Quantum statistical mechanics over function fields

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General philosophy:

Moduli spaces in arithmetic geometry:

Enrich the boundary structure with "invisible" degenerations

Quantum mechanical interpretation:

- L-functions as partition functions

- Galois actions as symmetries of quantum systems

Function field arithmetic:

$\mathbb{F}_q =$ finite field char $p > 0$, with $\#\mathbb{F}_q = q = p^r$

$C =$ smooth projective curve over $\mathbb{F}_q$

$\mathbb{K} = \mathbb{F}_q(C) =$ function field of C

$\infty \in C$ a chosen point of degree d_∞

$v_\infty =$ valuation: $|x|_\infty = q^{\deg(x)} = q^{-d_\infty v_\infty(x)}$, $x \in \mathbb{K}$

$\mathbb{K}_\infty =$ completion of $\mathbb{K}$ in v_∞

$\overline{\mathbb{K}}_\infty =$ a fixed algebraic closure of $\mathbb{K}_\infty$

$\mathbb{C}_\infty =$ completion of $\overline{\mathbb{K}}_\infty$ (in extension of v_∞ to $\overline{\mathbb{K}}_\infty$)
$\mathbb{C}_\infty$ also algebraically closed

$\mathbb{L} =$ a complete subfield of $\mathbb{C}_\infty$ containing $\mathbb{K}_\infty$

$\mathbf{A} \subset \mathbb{K}$ ring of functions regular outside ∞

$\mathcal{F} =$ a $\mathbf{A}$-field: fixed homomorphism $\iota : \mathbf{A} \to \mathcal{F}$
generic characteristic: $\mathrm{Ker}(\iota) = (0)$

$\mathcal{F}\{\tau\} =$ (non-commutative) ring

$$f(\tau) = \sum_{i=0}^{\nu} a_i \tau^i \quad \text{with} \quad \tau a = a^q \tau$$

$\tau(a) = a^q$ Frobenius

$\Sigma_{\mathbb{K}} =$ set of places $v \in \mathbb{K}$
$\Sigma_{\mathbf{A}} = \{v \in \Sigma_{\mathbb{K}} | v \neq \infty\}$

For $v \in \Sigma_{\mathbf{A}}$, $\mathbf{A}_v = v$-adic completion of $\mathbf{A}$
$\mathbb{K}_v = v$-adic completion of $\mathbb{K}$, with $\mathbf{A}_v \subset \mathbb{K}_v$

$\mathbb{A}_{\mathbb{K}} = \prod_{v \in \Sigma_{\mathbb{K}}}{}' \, \mathbb{K}_v$ adèles $\quad (a_v) \in \prod_{v \in \Sigma_{\mathbb{K}}} \mathbb{K}_v$
with $a_v \in \mathbf{A}_v$ for all but finitely many places v

$\mathbb{A}_{\mathbb{K},f} = \prod_{v \in \Sigma_{\mathbf{A}}}{}' \, \mathbb{K}_v$ finite adèles

$R = \prod_{v \in \Sigma_{\mathbf{A}}} \mathbf{A}_v$ ring of finite integral adeles
maximal compact subring of $\mathbb{A}_{\mathbb{K},f}$

Analogy: between $\mathbb{K} = \mathbb{F}_q(C)$ and $\mathbb{Q}$

$\mathbb{Q}$	$\mathbb{K}$
$\mathbb{R}$	$\mathbb{K}_\infty$
$\mathbb{C}$	$\mathbb{C}_\infty$
$\mathbb{Z}$	$\mathbf{A}$
$\mathbb{Z}_p$	$\mathbf{A}_v$
$\mathbb{Q}_p$	$\mathbb{K}_v$

<u>Number fields:</u> finite extensions of $\mathbb{Q}$

<u>Function fields:</u> finite extensions of $\mathbb{F}_q(\mathbb{P}^1)$

- Unramified: $\mathbb{F}_{q^m} \to \mathbb{F}_q$

- Ramified: $C \to \mathbb{P}^1$ branched cover

Drinfeld modules

$\mathbf{A}$-field $\mathcal{F}$: endomorphism ring

$$\mathrm{End}_{\mathcal{F}}(\mathbb{G}_a) = \mathcal{F}\{\tau\}$$

Drinfeld $\mathbf{A}$-module over $\mathcal{F}$:
$\mathbb{F}_q$-algebra homomorphism

$$\Phi : \mathbf{A} \to \mathrm{End}_{\mathcal{F}}(\mathbb{G}_a), \quad a \mapsto \Phi_a(\tau) \in \mathcal{F}\{\tau\}$$

with $D \circ \Phi = \iota$ and $\Phi_a \neq \iota(a)\tau^0$ some $a \in \mathbf{A}$
(derivation $Df := a_0 = f'(\tau)$)

rank n: $\forall a \in \mathbf{A}$

$$\deg \Phi_a(\tau) = n \deg(a) = -n d_\infty v_\infty(a)$$

Case of $\mathbb{P}^1$: $\mathbb{K} = \mathbb{F}_q(T)$ and $\mathbf{A} = \mathbb{F}_q[T]$
$\deg(a)$ = degree as polynomial in $\mathbf{A} = \mathbb{F}_q[T]$

Torsion points

$\mathbf{A}$ Dedekind domain, ideal $I \subset \mathbf{A}$ at most 2 gen $\{i_1, i_2\}$

$\Phi_I =$ monic generator of left ideal in $\mathcal{F}\{\tau\}$ gen by Φ_{i_1}, Φ_{i_2}

$\Phi[I] =$ roots of Φ_I (finite group)
$\bar{\mathcal{F}}$-points of subgroup scheme Φ_I of $\mathbb{G}_a$
(For $a \in \mathbf{A}$, notation $\Phi[a] = \Phi[(a)]$)

<u>a-torsion points of Φ</u>: roots $\Phi[a]$ of polyn Φ_a
$\Phi[a]$ finite $\mathbf{A}$-module

Generic characteristic: $\Phi[a] \simeq (\mathbf{A}/(a))^n$
if Φ of rank n

Tate modules (generic characteristic)

For $v \in \Sigma_{\mathbf{A}}$, v-adic Tate module $T_v\Phi$

$$T_v\Phi := \mathrm{Hom}_{\mathbf{A}}(\mathbb{K}_v/\mathbf{A}_v, \Phi[v^\infty])$$

$\mathbf{A}_v$-module

$$\Phi[v^\infty] := \bigcup_{m \geq 1} \Phi[v^m]$$

$$T_v\Phi = \varprojlim_{m \in \mathbb{N}} \Phi[v^m]$$

generic characteristic: $T_v\Phi$ free $\mathbf{A}_v$-module rank n

<u>Adelic Tate module</u>

$$T\Phi = \prod_{v \in \Sigma_{\mathbf{A}}} T_v\Phi$$

(like total Tate module of an elliptic curve)

$T\Phi$ free module of rank n over R

Isogenies

Φ and Ψ of rank n isogenous iff $\exists P(\tau) \in \mathcal{F}\{\tau\}$

$$P\Phi_a = \Psi_a P, \qquad \forall a \in \mathbf{A}$$

Isomorphism: isogeny P of degree zero
(*i.e.* $\exists\, Q \in \mathcal{F}\{\tau\}$ such that $P \cdot Q = \tau^0$)

Isogeny: equiv rel on Drinfeld modules

Category:
Objects = Drinfeld modules
Morphisms = isogenies

$$\Phi \mapsto T\Phi$$

covariant functor to R-modules

Isogeny P determined by action $T_v(P)$ on Tate modules (generic characteristic)

Pointed Drinfeld modules

$(\Phi, \zeta_1, \ldots, \zeta_n)$ n-pointed Drinfeld $\mathbf{A}$-module over $\mathcal{F}$: rank n Drinfeld module Φ with $\zeta_i \in T\Phi$

Commensurability

$$(\Phi, \zeta_1, \ldots, \zeta_n) \sim (\Psi, \eta_1, \ldots, \eta_n)$$

if isogeny P such that $P\Phi_a = \Psi_a P$ and

$$(\eta_i)_v = T_v(P)(\zeta_i)_v$$

action on v-adic Tate modules

Equivalence relation

$\mathcal{D}^{\mathcal{F}}_{\mathbb{K},n}$ = commensurability classes

Level structures and degenerations

Φ be a Drinfeld $\mathbf{A}$-module of rank n over $\mathcal{F} = \mathbb{L}$, gen.char.

level I structure $I \subset \mathbf{A}$ non-zero ideal isomorphism:

$$\rho_I : (I^{-1}\mathbf{A}/\mathbf{A})^n \stackrel{\simeq}{\to} \Phi[I]$$

Moduli spaces:

$$\mathcal{M}^n = \varprojlim_I \mathcal{M}^n_I$$

$\mathcal{M}^n_I =$ isom classes of Drinfeld mods w/ level structure

Degenerate level structures: ρ_I not necessarily isomorphisms

$$\mathcal{M}^n_{nc} = \mathcal{D}^{\mathbb{L}}_{\mathbb{K},n}$$

Analog of $\mathbb{Q}$-lattices and commensurability

$\mathcal{D}^{\mathbb{L}}_{\mathbb{K},n} =$ pairs (Φ, ζ) Drinfeld module and

$$\zeta : R^n \to T\Phi \simeq R^n$$

R-module homomorphism, up to isogenies

with $T\Phi \simeq R^n$ in generic char

$$\begin{array}{ccc} & & T\Phi \\ & \overset{u_\Phi}{\nearrow} & \\ R^n & & \downarrow TP \\ & \underset{u_\Psi}{\searrow} & \\ & & T\Psi \end{array}$$

$\mathcal{D}^{\mathbb{L}}_{\mathbb{K},n} =$ moduli space of isogeny classes of Drinfeld modules with degenerate level structure

$$\rho_I : (I^{-1}/\mathbf{A})^n \to \Phi[I]$$

obtained from the

$$\zeta_{v^m} : (\mathbf{A}/v^m\mathbf{A})^n \to \Phi[v^m]$$

Classical case: only isomorphisms

$$(\mathbf{A}/a\mathbf{A})^n \xrightarrow{\zeta_{(a)}} \Phi[a] \xrightarrow{P} \Psi[a]$$

has a nontrivial kernel: not a level structure

Lattices Various notions of lattices in function fields

(1)- lattice Λ of rank n: finitely generated $\mathbf{A}$-submodule of $\mathbb{K}^n$

$$\Lambda \otimes \mathbb{K}_\infty \simeq \mathbb{K}_\infty^n$$

(2)- lattice Λ in $\mathbb{C}_\infty$: discrete $\mathbf{A}$-submodule with $\mathbb{K}_\infty \Lambda$ finite dimensional $\mathbb{K}_\infty$-vector space

(3)- $\mathbb{L}$ = complete subfield of $\mathbb{C}_\infty$ containing $\mathbb{K}_\infty$. $\mathbb{L}$-lattice Λ: $\mathbf{A}$-submodule of $\mathbb{C}_\infty$ with
- Λ finitely generated as $\mathbf{A}$-module
- Λ discrete in topology of $\mathbb{C}_\infty$
- Λ in separable closure $\mathbb{L}^{sep}$ of $\mathbb{L}$ stable under Gal($\mathbb{L}^{sep}/\mathbb{L}$)

In rank 1 case: equivalent!

Lattices and Drinfeld modules

Equivalence of categories:

$$\left\{ \begin{array}{c} \text{Drinfeld modules} \\ \text{rk } n \text{ over } \mathbb{L} \end{array} \right\} \Leftrightarrow \{\text{rk } n \ \mathbb{L}\text{-lattices}\}$$

$\mathbb{L} =$ complete subfield of $\mathbb{C}_\infty$ containing $\mathbb{K}_\infty$

Exponential function of Λ

$$e_\Lambda(az) = \Phi_a^\Lambda(e_\Lambda(z))$$

for $a \in \mathbf{A}$ and with $\Phi_a^\Lambda \in \mathbb{L}\{\tau\}$

(e_Λ entire surjective function on $\mathbb{C}_\infty$)

$$a \mapsto \Phi_a^\Lambda$$

Drinfeld module; conversely given Φ

$$\Lambda = \text{Ker}(e_\Phi)$$

Isogeny $P \Rightarrow e_\Psi^{-1} P e_\Phi = \lambda \in \mathbb{L}$ with $\lambda\Lambda \subset \Lambda'$

Tate module $T\Phi_\Lambda \cong \Lambda \otimes_{\mathbf{A}} R$

- Scaling action on lattices $\Lambda \sim \Lambda'$ for $\Lambda = \lambda\Lambda'$

Drinfeld modules and noncommutative tori

Characteristic zero analogs:

Drinfeld modules of rank 1 $\Leftrightarrow \mathbb{G}_m$

$$\mathbb{G}_m(\mathbb{C}) = \mathbb{C}/\mathbb{Z} = \mathbb{C}^*$$

$\mathbb{C}_\infty$ mod rank one Λ

Drinfeld modules of rank 2 $\Leftrightarrow$ elliptic curves

$$E_\tau(\mathbb{C}) = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$$

$\mathbb{C}_\infty$ mod rank two Λ

Drinfeld modules of rank $n \geq 2 \Leftrightarrow$ noncommutative tori

$$\mathcal{A}_\theta = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau + \mathbb{Z}\theta)$$

$\mathbb{C}_\infty$ mod rank n lattice Λ

In function fields $\mathbb{C}_\infty$ infinite dimensional over $\mathbb{K}_\infty$: contains discrete Λ of arbitrary rank!

Anderson t-motives: analog of abelian varieties

$\mathbb{K}$-lattices

n-dimensional $\mathbb{K}$-rational lattice (Λ, ϕ)

$\Lambda =$ lattice (version (1))

$$\phi : (\mathbb{K}/\mathbf{A})^n \to \mathbb{K}\Lambda/\Lambda$$

$\mathbf{A}$-modules homomorphism

n-dimensional $\mathbb{K}$-rational $\mathbb{L}$-lattice (Λ, ϕ)

same with Λ as in version (3)

On torsion points

$$\phi_{|\mathrm{a-tor}} : (\mathbf{A}/a\mathbf{A})^n \to a^{-1}\Lambda/\Lambda$$

Invertible (Λ, ϕ): if ϕ isomorphism

Isomorphism classes $\mathcal{K}_{\mathbb{K},n}$, $\mathcal{K}^{\mathbb{L}}_{\mathbb{K},n}$, $\mathcal{K}^{\mathbb{C}_\infty}_{\mathbb{K},n}$

Parameter spaces

$\tilde{\Omega}^n = \{\tilde{\omega} = (\omega_1, \ldots, \omega_n) \in \mathbb{C}_\infty^n | \omega_i \text{ lin.ind.over } \mathbb{K}_\infty\}$

$$\Omega^n = \tilde{\Omega}^n / \mathbb{C}_\infty^*$$

complement of $\mathbb{K}_\infty$-hyperplanes in $\mathbb{P}^{n-1}(\mathbb{C}_\infty)$

point $z \in \Omega^n \Leftrightarrow \mathbb{K}_\infty$-monomorphism up to $\mathbb{C}_\infty^*$

$$\iota_z : \mathbb{K}_\infty^n \to \mathbb{C}_\infty$$

adèlic description of lattices $\Lambda(g) = R^n g^{-1} \cap \mathbb{K}$

$$g \in GL_n(\mathbb{A}_{\mathbb{K},f}) \Rightarrow \Lambda = \Lambda(g) \subset \mathbb{K}^n$$

$$\Lambda \cdot R = R^n g^{-1} \subset \mathbb{A}_{\mathbb{K},f}^n$$

Identifications

$$\mathcal{K}_{\mathbb{K},n} = \mathrm{GL}_n(\mathbb{K}) \backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times_{\mathrm{GL}_n(R)} M_n(R)$$
$$(\Lambda, \phi) = (R^n g^{-1} \cap \mathbb{K}, \rho g^{-1})$$

$$\mathcal{K}_{\mathbb{K},n}^{\mathbb{C}_\infty} = \mathrm{GL}_n(\mathbb{K}) \backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n \times_{\mathrm{GL}_n(R)} M_n(R)$$
$$(\Lambda, \phi) = (\iota_z(R^n g^{-1} \cap \mathbb{K}), \iota_z(\rho g^{-1}))$$

Combines isomorphism classes of lattices

$$\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})/\mathrm{GL}_n(R)$$

$$\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n/\mathrm{GL}_n(R)$$

with data for ϕ using $R = \mathrm{Hom}(\mathbb{K}/\mathbf{A}, \mathbb{K}/\mathbf{A})$

$$\begin{array}{ccccc} (\mathbb{K}/\mathbf{A})^n & \xrightarrow{\rho} & (\mathbb{K}/\mathbf{A})^n & \longrightarrow & \mathbb{K}^n/(R^n g^{-1} \cap \mathbb{K}) \\ \downarrow\cong & & \downarrow\cong & & \downarrow\cong \\ (\mathbb{A}_{\mathbb{K},f}/R)^n & \xrightarrow{\rho} & (\mathbb{A}_{\mathbb{K},f}/R)^n & \xrightarrow{g^{-1}} & \mathbb{A}^n_{\mathbb{K},f}/R^n g^{-1} \end{array}$$

<u>Rank one case:</u>

$$\mathcal{K}_{\mathbb{K},1} \simeq R \times_{R^*} (\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*) \simeq \mathcal{K}^{\mathbb{C}_\infty}_{\mathbb{K},1}$$

$\Omega^1 =$ point

Version with scaling: $\tilde{\mathcal{K}}_{\mathbb{K},1}$ of (Λ, ϕ)
with $\Lambda = \xi I$ with $\xi \in \mathbb{K}^*_\infty$ and $I \subset \mathbf{A}$ ideal

$$\phi : \mathbb{K}/\mathbf{A} \to \mathbb{K}\Lambda/\Lambda$$

$\mathbf{A}$-module homomorphism

$$\tilde{\mathcal{K}}_{\mathbb{K},1} \simeq R \times_{R^*} (\mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*)$$

$\mathbb{A}^*_{\mathbb{K}} = \mathbb{A}^*_{\mathbb{K},f} \times \mathbb{K}^*_\infty$

Commensurability of $\mathbb{K}$-lattices $(\Lambda, \phi) \sim (\Lambda', \phi')$

$\exists \gamma \in \mathrm{GL}_n(\mathbb{K})$ with $\Lambda' = \Lambda\gamma$ and $\phi' = \gamma \circ \phi$

$$\begin{array}{ccccc} \phi : (\mathbb{K}/\mathbf{A})^n & \xrightarrow{\rho} & (\mathbb{K}/\mathbf{A})^n & \xrightarrow{g^{-1}} & \mathbb{K}^n/(R^n g^{-1} \cap \mathbb{K}) \\ \downarrow \cong & & \downarrow \cong & & \downarrow \gamma^{-1} \\ \phi' : (\mathbb{A}_{\mathbb{K},f}/R)^n & \xrightarrow{\rho} & (\mathbb{A}_{\mathbb{K},f}/R)^n & \xrightarrow{g^{-1}\gamma^{-1}} & \mathbb{A}^n_{\mathbb{K},f}/R^n g^{-1}\gamma^{-1} \end{array}$$

Quotient by commensurability: $\mathcal{L}_{\mathbb{K},n}$, $\mathcal{L}^{\mathbb{C}_\infty}_{\mathbb{K},n}$, $\tilde{\mathcal{L}}_{\mathbb{K},1}$

$$\mathcal{L}_{\mathbb{K},n} = \mathrm{GL}_n(\mathbb{K}) \backslash M_n(\mathbb{A}_{\mathbb{K},f})$$

$$\Theta : \mathrm{GL}_n(\mathbb{K}) \backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times_{\mathrm{GL}_n(R)} M_n(R) \to \mathrm{GL}_n(\mathbb{K}) \backslash M_n(\mathbb{A}_{\mathbb{K},f})$$

$$(g, \rho) \mapsto \rho g^{-1}$$

$\gamma \in \mathrm{GL}_n(\mathbb{K})$ acts by $u \mapsto u\gamma^{-1}$, for $u \in M_n(\mathbb{A}_{\mathbb{K},f})$

$$\mathcal{L}^{\mathbb{C}_\infty}_{\mathbb{K},n} = \mathrm{GL}_n(\mathbb{K}) \backslash M_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n$$

$$\Upsilon : \mathcal{K}^{\mathbb{C}_\infty}_{\mathbb{K},n} \to \mathrm{GL}_n(\mathbb{K}) \backslash M_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n$$

$$\mathcal{K}^{\mathbb{C}_\infty}_{\mathbb{K},n} = \mathrm{GL}_n(\mathbb{K}) \backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n \times_{\mathrm{GL}_n(R)} M_n(R)$$

$$\Upsilon(g, z, \rho) = (\rho g^{-1}, z)$$

Noncommutative spaces:

The classical quotients (lattices up to isomorphism)

$$\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f})$$

$$\mathrm{GL}_n(\mathbb{K})\backslash \mathrm{GL}_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n$$

are "good quotients"

The spaces of commensurability classes of $\mathbb{K}$-lattices

$$\mathrm{GL}_n(\mathbb{K})\backslash M_n(\mathbb{A}_{\mathbb{K},f})$$

$$\mathrm{GL}_n(\mathbb{K})\backslash M_n(\mathbb{A}_{\mathbb{K},f}) \times \Omega^n$$

are "bad quotient": action of $\mathrm{GL}_n(\mathbb{K})$ on $M_n(\mathbb{A}_{\mathbb{K},f})$

$\Rightarrow$ noncommutative spaces, convolution algebras

Compatible commensurability relations

n-pointed Drinfeld modules and $\mathbb{K}$-lattices of rank n up to commensurability

$$\mathcal{L}^{\mathbb{C}_\infty}_{\mathbb{K},n} = \mathcal{D}^{\mathbb{C}_\infty}_{\mathbb{K},n}$$

(equivalence of categories between lattices and Drinfeld modules)

$\mathcal{K}^{\mathbb{C}_\infty}_{\mathbb{K},n} \to \mathcal{D}^{\mathbb{C}_\infty}_{\mathbb{K},n}$ sending (Λ, ϕ) to $(\Phi, \zeta_1, \ldots, \zeta_n)$

$$\Phi = \Phi^\Lambda \quad \zeta_i = \widehat{\phi}(e_i)$$

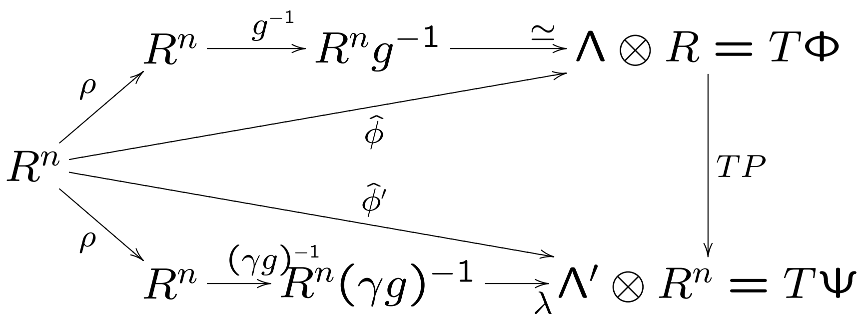
$\{e_i\}$ standard basis of R^n, taking $\mathrm{Hom}(-, \mathbb{K}/\mathbf{A})$ identify $\mathbf{A}$-homomorphism $\phi : \mathbb{K}/\mathbf{A} \to \mathbb{K}\Lambda/\Lambda$ with R-homomorphism $\widehat{\phi} : R^n \to \Lambda \otimes_{\mathbf{A}} R$

$(\Lambda,\phi) \sim (\Lambda',\phi') \Rightarrow$ same in $\mathcal{D}^{\mathbb{C}_\infty}_{\mathbb{K},n}$:
$\Lambda = \iota_z(R^n g^{-1} \cap \mathbb{K})$ and

$$\Lambda' = \iota_{z'}(R^n {g'}^{-1} \cap \mathbb{K}) = \lambda \iota_z(R^n(\gamma g)^{-1} \cap \mathbb{K})$$

(g,z,ρ) and (g',z',ρ) with $g' = \gamma g$ and $z' = \gamma z \lambda$ for some $g \in \mathrm{GL}_n(\mathbb{K})$ and $\lambda \in \mathbb{C}_\infty^*$.

$\lambda \in \mathbb{C}_\infty^* \Rightarrow$ isogeny P
R-homomorphisms related by



Conversely, $(\Phi,\zeta_1,\ldots,\zeta_n) \sim (\Psi,\xi_1,\ldots,\xi_n)$:
$P \Rightarrow$ morphism of lattices $\lambda \in \mathbb{C}_\infty$ with $\lambda\Lambda \subset \Lambda'$

$\mathbb{K}\lambda\Lambda$ and $\mathbb{K}\Lambda'$: $\exists \gamma \in \mathrm{GL}_n(\mathbb{K})$ with $\Lambda' = \lambda\Lambda\gamma$
$(\xi_i) = TP(\zeta_i)$ gives $\widehat{\phi}' = \lambda(\gamma \circ \widehat{\phi})$

Rank one case:

$\mathcal{L}_{\mathbb{K},1} = \mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$ and $\tilde{\mathcal{L}}_{\mathbb{K},1} = \mathbb{A}^{\cdot}_{\mathbb{K}}/\mathbb{K}^*$
(where $\mathbb{A}^{\cdot}_{K} = \mathbb{A}_{\mathbb{K},f} \times \mathbb{K}^*_\infty$)

$$\tilde{\Theta} : R \times_{R^*} \mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^* \to \mathbb{A}^{\cdot}_{\mathbb{K}}/\mathbb{K}^*$$

$$\tilde{\Theta}(\rho, s, \xi) = \rho s^{-1} \xi^{-1}$$

Drinfeld: action of $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ (class field theory)

$$\mathcal{M}^1(\mathbb{K}_\infty) = \mathcal{M}^1(\overline{\mathbb{K}}_\infty) = \mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$$

covering $\tilde{\mathcal{M}}^1$ of $\mathcal{M}$ with $\mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*$ action

Noncommutative: $(\mathcal{M}^1)^{nc} = \mathcal{L}_{\mathbb{K},1} = \mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$
and $(\tilde{\mathcal{M}}^1)^{nc} = \tilde{\mathcal{L}}_{\mathbb{K},1} = \mathbb{A}^{\cdot}_{\mathbb{K}}/\mathbb{K}^*$

Adèle class space: $\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$
(Connes trace formula, spectral realization)

Quantum statistical mechanics:

$(\mathcal{A}, \sigma_t)$ C^*-algebra and time evolution

State: $\varphi : \mathcal{A} \to \mathbb{C}$ linear $\varphi(1) = 1$, $\varphi(a^*a) \geq 0$

Representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$: Hamiltonian

$$\pi(\sigma_t(a)) = e^{itH}\pi(a)e^{-itH}$$

Partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$

KMS states: $\varphi(a\, \sigma_{i\beta}(b)) = \varphi(ba)$
for all $a, b \in \mathcal{A}^{an} \subset \mathcal{A}$ analytic elements

For function fields and rk 1 Drinfeld modules: Benoit Jacob

<u>Problem with $\mathbb{C}$-valued functions:</u> No possible intertwining of symmetries and Galois action

Remain in positive characteristic!
(Renounce C^*-algebras and Hilbert spaces)

Convolution algebras: (in characteristic $p > 0$)
$\mathbb{L}$ complete subfield of $\mathbb{C}_\infty$ containing $\mathbb{K}_\infty$

$\mathcal{A}_{\mathbb{L}}(\mathcal{L}_{\mathbb{K},1})$ = continuous, compactly supported, $\mathbb{L}$-valued functions $f(L, L')$ of $L = (\Lambda, \phi) \sim L' = (\Lambda', \phi')$ commensurable $\mathbb{K}$-lattices

convolution product

$$(f_1 * f_2)(L, L') = \sum_{L \sim L'' \sim L'} f_1(L, L'') f_2(L'', L')$$

$\mathcal{A}_{\mathbb{L}}(\mathcal{L}^{\mathbb{C}_\infty}_{\mathbb{K},1})$ and $\mathcal{A}_{\mathbb{L}}(\tilde{\mathcal{L}}_{\mathbb{K},1})$ similar

$\mathcal{A}_{\mathbb{L}}(\mathcal{D}_{\mathbb{K},1})$ = comp supp, $\mathbb{L}$-valued functions

$$f((\Phi, \zeta), (\Psi, \xi)), \qquad (\Phi, \zeta) \sim (\Psi, \xi)$$

convolution

$$(f_1 * f_2)((\Phi, \zeta), (\Psi, \xi)) = \sum f_1((\Phi, \zeta), (\Xi, \eta)) f_2((\Xi, \eta), (\Psi, \xi))$$

for $(\Phi, \zeta) \sim (\Xi, \eta) \sim (\Psi, \xi)$

Representations:

$$c(L) = \{L' \in \mathcal{K}_{\mathbb{K},1} \,|\, L' \sim L\}$$

commensurability class $\mathcal{V}_L$ = compactly supported $\mathbb{L}$-valued functions on $c(L)$

$\|\xi\|$ non-archimedean Banach space completion (no Hilbert spaces)

$$\pi_L : \mathcal{A}_{\mathbb{L}}(\mathcal{L}_{\mathbb{K},1}) \to \mathrm{End}(\mathcal{V}_L)$$

$$\pi_L(f)(\xi)(L') = \sum_{L'' \in c(L)} f(L', L'')\xi(L'')$$

$$\|f\|_{\pi_L} = \sup_{\xi \neq 0 \in \mathcal{V}_L} \frac{\|\pi_L(f)(\xi)\|}{\|\xi\|}$$

non-archimedean Banach algebra

The problem with involutions:

- Complex numbers $\mathbb{C}$: polar decomposition $z = |z|e^{i\theta}$ with $\mathbb{C}^* = \mathbb{R}^*_+ \times U(1)$
$\Rightarrow z \mapsto \bar{z} = |z|e^{-i\theta}$ with $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

- Function fields: sign functions

$$\text{sign} : \mathbb{K}^*_\infty \to \mathbb{F}^*_{q^{d_\infty}}, \quad \text{sign}|_{\mathbb{F}^*_{q^{d_\infty}}} = id$$

extended to sign(0) = 0 to $\mathbb{K}_\infty = \mathbb{F}_{q^{d_\infty}}((u_\infty))$
$\Rightarrow$ positivity: $x \in \mathbb{K}^*_\infty$ with sign$(x) = 1$
$\#\mathbb{F}^*_{q^{d_\infty}} = q^{d_\infty} - 1$ choices of sign

$$\text{sign}'(x) = \text{sign}(x)\xi^{\deg(x)/d_\infty}, \quad \xi \in \mathbb{F}^*_{q^{d_\infty}}$$

u_∞ = uniformizer at $\infty \Rightarrow$ sign$(x) = \zeta$

$$\mathbb{K}^*_\infty \ni x = \zeta u^m_\infty \gamma \in \mathbb{F}^*_{q^{d_\infty}} \times u^{\mathbb{Z}}_\infty \times U_1$$

U_1 = group of 1-units: $u \in \mathcal{O}_\infty$ with $u \equiv 1$ mod max ideal $\mathfrak{m}_\infty$ of ring of integers $\mathcal{O}_\infty$ of $\mathbb{K}_\infty$

But: $x = \text{sign}(x) u_\infty^{v_\infty(x)} \langle x \rangle$ not additive

Exponentiation in function fields:

- In complex numbers $\lambda \in \mathbb{R}_+^*$ and $s = x+iy \in \mathbb{C}$ exponential $\lambda^s = \lambda^x e^{iy \log \lambda}$

- In function fields: $\lambda \in \mathbb{K}_\infty^*$ positive ($\text{sign}(\lambda) = 1$) and $s = (x, y) \in S_\infty = \mathbb{C}_\infty^* \times \mathbb{Z}_p$

$$\lambda^s = x^{\deg(\lambda)} \langle \lambda \rangle^y$$

with $\deg(\lambda) = -d_\infty v_\infty(\lambda)$ and

$$\langle \lambda \rangle^y = \sum_{j=0}^{\infty} \binom{y}{j} (\langle \lambda \rangle - 1)^j$$

For $y \in \mathbb{Z}_p$ binomial coefficients

$$\binom{y}{k} = \frac{y(y-1)\cdots(y-k+1)}{k!}$$

mod p: continuous functions with values in $\mathbb{F}_p$

Exponentiation $s \mapsto \lambda^s$ entire function $S_\infty \to \mathbb{C}_\infty^*$ with $\lambda^{s+t} = \lambda^s \lambda^t$

Goss L-functions:

- Exponentiation of ideals: principal $I = (a)$

$$I^s = x^{-v_\infty(a) d_\infty} \langle a \rangle^y$$

$\exists$ unique extension of $a \mapsto \langle a \rangle$ to fractional ideals $I \mapsto \langle I \rangle$

$$I^s = x^{\mathsf{deg}(I)} \langle I \rangle^y, \ s = (x, y) \in S_\infty$$

- Goss L-function ($s \in S_\infty$)

$$L(s) = \sum_{I \subset \mathbf{A}} I^{-s}$$

convergence in the "half-plane"

$$\{s = (x, y) \in S_\infty : |x|_\infty > q\} \subset S_\infty$$

(for $\mathbb{K} = \mathbb{F}_q(C)$)

Time evolution:

$$\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$$

continuous homomorphism with $y \mapsto \pi(\sigma_y(a))\xi$ continuous for all $a \in \mathcal{A}$ and $\xi \in \mathcal{V}$

On the "line"

$$S_\infty \supset \{s = (1, y) \in S_\infty \,:\, y \in \mathbb{Z}_p\} \cong \mathbb{Z}_p$$

Note: switch of real and imaginary directions

Lattice $L = (\Lambda, \phi)$: ideal $\Lambda = \xi I$, $\xi \in \mathbb{K}_\infty^*$,
$I = sR \cap \mathbb{K}$, $s \in \mathrm{GL}_1(\mathbb{A}_{\mathbb{K},f})$

$$(\sigma_y f)(L, L') = \frac{\langle I \rangle^y}{\langle J \rangle^y} f(L, L'), \quad \forall y \in \mathbb{Z}_p$$

time evolution on $\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$

Extends analytically to

$$(\sigma_s f)(L, L') = \frac{I^s}{J^s} f(L, L'), \quad \text{for} \quad s = (x, y) \in S_\infty$$

Partition function:

$\mathbb{L}$-vector space V, w/non-archimedean norm,
linear basis $\{\epsilon_\alpha\}$; $\langle \epsilon_\beta, T\epsilon_\alpha \rangle \in \mathbb{L}$

$$\mathrm{Tr}_V(T) = \sum_\alpha \langle \epsilon_\alpha, T\epsilon_\alpha \rangle$$

(non-archimedean nuclear spaces $\Rightarrow$ indep of basis)

Time evolution in a representation (Hamiltonian)

$$\pi(\sigma_s(f)) = U(s)\pi(f)U(s)^{-1}, \quad \forall f \in \mathcal{A} \ \forall s \in S_\infty$$

$U(s_0) \in \mathrm{Aut}(V)$, with $s_0 = (0, 1)$: "exp of Hamiltonian"

Partition function:

$$Z(s) = \mathrm{Tr}_V(U(s)^{-1})$$

Goss L-function as a partition function

For time evolution $(\sigma_y f)(L, L') = \langle I \rangle^y / \langle J \rangle^y f(L, L')$

Representation π_L on $\mathcal{V}_L$ functions on $c(L)$
$L = (\Lambda, \phi)$ invertible $\Rightarrow$ $c(L) =$ ideals $J \subset \mathbf{A}$

$$(U(s)\xi)(L') = J^s \, \xi(L')$$

$L' = (\Lambda', \phi') \in c(L)$ with $\Lambda' \Rightarrow$ ideal $J \subset \mathbf{A}$

$$\langle \epsilon_{J'}, \pi_L(f) \epsilon_J \rangle = f({J'}^{-1} L, J^{-1} L)$$

$\Lambda = R(su)^{-1} \cap \mathbb{K}$, for $u \in R \cap \mathbb{A}^*_{\mathbb{K},f}$ and $J = Ru \cap \mathbb{K}$

Partition function:

$$Z(s) = L(s) = \sum_{I \subset \mathbf{A}} I^{-s}$$

KMS functionals: inverse temperature $x \in \mathbb{C}_\infty^*$

Given $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$ which extends analytically to $\sigma : S_\infty \to \mathrm{Aut}(\mathcal{A})$

continuous linear functional

$$\varphi(f_1 \sigma_x(f_2)) = \varphi(f_2 f_1), \qquad \forall f_1, f_2 \in \mathcal{A}$$

with $\sigma_x = \sigma_{s=(x,0)}$, normalization $\varphi(1) = 1$

For time evolution $(\sigma_y f)(L, L') = \langle I \rangle^y / \langle J \rangle^y f(L, L')$

$$\varphi_{x,L}(f) = Z(x)^{-1} \sum_{J \subset \mathbf{A}} f(J^{-1}L, J^{-1}L)\, J^{-x} \in \mathbb{C}_\infty$$

KMS$_x$-functional for $|x|_\infty > q$
with $L = (\Lambda, \phi)$ invertible

KMS condition:

$$Z(x)\varphi_{x,L}(f_1 * \sigma_x(f_2)) =$$

$$\sum_J \sum_{\tilde{J}} f_1(J^{-1}L, \tilde{J}^{-1}L) f_2(\tilde{J}^{-1}L, J^{-1}L) \frac{J^x}{\tilde{J}^x} J^{-x} =$$

$$\sum_{\tilde{J}} f_2 * f_1(\tilde{J}^{-1}L, \tilde{J}^{-1}L)\tilde{J}^{-x} = Z(x)\varphi_{x,L}(f_2 * f_1).$$

Continuity:

$$|f(J^{-1}L, J^{-1}L)| \le \sup_{L' \in c(L)} |(\pi_L(f)\epsilon_J)(L')| \le \sup_{\xi \neq 0} \frac{\|\pi_L(f)\xi\|}{\|\xi\|}$$

$$\left| \sum_{J \subset \mathbf{A}} f(J^{-1}L, J^{-1}L)\, J^{-x} \right| \le \|\pi_L(f)\|\, |Z(x)|$$

Combinations:

$$\varphi_{x,\mu}(f) = \int \varphi_{x,L}(f)\, d\mu(L)$$

normalized $\mathbb{C}_\infty$-valued non-archimedean measure μ on set of isom classes of invertible $\mathbb{K}$-lattices

Classical points and KMS functionals

Fits with general philosophy:

Classical moduli space $\mathcal{M}^1$

Noncommutative space $(\mathcal{M}^1)^{nc} = \mathcal{L}_{\mathbb{K},1}$
$\mathcal{A}(\mathcal{L}_{\mathbb{K},1})$ with time evolution σ_y

Points of classical moduli space $\Rightarrow$
low temperature (large $|x|_\infty > q$)
KMS states of $(\mathcal{A}(\mathcal{L}_{\mathbb{K},1}), \sigma_y)$

Questions: Phase transitions? Classification?

Symmetries of $(\mathcal{A}, \sigma)$

algebra homomorphism $U : \mathcal{A} \to \mathcal{A}$
with $U\sigma_s = \sigma_s U$ (Note: $U(1) \neq 1$ idempotent)

$$U^* : \varphi \mapsto \varphi(U(1))^{-1}\, \varphi \circ U$$

if $\varphi(U(1)) \neq 0$

If $u \in \mathcal{A}$ has left inverse $v \in \mathcal{A}$ inner

$$U(f) = ufv$$

with $\sigma_s(u) = \lambda^s u$ act trivially on KMS$_x$ states:

$$U^*(\varphi)(f) = \frac{\varphi(U(f))}{\varphi(U(1))} = \lambda^{-x}\, \varphi(ufv) = \lambda^{-x}\, \varphi(fv\sigma_x(u)) = \varphi(f)$$

For time evolution $(\sigma_y f)(L,L') = \langle I\rangle^y/\langle J\rangle^y f(L,L')$

Symmetries: semigroup $R \cap \mathbb{A}^*_{\mathbb{K},f}$

$$\theta_u(f)(L,L') = \begin{cases} f(L_u, L'_u) & L, L' \text{ divisible by } J \\ 0 & \text{otherwise.} \end{cases}$$

$L = (\Lambda, \phi) = (s, \rho)$ divisible by J: $s = s_u u \in \mathbb{A}^*_{\mathbb{K},f}$ and $\rho = \rho_u u \in R$, $J = Ru \cap \mathbb{K}$, $L_u = (s_u, \rho_u)$

$$\sigma_s(\theta_u(f))(L,L') = \begin{cases} \frac{I_u^s}{J_u^s} f(L_u, L'_u) & L, L' \text{ divisible by } J \\ 0 & \text{otherwise} \end{cases}$$

Sub-semigroup: $\mathbf{A} \setminus \{0\}$ inner

$$\mu_J(L,L') = \begin{cases} 1 & L = L'_u \\ 0 & \text{otherwise.} \end{cases}$$

$$U_J(f) = \mu_J * f * \tilde{\mu}_J$$

$\Rightarrow$ induced action $\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$ on KMS$_x$ states

Class field theory action on classical moduli space $\mathcal{M}^1$

v-**adic time evolutions**

$$\mathbb{A}_{\mathbb{K},v} = \{a = (a_w)_{w \in \Sigma_{\mathbb{K}}} \in \mathbb{A}_{\mathbb{K}} \,|\, a_v = 0\}$$

$\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*$ strata contributing to trace formula

Time evolutions

$$S_v = \mathbb{C}_v^* \times \mathbb{Z}_p \times \mathbb{Z}/(q^{d_v f_\tau} - 1)\mathbb{Z}$$

exponentiation of ideals extends to I^{s_v}

$$\sigma^v : \mathbb{Z}_p \times \mathbb{Z}/(q^{d_v f_\tau} - 1)\mathbb{Z} \to \mathrm{Aut}(\mathcal{A})$$

on $\mathcal{A}_{\mathbb{C}_v}(\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*)$

$$(\sigma_y^v f)(L, L') = \frac{I^y}{J^y} f(L, L'), \quad \forall y \in \mathbb{Z}_p \times \mathbb{Z}/(q^{d_v f_\tau} - 1)\mathbb{Z}$$

Analogous to the systems

$$\sigma_t^v(f)(r, \rho, \lambda) = |r|_v^{it} f(r, \rho, \lambda)$$

for $\mathbb{Q}$-lattices case

The dual system

Usual C^*-algebra setting: $(\mathcal{A}, \sigma)$ dual $(\hat{\mathcal{A}}, \theta)$

$$\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}, \quad f = \int \ell(t) U_t dt$$

Scaling action

$$\theta_\lambda(f) = \int \lambda^{it} \ell(t) U_t dt$$

For $\mathcal{A}(\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q}^*)$ 1-dim $\mathbb{Q}$-lattices, dual system $\mathcal{A}(\mathbb{A}^{\cdot}_{\mathbb{Q}}/\mathbb{Q}^*) \Rightarrow$ adèles class space $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$

For function field case $(\mathcal{A}(\mathcal{L}_{\mathbb{K},1}), \sigma))$ also $\hat{\mathcal{A}}$ generated by $f = \int \ell(s) U_s d\mu(s)$ with

$$\theta_\lambda(f) = \int \lambda^s \ell(s) U_s d\mu(s)$$

$\mu =$ non-archimedean measure, $\lambda \in \mathbb{K}^*_{\infty,+}$ and $s \in H \subset S_\infty$

Non-archimedean measures: momenta

$$\int_{\mathbb{Z}_p} \binom{y}{k} d\mu(y) = X^{-k}$$

transform:

$$\hat{f}(X) = \sum_{k=0}^{\infty} f_k X^{-k} = \int_{\mathbb{Z}_p} f(y)\, d\mu(y)$$

$$f(y) = \sum_{k=0}^{\infty} f_k \binom{y}{k}$$

Time evolution $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$

$$\sigma_y(a) = \sum_{k=0}^{\infty} \sigma_k(a) \binom{y}{k}$$

$\sigma_k(a) \in \mathcal{A}$ for $k \in \mathbb{Z}_{\geq 0}$

Properties of time evolution:

$$\sigma_{k+m}(a) = \sigma_k(\sigma_m(a)), \quad \forall k, m \in \mathbb{Z}_{\geq 0}, \quad \forall a \in \mathcal{A}$$

$$\sigma_k(ab) = \sum_{j=0}^{k} \sigma_j(a)\sigma_{k-j}(b), \quad \forall k \in \mathbb{Z}_{\geq 0}, \quad \forall a, b \in \mathcal{A}$$

$$\sigma_y(\sigma_x(a)) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \sigma_{k-j}(\sigma_j(a)) \binom{y}{k-j}\binom{x}{j}$$

gives $\sigma_k(a) = \sigma_{k-j}(\sigma_j(a))$ from

$$\binom{y+x}{k} = \sum_{j=0}^{k} \binom{y}{k-j}\binom{x}{j}$$

gives form of $\sigma_{y+x}(a) = \sigma_y(\sigma_x(a))$

$$\Sigma_a(X) = \widehat{\sigma_{\cdot}(a)}(X) = \int_{\mathbb{Z}_p} \sigma_y(a) d\mu(y)$$

$$\Sigma_{ab}(X) = \sum_{k=0}^{\infty} \sigma_k(ab) X^{-k}$$

$$\Sigma_a(X)\Sigma_b(X) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} \sigma_j(a)\sigma_{k-j}(b) X^{-k}$$

gives form of $\sigma_y(ab) = \sigma_y(a)\sigma_y(b)$

Dual system:

$$\ell(y) = \sum_{k=0}^{\infty} \ell_k \binom{y}{k}, \quad \widehat{\ell}(X) = \sum_{k=0}^{\infty} \ell_k X^{-k}$$

$$\sigma_y(\ell(x)) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \sigma_{k-j}(\ell_j) \binom{y}{k-j} \binom{x}{j}$$

Convolution product:

$$(\widehat{\ell}_1 *_\sigma \widehat{\ell}_2)(X) = \sum_{r=0}^{\infty} \sum_{k=0}^{r} \sum_{j=0}^{r-k} a_k \sigma_{r-k-j}(b_j)\, X^{-r}$$

for $\ell_1(y) = \sum_k a_k \binom{y}{k}$ and $\ell_2(y) = \sum_k b_k \binom{y}{k}$, $a_k, b_k \in \mathcal{A}$

Scaling action: $\lambda = u_\infty^m \langle \lambda \rangle$

$$\langle \lambda \rangle^y = \sum_{j=0}^{\infty} \alpha_\lambda^j \binom{y}{j}$$

$$\theta_\lambda(\widehat{\ell}) = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \ell_j \alpha_\lambda^{k-j}\, X^{-k}$$

The Artin map and class field theory

local class field theory, non-archimedean local field K

$$\Theta : K^* \to \mathrm{Gal}(K^{ab}/K)$$

Artin homomorphism: injective

$$\Theta(\mathcal{O}^*) = \mathrm{Gal}(K^{ab}/K^{un})$$

inertia group

$$\Theta(u) = Fr$$

($u \in \mathcal{O}$ chosen uniformizer)

$$u^{\hat{\mathbb{Z}}} \simeq \mathrm{Gal}(K^{un}/K) \simeq \mathrm{Gal}(k^s/k)$$

$k =$ residue field, Fr generator of $\mathrm{Gal}(k^s/k)$

- Subgroup $u_\infty^{\mathbb{Z}}$ of $\mathbb{K}^*_{\infty,+}$ mapped to $Fr^{\mathbb{Z}}$

- Subgroup U_1 of $\mathbb{K}^*_{\infty,+}$ mapped to inertia group

Here $K = \mathbb{K}_\infty$, $\mathcal{O} = \mathbf{A}_\infty$, $k = \mathbb{F}_{q^{d_\infty}}$

Frobenius and scaling

$X = \int_H \ell(s) U_s d\mu(s)$ with scaling

$$\theta_\lambda(X) = \int_H \ell(s) \lambda^s U_s d\mu(s)$$

For $H = G \times \mathbb{Z}_p$, $G \subset \mathbb{C}_\infty^*$

$$\theta_\lambda|_G(X) := \theta_m(X) = \int_H \ell(s) x^{-d_\infty m} U_s d\mu(s)$$

$$\theta_\lambda|_{\mathbb{Z}_p}(X) := \theta_{\langle\lambda\rangle}(X) = \int_H \ell(s) \langle\lambda\rangle^y U_s d\mu(s)$$

Algebra $\hat{\mathcal{A}}$ maps to $\mathcal{A}(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*)$

$$f = \int_H \ell(s) U_s d\mu(s) \mapsto X_f(\lambda L, \lambda L')$$

$$X_f(\lambda L, \lambda L') = \int_H \ell(s)(L, L')\, \lambda^s\, U(s)\, d\mu(s)$$

- $\theta_\lambda|_G \mapsto Fr^{\mathbb{Z}}$ Frobenius
- $\theta_\lambda|_{\mathbb{Z}_p} \mapsto \mathrm{Gal}(K^{ab}/K^{un})$ inertia

when viewed as scaling action on lattices in $\tilde{\mathcal{L}}_{\mathbb{K},1}$ by $\lambda \in \mathbb{K}^*_{\infty,+} = u_\infty^{\mathbb{Z}} \times U_1$

and image under Artin homomorphism Θ

Question: intertwining of symmetries and Galois via states?