} \frac{\alpha_\xi}{df}$$

Gelfand–Leray function ($f = \Psi_\Gamma|_{\Pi_\xi}$, $\alpha_\xi = \chi_\xi P_\Gamma^\ell \Omega_\xi$)

$$F_{\Gamma,\xi}(z) = \int_0^\infty s^z J_{\Gamma,\xi}(s) ds$$

Mellin transform (Note: $\Psi_\Gamma > 0$ and $P_\Gamma \in \mathbb{R}$ on σ)

Regular singular Picard–Fuchs equation

$$J_{\Gamma,\xi}^{(\ell)}(s) + p_1(s) J_{\Gamma,\xi}^{(\ell-1)}(s) + \cdots + p_\ell(s) J_{\Gamma,\xi}(s) = 0$$

Regular to irregular singular connections

$\mathcal{S}_\Gamma$ = manifold of planes Π_ξ of dim $\leq$ codim Sing X_Γ

Distributions $\sigma_\Gamma \in \mathcal{C}^{-\infty}(\mathcal{S}_\Gamma)$ induce σ_γ and $\sigma_{\Gamma/\gamma}$

Hopf algebra: (Γ, σ) with coproduct

$$\Delta(\Gamma, \sigma_\Gamma) = (\Gamma, \sigma_\Gamma) \otimes 1 + 1 \otimes (\Gamma, \sigma_\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} (\gamma, \sigma_\gamma) \otimes (\Gamma/\gamma, \sigma_{\Gamma/\gamma})$$

Dual affine group scheme G with $\mathfrak{g} = \mathrm{Lie} G$

- A solution $J_{\Gamma,\xi}$ of Picard–Fuchs equation

$\Rightarrow$ flat $\mathfrak{g}(K)$-valued equisingular connection

$$\phi_\mu(\Gamma, \sigma)(\epsilon) = \mu^{-\epsilon b_1(\Gamma)} \sigma(F_{\Gamma,\xi}(z))|_{z=-(D+\epsilon)/2}$$

$$a_\mu(\epsilon) = (\phi_\mu \circ S) * \frac{d}{d\epsilon} \phi_\mu$$

$$b_\mu(\epsilon) = (\phi_\mu \circ S) * Y(\phi_\mu)$$

$$\omega(\epsilon, u) = u^Y (a_\mu(\epsilon)) d\epsilon + u^Y (b_\mu(\epsilon)) \frac{du}{u}$$

flat $\frac{db}{d\epsilon} - Y(a) + [a, b] = 0$

The geometry of Dim Reg

Dim Reg basic rule: Gaussian integral in dim $z \in \mathbb{C}^*$

$$\int e^{-\lambda t^2} d^z t := \pi^{z/2} \lambda^{-z/2}$$

What is a space of dim $z \in \mathbb{C}^*$?

Two possible models:

- Noncommutative Geometry (Connes-M. 2008)
- Motives (M. 2008)

In both cases: product of an ordinary geometry by something else

- NCG: of spacetime by an NC space X_z
- Motives: of individual graph motives by Log^∞ motive

Noncommutative geometry of DimReg

Spectral triples $X = (\mathcal{A}, \mathcal{H}, \mathcal{D})$ (metric NC spaces)

$\text{Dim} = \{s \in \mathbb{C} | \zeta_a(s) = \text{Tr}(a|D|^{-s})$ have poles $\}$

Exists (type II) spectral triple X_z with:

- $\text{Dim} = \{z\}$
- $\text{Tr}(e^{-\lambda D_z^2}) = \pi^{z/2} \lambda^{-z/2}$

$$D_z = \rho(z) F |Z|^{1/z}$$

$Z = F|Z|$ self-adj affiliated to a type II_∞ $\mathcal{N}$
$\rho(z) = \pi^{-1/2} (\Gamma(1 + z/2))^{1/z}$ and spectral measure

$$\text{Tr}(\chi_{[a,b]}(Z)) = \frac{1}{2} \int_{[a,b]} dt$$

(Tr = type II)

Ordinary spacetime = commutative

$$(\mathcal{A},\mathcal{H},\mathcal{D}) = (\mathcal{C}^\infty(X), L^2(X,S), \not{D}_X)$$

Product $X \times X_z$= cup product

$$(\mathcal{A},\mathcal{H},\mathcal{D}) \cup (\mathcal{A}_z,\mathcal{H}_z,D_z)$$

$$= (\mathcal{A}\otimes\mathcal{A}_z, \mathcal{H}\otimes\mathcal{H}_z, \mathcal{D}\otimes 1 + \gamma\otimes D_z)$$

(adapted to type II case)

$\Rightarrow$ Breitenlohner–Maison prescription
γ_5 problem in DimReg

$$\mathcal{D}\otimes 1 + \gamma\otimes D_z$$

Example of X_z: adèle class space

$$\mathcal{N} = L^\infty(\hat{\mathbb{Z}}\times\mathbb{R}^*) \rtimes \mathrm{GL}_1(\mathbb{Q})$$

(partially defined action)

$$\mathrm{Tr}(f) = \int_{\hat{\mathbb{Z}}\times\mathbb{R}^*} f(1,a)\, da$$

$$Z(1,\rho,\lambda) = \lambda, \quad Z(r,\rho,\lambda) = 0,\ r \neq 1 \in \mathbb{Q}^*$$

Motivic interpretation of DimReg

Kummer motive

$$M = [u : \mathbb{Z} \to \mathbb{G}_m] \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{K})}(\mathbb{Q}(0), \mathbb{Q}(1))$$

with $u(1) = q \in \mathbb{K}^*$ and period matrix

$$\begin{pmatrix} 1 & 0 \\ \log q & 2\pi i \end{pmatrix}$$

Kummer extension of Tate sheaves

$$\mathcal{K} \in \mathrm{Ext}^1_{\mathcal{DM}(\mathbb{G}_m)}(\mathbb{Q}_{\mathbb{G}_m}(0), \mathbb{Q}_{\mathbb{G}_m}(1))$$

$$\mathbb{Q}_{\mathbb{G}_m}(1) \to \mathcal{K} \to \mathcal{Q}_{\mathbb{G}_m}(0) \to \mathbb{Q}_{\mathbb{G}_m}(1)[1]$$

Logarithmic motives $\mathrm{Log}^n = \mathrm{Sym}^n(\mathcal{K})$

$$\mathrm{Log}^\infty = \varprojlim_n \mathrm{Log}^n$$

$$\begin{pmatrix}
1 & 0 & 0 & \cdots & 0 & \cdots \\
\log(s) & (2\pi i) & 0 & \cdots & 0 & \cdots \\
\frac{\log^2(s)}{2!} & (2\pi i)\log(s) & (2\pi i)^2 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots \\
\frac{\log^n(s)}{n!} & (2\pi i)\frac{\log^{n-1}(s)}{(n-1)!} & (2\pi i)^2\frac{\log^{n-2}(s)}{(n-2)!} & \cdots & (2\pi i)^{n-1} & \cdots \\
\vdots & \vdots & \vdots & \cdots & \vdots & \cdots
\end{pmatrix}$$

Graph polynomials and motivic sheaves M_Γ

$(\Psi_\Gamma : \mathbb{P}^{n-1} \smallsetminus X_\Gamma \to \mathbb{G}_m, \Sigma_n \smallsetminus X_\Gamma \cap \Sigma_n, n-1, n-1)$

Arapura's category of motivic sheaves $(f : X \to S, Y, i, w)$

DimReg = product $M_\Gamma \times \mathrm{Log}^\infty$ fibered product

$$(X_1 \times_S X_2 \to S, Y_1 \times_S X_2 \cup X_1 \times_S Y_2, i_1 + i_2, w_1 + w_2)$$

$$\int \pi_{X_1}^*(\omega) \wedge \pi_{X_2}^*(\eta) = \int \omega \wedge f_1^*(f_2)_*(\eta)$$

on $\Sigma_1 \times_S \Sigma_2$ for $\Sigma_i \subset X_i$, $\partial \Sigma_i \subset Y_i$

$$\begin{array}{ccccc} & & X_1 \times_S X_2 & & \\ & \swarrow^{\pi_{X_1}} & & \searrow^{\pi_{X_2}} & \\ X_1 & & & & X_2 \\ & \searrow_{f_1} & & \swarrow_{f_2} & \\ & & S & & \end{array}$$

DimReg integral $\int_\sigma \Psi_\Gamma^z \alpha$ period on $M_\Gamma \times \mathrm{Log}^\infty$