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Equivariant Seiberg-Witten Floer homology. (English summary)
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For almost half a century the homology three-spheres have occupied a special place in low-dimensional topology, as they are responsible for many exotic phenomena. Since the pioneering work of Andreas Floer in the early 1980s, the researchers in this field have adopted a new technology for the investigation of these elusive objects. We are referring of course to the various incarnations of Floer homology, the instanton (Yang-Mills-Donaldson) and the monopole (Seiberg-Witten) homology.

These two theories have many similar structural features: they are variational gauge theories associated to a functional on an infinite-dimensional space with the property that its critical points have infinite Morse indices but the relative ones are finite. To put it differently, although the (un)stable manifolds of the various critical points are infinite-dimensional, the intersection of the stable manifold of one critical point with the unstable manifold of another is a finite-dimensional orientable manifold. As observed by Floer and E. Witten, this is all one needs to transplant the constructions in the classical Morse theory to this "pathological" situation.

There are however important differences between the instanton and the monopole homology. The nonlinear analysis of the instanton homology is much more complicated due to the non-abelian nature of the gauge group, the presence of critical Sobolev exponents and the corresponding lack of compactness. To compensate for these difficulties, the nature and the structure of the critical points of this theory are very transparent and are intimately related to the topology of the background homology three-sphere: they are certain representations of the fundamental group of the manifold.

In the monopole case the nonlinear analysis is less sophisticated since the gauge group is abelian, and there is no extra lack of compactness due to critical Sobolev exponents. There are several new difficulties which balance out these technical gifts. The nature of the critical points is quite obscure, and to make matters worse, the resulting homology theory depends on the various parameters needed in its definition.

The root cause of this phenomenon is easy to explain. The energy functional whose Morse theory is used to define monopole homology is defined on an infinite-dimensional variety $\mathcal{C}$ which has a singular point (the so-called reducible monopole) whose link is the infinite-dimensional complex projective space $\mathbb{C}\mathrm{P}^\infty$. To formulate a Morse theory we need to remove this point from the picture. Unfortunately, there could still be gradient lines flowing towards or out of this "hole", and the nature and "number" of creeping lines depend on the choice of defining parameters, whence the dependence of the homology on these parameters. Fortunately, the source of this singularity is benign. The variety $\mathcal{C}$ is the S^1-quotient of a *smooth* manifold $\widetilde{\mathcal{C}}$, the so-called space of framed (Seiberg-Witten) configurations. This suggests that working equivariantly might lead to a topological object.

The reviewer is aware of three different approaches to this equivariant point of view. There is the approach adopted by Ozsváth and Szabó which abandons the gauge-theoretic interpretation and instead uses a version of the symplectic Floer theory naturally associated to a handlebody decomposition of the manifold. Although the relationship to the gauge-theoretic description is not apparent, numerical evidence suggests that the Ozsváth-Szabó homology ought to be isomorphic to the equivariant monopole homology.

Yet another approach in the gauge-theoretic context is due to Manolescu, but it is quite different in flavor from the typical constructions in Floer theory. More precisely, Manolescu constructs a sort of inductive limit of finite-dimensional S^1-manifolds, and the Floer theory is a sort of stable limit of the S^1-equivariant Morse-Conley theories of these spaces.

In this paper the authors follow an older approach pioneered by D. M. Austin and P. J. Braam [Math. Proc. Cambridge Philos. Soc. **118** (1995), no. 1, 125–139; MR 96f:55007; in *The Floer memorial volume*, 123–183, Progr. Math., 133, Birkhäuser, Basel, 1995; MR 96i:57037; Topology **35** (1996), no. 1, 167–200; MR 97e:57037] and work directly with the monopole energy on the space $\widetilde{C}$ of framed configurations. This approach, although conceptually the most transparent one, is burdened by many formidable technical obstacles too numerous to list here. The considerable length of this paper is due to the large amount of work required to establish "genericity", where the genericity attribute should be understood in the broader sense that we can cleverly place ourselves in the best of all possible worlds. One of the end products of this paper is that the S^1-equivariant monopole homology is a well-defined topological invariant of a three-manifold, in particular of a homology three-sphere. A secondary yet very important product of this work is the wealth of techniques and ideas developed in order to overcome all the above-mentioned difficulties. I believe these techniques will find many uses in other related instances.

It is too early to judge which of the above three points of view will "win" in the end. Each of them reflects one facet of the same concept, and the reviewer strongly believes that the more descriptions are available to researchers the greater the chances of progress on the topological/geometric front. Gauge theory is notorious for its technical demands, and it is still plagued by a lack of foundational writings. This paper will certainly fill in some of this foundational gap.