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Continued fractions, modular symbols, and noncommutative geometry. (English summary)
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FEATURED REVIEW.

The rotation algebra is a well-studied C^*-algebra, providing a ready example on which to test notions related to foliations and crossed products. It is defined as the C^*-algebra A_θ, where $\theta \in \mathbb{R}$, generated by two unitaries U and V satisfying

$$VU = \exp(2\pi i\theta)UV.$$

By an important result of M. A. Rieffel [Pacific J. Math. **93** (1981), no. 2, 415–429; MR 83b:46087; J. Pure Appl. Algebra **5** (1974), 51–96; 51 #3912], the algebras A_θ and $A_{\theta'}$, $\theta, \theta' \in \mathbb{R}$, are strongly Morita equivalent if and only if θ and θ' are in the same orbit of the fractional linear action of $\mathrm{PSL}(2,\mathbb{Z})$ on the projective real line $\mathrm{P}_1(\mathbb{R})$. Strongly Morita equivalent C^*-algebras have the same space of classes of irreducible representations, canonically isomorphic K-theory groups, and share many other properties [see A. Connes, *Noncommutative geometry*, Academic Press, San Diego, CA, 1994; MR 95j:46063(Chapter II, Appendix A)]. On putting $U = \exp(2\pi ix)$, $V = \exp(2\pi iy)$, $x, y \in \mathbb{R}$ and $\theta = 0$, we see that A_0 is the algebra of continuous functions on the 2-torus $\mathbb{R}^2/\mathbb{Z}^2$. In Connes's theory of noncommutative geometry, a dense subalgebra $\mathcal{A}_\theta$ of "smooth elements" of the rotation algebra A_θ is known as the "noncommutative 2-torus $\mathbb{T}_\theta$" and is an important case study in this theory. A generic element of $a \in \mathcal{A}_\theta$ is a formal sum

$$a = \sum_{(m,n)\in\mathbb{Z}^2} a(m,n)U^mV^n,$$

where the sequence $(a(m,n))_{(m,n)\in\mathbb{Z}^2}$ is of rapid decay. The algebra A_θ has a canonical trace function $\tau(\cdot)$ determined by

$$\tau(a) = a(0,0).$$

In [Inst. Hautes Études Sci. Publ. Math. No. 62 (1985), 257–360; MR 87i:58162] Connes computed the Hochschild and periodic cyclic cohomology of $\mathcal{A}_\theta$. He showed that the dimension of the Hochschild cohomology spaces depends on the Diophantine properties of θ, whereas the periodic cyclic cohomology is complex 2-dimensional in both odd and even degrees. The bases of the periodic cyclic cohomology can be described in terms of the trace τ and the natural derivations on $\mathcal{A}_\theta$ determined by

$$\delta_1(U^mV^n) = 2\pi i m U^m V^n, \quad \delta_2(U^mV^n) = 2\pi i n U^m V^n.$$

The noncommutative torus $\mathbb{T}_\theta$ was used by J. Bellissard [in *Statistical mechanics and field theory : mathematical aspects (Groningen, 1985)*, 99–156, Lecture Notes in Phys., 257, Springer, Berlin, 1986; MR 88e:46053; in *Localization in disordered systems (Bad Schandau, 1986)*, 61–74, Teubner, Leipzig, 1988; MR 91c:82005] in an application of noncommutative geometry to the Quantum Hall Effect (QHE) in physics. Bellissard gave a complete explanation of certain stable numerical quantities described by the QHE in terms of noncommutative topological invariants of $\mathbb{T}_\theta$, thereby completing earlier work of Novikov and Thouless.

The 2-torus $\mathbb{R}^2/\mathbb{Z}^2$ can be given a complex structure. The different possibilities for this structure are parametrized by the Poincaré upper half plane $\mathcal{H}$ of complex numbers with positive imaginary part. To $z \in \mathcal{H}$, we may associate the complex 1-dimensional torus

$$T_z = \mathbb{C}^2/(\mathbb{Z} + z\mathbb{Z}),$$

given by the complex numbers modulo translation by the lattice $\mathbb{Z} + z\mathbb{Z}$. The complex isomorphism classes of these tori, also known as elliptic curves from their geometric description, are in bijective correspondence with the orbits of the fractional linear action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathcal{H}$. The quotient space $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathcal{H}$ can be compactified by adding a point at infinity corresponding to the 1-point quotient or "cusp" $\mathrm{PSL}(2,\mathbb{Z})\backslash \mathrm{P}_1(\mathbb{Q})$. Adding extra structure to the isomorphism classes of elliptic curves leads to replacing $\mathrm{PSL}(2,\mathbb{Z})$ by certain of its finite index subgroups G_0. The corresponding modular curves $G_0\backslash\mathcal{H}$ can be compactified by adding the finite set of cusps $G_0\backslash \mathrm{P}_1(\mathbb{Q})$.

The central point of the paper under review is that this traditional picture bypasses the Morita classes of noncommutative tori that would appear if the boundary of the modular curve $G_0\backslash\mathcal{H}$ were considered instead to be the "noncommutative modular curve" $\mathrm{PSL}(2,\mathbb{Z})\backslash \mathrm{P}_1(\mathbb{R})$. This is in the spirit of J. Bost and Connes [Selecta Math. (N.S.) **1** (1995), no. 3, 411–457; MR 96m:46112], Connes [Selecta Math. (N.S.) **5** (1999), no. 1, 29–106; MR 2000i:11133], Y. Soibelman [Lett. Math. Phys. **56** (2001), no. 2, 99–125 MR1854130] and others, as discussed in the paper under review and its references. The paper contributes completely new concrete examples of how mathematics usually applied to the commutative case relates to that traditionally applied to the noncommutative case. It opens up a new field in "noncommutative number theory", aimed at combining the mathematics of classical spaces of automorphic functions with that of noncommutative algebras. For further work in this direction by the authors see [Yu. I. Manin, "Real multiplication and noncommutative geometry", preprint, arXiv.org/abs/math/0202109; "Von Zahlen und Figuren", preprint, arXiv.org/abs/math/0201005; M. Marcolli, J. Number Theory **98** (2003), no. 2, 348–376; MR 2004b:11062].

We now outline the main results of the paper. A matrix $A \in \mathrm{PSL}(2,\mathbb{R})$ is hyperbolic if its trace has absolute value greater than 2. In this case, it has two hyperbolic fixed points θ, with $A'(\theta) < 1$, and θ', with $A'(\theta') > 1$. The oriented geodesic in $\mathcal{H}$ from θ' to θ is invariant under the action of A and is called the axis of A. If $A \in G_0$, then the axis of A becomes a closed geodesic in $G_0\backslash\mathcal{H}$. Moreover θ and θ' are then irrational conjugates in a real quadratic field. Conversely, every closed geodesic in $G_0\backslash\mathcal{H}$ represents the conjugacy class of a primitive hyperbolic transformation in G_0. Furthermore, closed geodesics for the modular group are known to be coded by "minus" continued fractions [D. B. Zagier, *Zetafunktionen und quadratische Körper*, Springer, Berlin, 1981; MR 82m:10002].

For $\theta \in \mathbb{R}$, we may try to understand in what sense $\mathbb{T}_\theta$ is a limit of T_z as z tends to θ along a geodesic in $\mathcal{H}$. In this vein, the authors extend the classical definition of modular symbols to "limiting modular symbols", with limits along geodesics in the upper half plane ending at points on the "noncommutative boundary". They show that quadratic irrationalities give rise to limiting cycles whereas generic irrational points give rise to cycles vanishing in a suitable averaged sense.

Let $X_{G_0} = X_{G_0}(\mathbb{C})$ denote the smooth compactification of $G_0\backslash\mathcal{H}$ by the finite number of cusps in bijection with $G_0\backslash\mathrm{P}_1(\mathbb{Q})$ and let φ be the corresponding covering map. As in [Yu. I. Manin, Izv. Akad. Nauk SSSR Ser. Mat. **36** (1972), 19–66; 47 #3396; also in *Selected papers of Yu. I. Manin*, World Sci. Publishing, River Edge, NJ, 1996; MR 97m:01108(pp. 202–247); L. Merel, Manuscripta Math. **80** (1993), no. 3, 283–289; MR 94k:14014], for any two points α, β in $\mathcal{H} \cup \mathrm{P}_1(\mathbb{Q})$, we can define a real homology class or "modular symbol" $\{\alpha,\beta\} \in H^1(X_{G_0},\mathbb{R})$ by integrating lifts $\varphi^*(\omega)$ of differentials ω of the first kind on X_{G_0} along the geodesic path connecting α to β:

$$\int_{\alpha,\beta} \omega := \int_\alpha^\beta \varphi(\omega).$$

When α, β are cusps, the modular symbol represents a rational homology class. To extend the definition to "limiting modular symbols", when either endpoint is real irrational, the authors define

$$\{\{*,\beta\}\}_{G_0} := \lim \frac{1}{T(x,y)}\{x,y\}_{G_0} \in H^1(X_{G_0},\mathbb{R}),$$

where $x, y \in \mathcal{H}$ are two points on the geodesic joining α to β, x is arbitrary but fixed, $T(x,y)$ is the geodesic distance between them, and the limit is taken as y tends to β. If the limit exists, the authors show that it depends neither on x nor on α, which justifies the notation. These integrals can be related to finite (when α, β are cusps), stably periodic (when α, β are two fixed points of a hyperbolic element of G_0 as described above), or general infinite continued fractions. The different cases are treated using results from [Yu. I. Manin, op. cit., 1972] and [J. B. Lewis and D. B. Zagier, in *The mathematical beauty of physics (Saclay, 1996)*, 83–97, World Sci. Publishing, River Edge, NJ, 1997; MR 99c:11108]. Continued fractions that eventually agree up to a shift of index can be identified by $\mathrm{PGL}(2,\mathbb{Z})\backslash\mathrm{P}_1(\mathbb{R})$. In this paper all of this noncommutative boundary is considered.

A striking result of the paper (Theorem 0.2.2), which is derived from certain averaging techniques over successive convergents in infinite continued fractions, uses modular symbols to relate Mellin transforms of weight-two cusp forms for $G_0 = \Gamma_0(N)$ to quantities defined entirely on the noncommutative boundary of the corresponding modular curve. This gives a concrete example of how the two types of tori $\mathbb{T}_\theta$, $\theta \in \mathbb{R}$, and T_z, $z \in \mathcal{H}$, give information about each other.

These results on the limiting modular symbols rely on certain properties, involving spectral analysis, of the Ruelle transfer operator or Gauss-Kuzmin operator for the shift of the continued fraction expansion, generalized to subgroups G of finite index in $\mathrm{GL}(2, \mathbb{Z})$. In particular the authors generalize the Gauss-Kuzmin-Lévy formula (Theorem 0.1.2). This result gives a formula for the limit of the pullback of the Lebesgue measure on $(0, 1) \times \mathrm{GL}(2, \mathbb{Z})/G$ with respect to $g_n(\alpha)$ acting on α and t simultaneously, where

$$g_n(\alpha) = \begin{pmatrix} p_{n-1}(\alpha) & p_n(\alpha) \\ q_{n-1}(\alpha) & q_n(\alpha) \end{pmatrix},$$

and $p_n(\alpha)/q_n(\alpha)$ are the successive convergents to α.

A different direction, with a related philosophy, is the study of the K-theory of the noncommutative modular curves in the spirit of Connes noncommutative geometry [A. Connes, op. cit., 1985]. In this theory, the quotient space $G\backslash \mathrm{P}_1(\mathbb{R})$, where G is of finite index in $\mathrm{PSL}(2, \mathbb{Z})$, can be studied "topologically" via its associated crossed product algebra $C(\mathrm{P}_1(\mathbb{R})) \rtimes G$ or the strongly Morita equivalent $C(\widehat{X}) \rtimes \mathrm{PSL}(2, \mathbb{Z})$, where $\widehat{X} = \mathrm{P}_1(\mathbb{R}) \times \mathrm{PSL}(2, \mathbb{Z})/G$. M. V. Pimsner [Invent. Math. **86** (1986), no. 3, 603–634; MR 88f:22022] (see also [M. Laca and J. S. Spielberg, J. Reine Angew. Math. **480** (1996), 125–139; MR 98a:46085]) has studied the K-theory of $C(\widehat{X})$ and its crossed product with $\Gamma = \mathrm{PSL}(2, \mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$, $\Gamma_0 = \mathbb{Z}/2$ and $\Gamma_1 = \mathbb{Z}/3$. The K-theory in degrees 0 and 1 is related by a six-term exact sequence. On the other hand, in [Yu. I. Manin, op. cit., 1972] and [L. Merel, op. cit.] the homology groups $H_1(X_G; \mathbb{Z})$ and relative homology groups $H^{\text{cusps}} := H_1(X_G, \text{cusps}; \mathbb{Z})$ were studied via the "modular complex" and "relative modular complex" (with respect to the elliptic and parabolic (cuspidal) fixed points of $\mathrm{PSL}(2, \mathbb{Z})$). This homology is based on the n-cells, $n = 0, 1, 2$, of the $\mathrm{PSL}(2, \mathbb{Z})/G$-orbit of the fundamental region of $\mathrm{PSL}(2, \mathbb{Z})$ built from geodesics joining those fixed points. The authors show (Theorem 4.4.1) that there is a natural isomorphism between a four-term exact sequence derived from Pimsner's exact sequence and an exact sequence derived from the modular complexes. Essentially, this relates H^{cusps} to the noncommutative topology of $G\backslash \mathrm{P}_1(\mathbb{R})$. The authors also relate the modular complex to homological constructions of noncommutative geometry via the periodic cyclic cohomology of the "smooth" crossed product algebras associated to $G\backslash \mathrm{P}_1(\mathbb{R})$.

These innovative and ground-breaking results reveal the mutual influence of the mathematics of a commutative geometric object and that of its natural noncommutative boundary.

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