Spectral triples from Mumford curves

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Abstract

We construct spectral triples associated to Schottky–Mumford curves, in such a way that the local Euler factor can be recovered from the zeta functions of such spectral triples. We propose a way of extending this construction to the case where the curve is not k-split degenerate.

1 Introduction

Let X be a curve over a finite extension K of $\mathbb{Q}_p$, which is k-split degenerate, for k the residue field. It is well known that such curve admits a p-adic uniformization by a p-adic Schottky group Γ acting on the Bruhat-Tits tree Δ_K. We associate C^*-algebras to certain subgraphs Δ of the Bruhat-Tits tree and construct corresponding dynamical cohomologies $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$ that resemble the construction at arithmetic infinity given in [6]. We introduce a Dirac operator D, which depends on the graded structure of the dynamical cohomology, in such a way that the data

$$(\mathrm{C}^*(\Delta/\Gamma), \mathcal{H}^1_{dyn}(\Delta/\Gamma) \oplus \mathcal{H}^1_{dyn}(\Delta/\Gamma), D)$$

give a spectral triple in the sense of Connes.

We recover, from a certain family of zeta functions associated to the spectral triple, the local Euler factor $L_p(H^1(X), s)$ of the curve X, in the form of a regularized determinant as computed by Deninger. The advantage of this construction is that it provides, in the case we are considering, a natural geometric interpretation, in terms of dynamics of walks on the graph Δ/Γ, of the infinite dimensional cohomology theory introduced by Deninger.

We propose a way of extending the construction to the case where the curve is not k-split degenerate, by enlarging the dual graph of the special fiber by new edges, in such a way that we also obtain embeddings in the dynamical cohomology of the first cohomology group of the components of the special fiber.

2 Directed graphs

We begin by recalling some generalities about graphs that we will use throughout the paper.

A *directed graph* E consists of data $E = (E^0, E^1, E^1_+, r, s, \iota)$, where E^0 is the set of vertices, E^1 is the set of oriented edges $w = \{e, \epsilon\}$, where e is an edge of the graph and $\epsilon = \pm 1$ is a choice of orientation. The set E^1_+ consists of a choice of orientation for each edge, namely one element in each pair $\{e, \pm 1\}$. The maps $r, s : E^1 \to E^0$ are the range and source maps, and ι is the involution on E^1 defined by $\iota(w) = \{e, -\epsilon\}$.

A morphism f of directed graphs E and $\tilde{E}$ consists of maps $f^0 : E^0 \to \tilde{E}^0$ and $f^1 : E^1 \to \tilde{E}^1$ which satisfy $f^i \circ r = \tilde{r} \circ f^i$, $f^i \circ s = \tilde{s} \circ f^i$, and $f^i \circ \iota = \tilde{\iota} \circ f^i$, for $i = 0, 1$. The morphism f is a monomorphism if the f^i are invertible, for $i = 0, 1$. This defines the automorphism group of a directed graph, which we denote $Aut(E)$. A morphism f of directed graphs is a covering map if the f^i are onto, for $i = 0, 1$, and f^1 gives a bijection $f^1 : s^{-1}(v) \stackrel{\sim}{\to} \tilde{s}^{-1}(f(v))$ and the same with respect to the range map.

A directed graph is finite if E^0 and E^1 are finite sets. It is *row finite* if at each vertex $v \in E^0$ there are at most finitely many edges w in E^1_+ such that $s(w) = v$. The graph is *locally finite* if each vertex emits and receives at most finitely many oriented edges in E^1. A vertex v in a directed graph is a *sink* if there is no edge in E^1_+ with source v. We denote by $\sigma(E)$ the subset of E^0 given by the sinks.

A juxtaposition of oriented edges $w_1 w_2$ is said to be *admissible* if $w_2 \neq \iota(w_1)$ and $r(w_1) = s(w_2)$. A (finite, infinite, doubly infinite) *walk* in a directed graph E is an admissible (finite, infinite, doubly infinite) sequence of elements in E^1. A (finite, infinite, doubly infinite) *path* in E is a walk where all edges in the sequence are in E^1_+. We denote by $\mathcal{W}^n(E)$ the set of walks of length n, by $\mathcal{W}^*(E) = \cup_n \mathcal{W}^n(E)$, by $\mathcal{W}^+(E)$ the set of infinite walks, and by $\mathcal{W}(E)$ the set of doubly infinite walks. Similarly, we introduce the analogous notation $\mathcal{P}^n(E)$, $\mathcal{P}^*(E)$, and $\mathcal{P}^+(E)$ for paths. We shall drop the explicit mention of the graph E in the notation for walks and paths, when no confusion arises. We denote by $\sigma^*(E) \subset \mathcal{P}^*(E)$ the set of paths that end at a sink. A directed graph is a directed tree if, for any two vertices, there exists a unique walk in $\mathcal{W}^*(E)$ connecting them.

The universal cover Δ of a connected directed graph E, endowed with a choice of a base point $v_0 \in E^0$, is a directed tree obtained by setting $\Delta^0 = \mathcal{W}^*(v_0)$, the set of all walks in E that start at v_0, $\Delta^1 = \{(\omega, w) \in \mathcal{W}^*(v_0) \times E^1 : r(\omega) = s(w)\}$, $r(\omega, w) = \omega$, $s(\omega, w) = \omega w$, $\iota(\omega, w) = (\omega w, \iota(w))$. The fundamental group of E, with respect to the choice of base point v_0, is given by $\Gamma = \{\gamma \in \mathcal{W}^*(v_0) : r(\gamma) = v_0\}$.

Let G be a subgroup of $Aut(E)$. We can then form the quotient E/G, which is also a directed graph. In particular, if Δ is the universal cover of a directed graph E and Γ is the fundamental group, with respect to the choice of a base point $v_0 \in E^0$, then we have an isomorphism of directed graphs $E \simeq \Delta/\Gamma$. The map $\Delta \to E$ is a covering map.

Two paths $\omega, \tilde{\omega}$ in $\mathcal{P}^+$ are *shift-tail equivalent*, if there exists a $N \geq 1$ and a $k \in \mathbb{Z}$ such that $\omega_i = \tilde{\omega}_{i-k}$ for all $i \geq N$. The boundary of a directed tree Δ is given by $\partial\Delta = (\mathcal{P}^+ \cup \sigma^*)/\sim$, where the shift-tail equivalence is extended to the set of paths ending at a sink by the condition $\omega \sim \tilde{\omega}$, for $\omega, \tilde{\omega} \in \sigma^*$, if and only if $r(\omega) = r(\tilde{\omega})$. This definition extends to a functorial notion of boundary of directed graphs, as shown in [21].

3 Schottky-Mumford curves

Throughout this chapter we will denote by p a fixed prime number and by $\mathbb{Q}_p$ the field of p-adic numbers. The field K will be a given finite extension of $\mathbb{Q}_p$, with $\mathcal{O} \subset K$ its ring of integers, $\mathfrak{m} \subset \mathcal{O}$ the maximal ideal and $\pi \in \mathfrak{m}$ a uniformizer (*i.e.* $\mathfrak{m} = (\pi)$). Finally, we will denote by k the residue field $k = \mathcal{O}/\mathfrak{m}$. This is a finite field of cardinality $q = \mathrm{card}(\mathcal{O}/\mathfrak{m})$.

Let V be a two-dimensional vector space over K. We write $\mathbb{P}^1(K)$ for the set of K-rational points of $\mathbb{P}^1_K$, the projective line over K. The space $\mathbb{P}^1(K)$ is identified with the set of K-lines passing through the origin in V. Let $G = \mathrm{PGL}(2,K)$ be the group of automorphisms of $\mathbb{P}^1(K)$.

In this first chapter we collect some well known facts and properties on the tree of the group $\mathrm{PGL}(2,K)$ and on the action of a Schottky group on such tree. Detailed explanations for the statements we will recall here without proofs are contained in [14] and [18].

3.1 The tree of the group PGL(2,K)

The description of the vertices of the tree associated to $\mathrm{PGL}(2,K)$ is as follows. One considers the set of free $\mathcal{O}$-modules of rank 2: $M \subset V$. Two such modules are *equivalent* $M_1 \sim M_2$ if there exists an element $\lambda \in K^*$, such that $M_1 = \lambda M_2$. The group $\mathrm{GL(V)}$ of linear automorphisms of V operates on the set of such modules *on the left*: $gM = \{gm \mid m \in M\}$, $g \in \mathrm{GL(V)}$. Notice that the relation $M_1 \sim M_2$ is equivalent to the condition that M_1 and M_2 belong to the same orbit of the center $K^* \subset \mathrm{GL(V)}$. Hence, the group $G = \mathrm{GL(V)}/\mathrm{K}^*$ operates (on the left) on the set of classes of equivalent modules.

We denote by Δ^0_K the set of such classes and by $\{M\}$ the class of the module M. Because $\mathcal{O}$ is a principal ideals domain and every module M has two generators, it follows that

$$\{M_1\}, \{M_2\} \in \Delta^0_K, M_1 \supset M_2 \quad \Rightarrow \quad M_1/M_2 \simeq \mathcal{O}/\mathfrak{m}^l \oplus \mathcal{O}/\mathfrak{m}^k, \quad l,k \in \mathbb{N}.$$

The multiplication of M_1 and M_2 by elements of K preserves the inclusion $M_1 \supset M_2$, hence the natural number

$$d(\{M_1\},\{M_2\}) = |l-k| \tag{3.1}$$

is well defined.

Definition 3.1 *The graph Δ_K of the group $\mathrm{PGL}(2,K)$ is the infinite graph with set of vertices Δ^0_K, in which two vertices $\{M_1\},\{M_2\}$ are adjacent and hence connected by an edge if and only if $d(\{M_1\},\{M_2\}) = 1$.*

The following properties characterize completely Δ_K and are well known (*cf. e.g.* [14] and [18]).

Proposition 3.2 *1. The graph Δ_K is a connected, locally finite tree with $q+1$ edges departing from each of its vertices.*

2. The shortest walk in Δ_K connecting two vertices $\{M_1\}, \{M_2\}$ of Δ_K^0 without back-tracking has length $d(\{M_1\}, \{M_2\})$.

3. The group G acts (on the left) transitively on Δ_K^0 and it preserves the metric d.

The tree Δ_K is called the Bruhat–Tits tree associated to the group $G = \mathrm{PGL}(2, K)$.

A *half-line* in Δ_K is an infinite sequence $(\{M_n\})_{n\in\mathbb{N}}$ of vertices of Δ_K *without repetitions* such that $\{M_n\}$ is adjacent to $\{M_{n-1}\}$ for all n. Thus, a half-line is given by a sequence $M_0 \supset M_1 \supset \ldots$ of free $\mathcal{O}$-modules where $M_0/M_n \simeq \mathcal{O}/(\pi^n)$ for all n.

The subspace $K(\cap_{n\in\mathbb{N}} M_n) \subset V$ defines a point of $\mathbb{P}^1(K)$. Conversely, given a point of $\mathbb{P}^1(K)$ represented by a vector $v_1 \in V$, choose a second vector $v_2 \in V$ such that $\{v_1, v_2\}$ form a basis of V. Let M_n be the free $\mathcal{O}$-module with basis $\{v_1, \pi^n v_2\}$. Then, $K(\cap_{n\in\mathbb{N}} M_n)$ defines the point of $\mathbb{P}^1(K)$ we started with.

Two half-lines are said to be equivalent if they differ only by a finite number of vertices. An equivalence class of halflines is called an *end* of Δ_K. The set of ends of Δ_K will be denoted by $\partial\Delta_K$ (the "boundary" of Δ_K). We shall give Δ_K the structure of a directed graph, in such a way that this notion of boundary agrees with the one described in the previous section.

It is immediate to verify that the construction described above defines a one-to-one correspondence $\partial\Delta_K \rightleftarrows \mathbb{P}^1(K)$ between equivalence classes of half-lines and elements of $\mathbb{P}^1(K)$.

3.2 The action of a Schottky group on the tree Δ_K

A Schottky group Γ is a subgroup of $\mathrm{PGL}(2, K)$ which is finitely generated and whose elements $\gamma \neq 1$ are *hyperbolic* (*i.e.* the eigenvalues of γ in K have different valuation). The group Γ is discrete in $G = \mathrm{PGL}(2, K)$, torsion free and acts freely on the tree Δ_K. Furthermore, one can show that Γ acts discretely at some point $z \in \mathbb{P}^1(\bar{K})$.

Let $\Lambda_\Gamma \subset \mathbb{P}^1(K)$ be the closure of the set of points in $\mathbb{P}^1(K)$ that are fixed points of some $\gamma \in \Gamma \setminus \{1\}$. This is called the *limit set* of Γ. We have $\mathrm{card}(\Lambda_\Gamma) < \infty$ if and only if $\Gamma = (\gamma)^{\mathbb{Z}}$, for some $\gamma \in \Gamma$. We will denote by $\Omega_\Gamma = \Omega_\Gamma(K)$ the set of points on which Γ acts discretely; equivalently said Ω_Γ is the set of points which are not limits of fixed points of elements of Γ: $\Omega_\Gamma = \mathbb{P}^1(K) \setminus \Lambda_\Gamma$. This set is called the *domain of discontinuity* for the Schottky group Γ.

A path in Δ_K, infinite in both directions and with no back-tracking, is called an *axis* of Δ_K. Any two points $z_1, z_2 \in \mathbb{P}^1(K)$ uniquely define their connecting axis whose endpoints lie in the classes described by z_1 and z_2 in $\partial\Delta_K$.

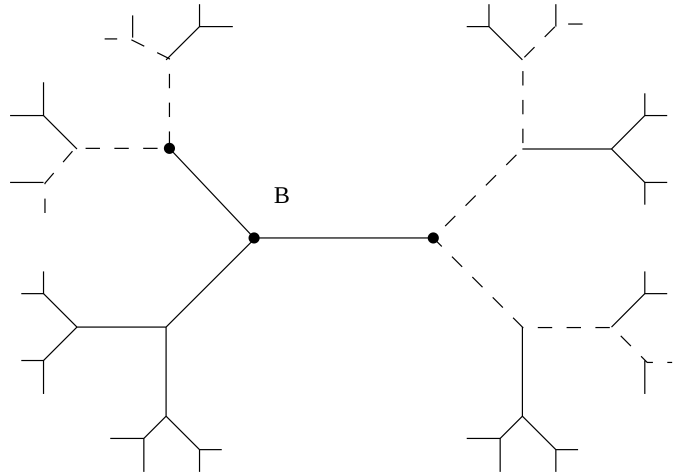

Figure 1: Two axes and the bridge B between them.

Any hyperbolic element $\gamma \in \Gamma$ has two fixed points in $\mathbb{P}^1(K)$. The unique axis of Δ_K whose ends are fixed by γ is called the axis of γ. The element γ acts on its axis as a translation.

Suppose two axes are given in Δ_K. Any path without back-tracking beginning on one axis and ending on the other and with no edges in common with either axis is said to be the *bridge* between them. A bridge may consist of a single point, else it is uniquely defined (Figure 1).

For any Schottky group $\Gamma \subset \mathrm{PGL}(2, K)$ there is a smallest *subtree* $\Delta'_\Gamma \subset \Delta$ containing the axes of all elements of Γ. Equivalently said, Δ'_Γ is the maximal connected subgraph of Δ_K containing the axes of all elements of Γ and the bridges between them.

The set of ends of Δ'_Γ in $\mathbb{P}^1(K)$ is Λ_Γ, the limit set of Γ. The group Γ carries Δ'_Γ into itself so that the quotient Δ'_Γ/Γ is a *finite graph.*

The graph Δ'_Γ/Γ has an important geometric interpretation as the dual graph of the closed fiber of the *minimal smooth model* over $\mathcal{O}$ (k-split degenerate semi-stable curve) of the algebraic curve C holomorphically isomorphic to $X_\Gamma := \Omega_\Gamma/\Gamma$ (*cf.* [18] p. 163).

Furthermore, there is a smallest tree Δ_Γ on which Γ acts, with vertices $\Delta^0_\Gamma \subset \Delta^0_K$, and such that the finite graph Δ_Γ/Γ is the dual graph of the specialization of C over $\mathcal{O}$. The set Δ^0_Γ is a subset of the set of vertices of Δ'_Γ.

The degenerating curve C describing the analytic uniformization $X_\Gamma \overset{\simeq}{\to} C$ is a k-*split degenerate, stable curve.* When the genus of the fibers is at least 2 - *i.e.* when the Schottky group has at least $g \geq 2$ generators - the curves X_Γ are called Schottky–Mumford curves.

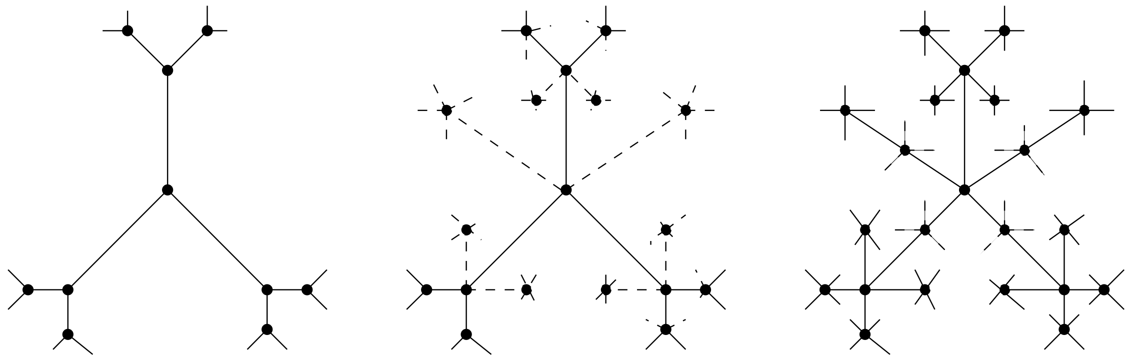

Figure 2: The tree Δ_K for $K = \mathbb{Q}_2$ and Δ_L for field extensions with $f = 2$ and $e_{L/K} = 2$

3.3 Field extensions: functoriality

Let $L \supset K$ be a finite fields extension with ramification index $e_{L/K}$ and rings of integers $\mathcal{O}_L$ and $\mathcal{O}_K$. If $M \subset V$ is a free $\mathcal{O}_K$-module of rank 2, then $M \otimes_{\mathcal{O}_K} \mathcal{O}_L \subset V \otimes_K L$ is a free $\mathcal{O}_L$-module of the same rank. It is obvious that equivalent modules remain equivalent, hence one gets a natural embedding of the sets of vertices $\Delta_K^0 \hookrightarrow \Delta_L^0$.

The isomorphism $(\mathcal{O}_K/\mathfrak{m}^r) \otimes \mathcal{O}_L \simeq \mathcal{O}_L/\mathfrak{m}^{re_{L/K}}$ shows that distance would not, in general, be preserved under field extensions. To eliminate this disadvantage, one introduces on graphs Δ_L, for all extensions $L \supset K$, a K-normalized distance

$$d_K(\{M_1\},\{M_2\}) = \frac{1}{e_{L/K}} d_L(\{M_1\},\{M_2\}); \qquad M_1, M_2 \subset V \otimes L.$$

This way, the embedding $\Delta_K^0 \hookrightarrow \Delta_L^0$ becomes isometric. When $L \supset K$ ramifies, $e_{L/K} - 1$ new vertices appear in Δ_L^0 between each couple of adjacent vertices in Δ_K^0 (*cf.* the third graph in Figure 2). Moreover, $q^f + 1$ edges each of which of length $\frac{1}{e_{L/K}}$, for $f = \frac{1}{e_{L/K}}[L : K]$, depart from each vertex in Δ_L^0 (*cf.* the second graph in Figure 2).

Because $\mathrm{PGL}(2, K) \subset \mathrm{PGL}(2, L)$ and the natural embedding $\mathbb{P}^1(K) \subset \mathbb{P}^1(L)$ is compatible with a concept of K-direction of exiting from any vertex of Δ_K^0, the construction is functorial under finite fields extensions and this process determines, for a fixed Schottky group $\Gamma \subset \mathrm{PGL}(2, K)$, a projective system $\{X_{L,\Gamma} \; : \; [L : K] < \infty\}$ of Schottky-Mumford curves.

3.4 Edge orientation

We now show how to endow the graphs Δ_Γ and Δ_K with the structure of directed graph. The choice of a coordinate $z \in \mathbb{P}^1(K)$ determines a base point $\tilde{v}_0$ in Δ_K. In fact, the points $\{0, 1, \infty\}$ on $\mathbb{P}^1(K)$ determine a unique *crossroad*: the unique vertex of Δ_K with the property that the walks without back-tracking from $\tilde{v}_0$ to the three points on $\mathbb{P}^1(K)$ start from $\tilde{v}_0$ in three different directions (Figure 3).

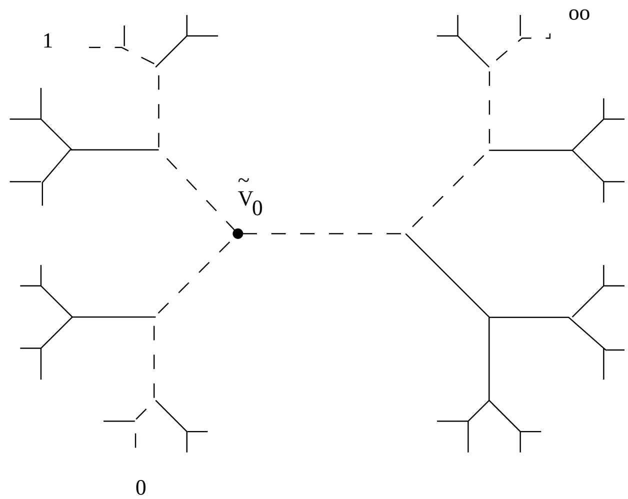

Figure 3: The crossroad $\tilde{v}_0$ of the points $\{0, 1, \infty\}$

In order to obtain a structure of directed graph on the tree Δ_K, for each $z \in \mathbb{P}^1(K)$ we consider the unique infinite chain of edges without backtracking in Δ_K that has initial vertex $\tilde{v}_0$ and whose equivalence class modulo shift–tail equivalence is the point z. We declare such chain of edges to be a path in $\mathcal{P}^+(\Delta_K)$. This determines on Δ_K the structure of a directed graph, with $\partial\Delta_K = \mathbb{P}^1(K)$. This agrees with the boundary as defined in §3.1.

We assume here that the coordinate $z \in \mathbb{P}^1(K)$ is chosen in such a way that the crossroad $\tilde{v}_0$ of $\{0, 1, \infty\}$ is a vertex of Δ_Γ. Then, by the same procedure, we can regard Δ_Γ as a directed graph with $\partial\Delta_\Gamma = \Lambda_\Gamma$.

4 C^*-algebras of graphs

In this section we recall the construction of C*-algebras associated to locally finite directed graphs. We follow mostly the references [1], [10], [11], [21]. We refer the reader to the bibliography of the aforementioned articles for more information. For simplicity, we state the following results in the special case of locally finite directed graph, though the theory extends to more general directed graphs (*cf. e.g.* [21]).

A *Cuntz–Krieger family* consists of a collection $\{P_v\}_{v\in E^0}$ of mutually orthogonal projections and $\{S_w\}_{w\in E^1_+}$ of partial isometries, satisfying the conditions: $S^*_w S_w = P_{r(w)}$ and, for all $v \in s(E^1_+)$, $P_v = \sum_{w:s(w)=v} S_w S^*_w$.

The *edge matrix* A_+ of a locally finite (or row finite) directed graph is an $(\#E^1_+) \times (\#E^1_+)$ (possibly infinite) matrix. The entries $A_+(w_i, w_j)$ satisfy $A_+(w_i, w_j) = 1$ if $w_i w_j$ is an admissible path, and $A_+(w_i, w_j) = 0$ otherwise. The Cuntz–Krieger elements $\{P_v, S_w\}$ satisfy the relation $S^*_w S_w = \sum A(w, \tilde{w}) S_{\tilde{w}} S^*_{\tilde{w}}$. The *directed edge matrix* of E

is a $\#E^1 \times \#E^1$ (possibly infinite) matrix with entries $A(w_i, w_j) = 1$ if $w_i w_j$ is an admissible walk and $A(w_i, w_j) = 0$ otherwise.

There is a universal C^*-algebra $C^*(E)$ generated by a Cuntz–Krieger family. If E is a finite graph with no sinks, we have $C^*(E) \simeq \mathcal{O}_{A_+}$, where $\mathcal{O}_{A_+}$ is the Cuntz-Krieger algebra of the edge matrix A_+. If the directed graph is a tree Δ, then $C^*(\Delta)$ is an AF algebra strongly Morita equivalent to the commutative C^*-algebra $C_0(\partial\Delta)$. A monomorphism of directed trees induces an injective $*$-homomorphism of the corresponding C^*-algebras.

If $G \subset Aut(E)$ is a group acting freely on the directed graph E, with quotient graph E/G, then the crossed product C^*-algebra $C^*(E) \rtimes G$ is strongly Morita equivalent to $C^*(E/G)$. In particular, if Δ is the universal covering tree of a directed graph E and Γ is the fundamental group, then the algebra $C^*(E)$ is strongly Morita equivalent to $C_0(\partial\Delta) \rtimes \Gamma$.

There is a *gauge action* of $U(1)$ on the graph algebra $C^*(E)$ given by $\lambda : \{P_v, S_w\} \mapsto \{P_v, \lambda S_w\}$. A subset H of the set of vertices E^0 of a directed graph is *saturated hereditary* if $v \in H$ implies that, for all $\omega \in \mathcal{P}^*(E)$ with $s(\omega) = v$, also $r(\omega) \in H$, and conversely, if any $\omega \in \mathcal{P}^*(E)$ with $s(\omega) = v$ satisfies $r(\omega) \in H$, then also $v \in H$. For a locally finite graph there is a bijective correspondence between saturated hereditary subsets of E^0 and gauge invariant closed ideals in $C^*(E)$. In the case of a tree Δ, there is a bijection between saturated hereditary subsets of Δ^0 and open sets in $\partial\Delta$. This is proved for the more general (non necessarily locally finite) case in [21].

It is convenient to consider also the Toeplitz extensions (*cf.* [21])

$$0 \to I_S \to T\mathcal{O}(E, S) \to C^*(E) \to 0,$$

where S is a subset of E^0 and $I_S = \oplus_{v \in S^c} \mathcal{K}_v$, where $\mathcal{K}_v$ is the algebra of compact operators on a Hilbert space of dimension $\#(\mathcal{P}^* \cap r^{-1}(v))$. The C^*-algebra $T\mathcal{O}(E, S)$ is generated by operators $\{S_w\}_{w \in E^1_+}$ and $\{P_v\}_{v \in E^0}$ satisfying the conditions: $S_w^* S_w = P_{r(w)}$ and, for all $v \in s(E^1_+)$, $P_v \geq \sum_{w : s(w) = v} S_w S_w^*$, with equality for $v \in S$.

If $j : E \hookrightarrow \tilde{E}$ is an inclusion of directed graphs, the following functoriality property holds (*cf.* [21]): given $\tilde{S} \subset \tilde{E}$, consider the set S of vertices v in E^0 such that $j(v) \in \tilde{S}$ and the outgoing edges in $\tilde{E}^1_+$ with origin at $j(v)$ are all of the form $j(w)$, for some $w \in E^1_+$ with $s(w) = v$. Then this induces an injective $*$-homomorphism $J : T\mathcal{O}(E, S) \to T\mathcal{O}(\tilde{E}, \tilde{S})$.

In particular, for a family of subgraphs E of a directed graph $\tilde{E}$, ordered by inclusions, with $\cup E^0 = \tilde{E}^0$ and $\cup E^1 = \tilde{E}^1$, and for a choice of $\tilde{S} \subset \tilde{E}^0$ and corresponding S as above, we have

$$T\mathcal{O}(\tilde{E}, \tilde{S}) = \lim_{E} T\mathcal{O}(E, S). \tag{4.1}$$

4.1 Reduction graphs

In the theory of Mumford curves, it is important to consider also the reduction modulo powers $\mathfrak{m}^n$ of the maximal ideal $\mathfrak{m} \subset \mathcal{O}_K$, which provides infinitesimal neighborhoods of order n of the closed fiber. In the language of C^*-algebras, this corresponds to the following construction.

For each $n \geq 0$, we consider a subgraph $\Delta_{K,n}$ of the Bruhat-Tits tree Δ_K defined by setting

$$\Delta^0_{K,n} := \{v \in \Delta^0_K : d(v, \Delta'_\Gamma) \leq n\},$$

with respect to the distance (3.1), with $d(v, \Delta'_\Gamma) := \inf\{d(v, \tilde{v}) : \tilde{v} \in (\Delta'_\Gamma)^0\}$, and

$$\Delta^1_{K,n} := \{w \in \Delta^1_K : s(w), r(w) \in \Delta^0_{K,n}\}.$$

Thus, we have $\Delta_{K,0} = \Delta'_\Gamma$. For $n \geq 1$ the graph $\Delta_{K,n}$ has a non-empty set of sinks $\sigma_{K,n} \subset \partial\Delta_{K,n}$. We have $\Delta_K = \cup_n \Delta_{K,n}$.

For all $n \in \mathbb{N}$, the graph $\Delta_{K,n}$ is invariant under the action of the Schottky group Γ on Δ, and the finite graph $\Delta_{K,n}/\Gamma$ gives the dual graph of the reduction $X_K \otimes \mathcal{O}/\mathfrak{m}^{n+1}$. Thus, we refer to the $\Delta_{K,n}$ as *reduction graphs*. They form a directed family with inclusions $j_{n,m} : \Delta_{K,n} \hookrightarrow \Delta_{K,m}$, for all $m \geq n$, with all the inclusions compatible with the action of Γ.

For each reduction graph, we can consider corresponding C*-algebras $C^*(\Delta_{K,n})$ and $C^*(\Delta_{K,n}/\Gamma) \simeq C^*(\Delta_{K,n}) \rtimes \Gamma$ (Morita equivalence).

The following result, which is a direct application of the statements on graph C*-algebras listed in §4, describes the relation between the algebras $C^*(\Delta_{K,n}/\Gamma)$ and $C^*(\Delta_K/\Gamma)$.

Lemma 4.1 *We have injective $*$-homomorphisms $J_{n,m} : C^*(\Delta_{K,n}) \to C^*(\Delta_{K,m})$. Correspondingly, if we set $S_n = \Delta^0_{K,n} \backslash \sigma_{K,n}$, we obtain*

$$C^*(\Delta_K/\Gamma) = \lim_n T\mathcal{O}(\Delta_{K,n}/\Gamma, S_n/\Gamma),$$

where $T\mathcal{O}(\Delta_{K,n}/\Gamma, S_n/\Gamma)$ satisfies

$$0 \to \oplus_{v \in \sigma(\Delta_{K,n}/\Gamma)} \mathcal{K}_v \to T\mathcal{O}(\Delta_{K,n}/\Gamma, S_n/\Gamma) \to C^*(\Delta_{K,n}/\Gamma) \to 0.$$

5 Dynamics of walks on dual graphs

In this section we introduce a dynamical system associated to the space $\mathcal{W}(\Delta/\Gamma)$ of walks on the quotient of a directed tree Δ by a free action of Γ. In particular, we are interested in the cases where Δ is (a certain extension of) one of the graphs $\Delta_{K,n}$ for some $n \geq 0$. The same construction applies to the tree Δ_Γ, where this dynamical system is a subshift of finite type associated to the action of the Schottky group Γ on its limit set Λ_Γ, analogous to the one considered in [6].

Let $\bar{V} \subset \Delta$ be a finite subtree whose set of edges consists of one representative for each Γ-class. This is a fundamental domain for Γ in the weak sense (following the notation of [14]), since some vertices may be identified under the action of Γ. Correspondingly, we denote by V the set of ends of all infinite paths in Δ starting at points in $\bar{V}$.

We assume that, for the Γ-invariant directed tree Δ, the set $\bar{V}$ has *finitely many edges*. This is the case for Δ_Γ as well as for any of the trees $\Delta_{K,n}$.

Since, for $n \geq 1$, the graphs $\Delta_{K,n}$ have sinks, in order to consider the space of doubly infinite walks on these graphs, we need to complete each walk ending at a sink to an infinite walk obtained by repeating the last word. This is equivalent to extending the graph $\Delta_{K,n}$ by adding an infinite tail to each sink. Appending tails to sinks is standard technique in the theory of graph C*-algebras, in order to reduce the general case to the easier case of graphs with no sinks. We use the notation $\bar{\Delta}_{K,n}$ for the completed graph with infinite tails. These have an action of Γ obtained by extending the action on $\Delta_{K,n}$, by translating the whole tail in the same way as the corresponding sink in $\Delta_{K,n}$.

For $\Delta = \bar{\Delta}_{K,n}$, the set $\mathcal{W}(\Delta/\Gamma)$ of doubly infinite walks on the graph $\bar{\Delta}_{K,n}/\Gamma$ can be identified with the set of doubly infinite admissible sequences in the *finite alphabet* given by the edges in the fundamental domain $\bar{V}$ of the graph $\Delta_{K,n}$, with both possible orientations.

On $\mathcal{W}(\Delta/\Gamma)$ we consider the topology generated by the sets $\mathcal{W}^s(\omega,\ell) = \{\tilde{\omega} \in \mathcal{W}(\Delta/\Gamma) : \tilde{\omega}_k = \omega_k, k \geq \ell\}$ and $\mathcal{W}^u(\omega,\ell) = \{\tilde{\omega} \in \mathcal{W}(\Delta/\Gamma) : \tilde{\omega}_k = \omega_k, k \leq \ell\}$, for $\omega \in \mathcal{W}(\Delta/\Gamma)$ and $\ell \in \mathbb{Z}$. With this topology, the space $\mathcal{W}(\Delta/\Gamma)$ is a totally disconnected compact Hausdorff space. The invertible shift map T, given by $(T\omega)_k = \omega_{k+1}$, is a homeomorphism of $\mathcal{W}(\Delta/\Gamma)$.

We have just described the dynamical system $(\mathcal{W}(\Delta/\Gamma), T)$ in terms of subshifts of finite type, according to the following definition:

Definition 5.1 *A* subshift of finite type $(\mathcal{S}_A, T)$ *consists of all doubly infinite sequences in the elements of a given finite set W (alphabet) with the admissibility condition specified by a $\#W \times \#W$ elementary matrix,*

$$\mathcal{S}_A = \{(w_k)_{k\in\mathbb{Z}} : w_k \in W, A(w_k, w_{k+1}) = 1\},$$

with the action of the invertible shift $(Tw)_k = w_{k+1}$.

Lemma 5.2 *The space $\mathcal{W}(\Delta/\Gamma)$ with the action of the invertible shift T is a subshift of finite type, where $\mathcal{W}(\Delta/\Gamma) = \mathcal{S}_A$ with A the directed edge matrix of the finite graph Δ/Γ.*

We can consider the mapping torus of T,

$$\mathcal{W}(\Delta/\Gamma)_T := \mathcal{W}(\Delta/\Gamma) \times [0,1]/(Tx,0) \sim (x,1). \tag{5.1}$$

We consider the first cohomology group $H^1(\mathcal{W}(\Delta/\Gamma)_T, \mathbb{Z})$, identified with the group of homotopy classes of continuous maps of $\mathcal{W}(\Delta/\Gamma)_T$ to the circle. If we denote by $\mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})$ the $\mathbb{Z}$-module of integer valued continuous functions on $\mathcal{W}(\Delta/\Gamma)$, and by $\mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})_T$ the cokernel of the map $\delta(f) = f - f \circ T$, we obtain the following result (*cf.* [2], [19]).

Proposition 5.3 *The map $f \mapsto [\exp(2\pi i t f(x))]$, which associates to an element $f \in \mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})$ a homotopy class of maps from $\mathcal{W}(\Delta/\Gamma)_T$ to the circle, gives an isomorphism $\mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})_T \simeq H^1(\mathcal{W}(\Delta/\Gamma)_T, \mathbb{Z})$. Moreover, there is a filtration of $\mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})_T$ by free $\mathbb{Z}$-modules $F_0 \subset F_1 \subset \cdots F_n \cdots$, of rank $\theta_n - \theta_{n-1} + 1$, where θ_n is the number of admissible words of length $n+1$ in the alphabet, so that we have*

$$H^1(\mathcal{W}(\Delta/\Gamma)_T, \mathbb{Z}) = \varinjlim{}_n F_n.$$

The quotients F_{n+1}/F_n are also torsion free.

Proof. A continuous function $f \in \mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})$ depends on just finitely many coordinates ω_k of $\omega \in \mathcal{W}(\Delta/\Gamma)$. In particular, this implies that, for some k_0, the composite $f \circ T^{k_0}$ is a function of only 'future coordinates' (ω_k with $k \geq 0$). We denote by $\mathcal{P} \subset \mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})$ the submodule of functions of future coordinates. It is then clear that we have $\mathrm{C}(\mathcal{W}(\Delta/\Gamma), \mathbb{Z})_T \simeq \mathcal{P}/\delta\mathcal{P}$. We also have an identification $\mathcal{P} \simeq \mathrm{C}(\mathcal{W}^+(\Delta/\Gamma), \mathbb{Z})$. This gives a filtration $\mathcal{P} = \cup_n \mathcal{P}_n$, where $\mathcal{P}_n$ is identified with the submodule of $\mathrm{C}(\mathcal{W}^+(\Delta/\Gamma), \mathbb{Z})$ generated by characteristic functions of $\mathcal{W}^+(\Delta/\Gamma, \rho) \subset \mathcal{W}^+(\Delta/\Gamma)$, where $\rho \in \mathcal{W}^*(\Delta/\Gamma)$ is a finite walk $\rho = w_0 \cdots w_n$ of length $n+1$, and $\mathcal{W}^+(\Delta/\Gamma, \rho)$ is the set of infinite paths $\omega \in \mathcal{W}^+(\Delta/\Gamma)$, with $\omega_k = w_k$ for $0 \leq k \leq n+1$. We have $\delta : \mathcal{P}_n \to \mathcal{P}_{n+1}$, with kernel the constant functions. We set $F_n := \mathcal{P}_n/\delta(\mathcal{P}_{n-1})$, for $n \geq 1$ and $F_0 = \mathcal{P}_0$. The inclusions $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ induce injections $j : F_n \hookrightarrow F_{n+1}$, $j(f + \delta\mathcal{P}_{n-1}) = f + \delta\mathcal{P}_n$, such that $\mathcal{P}/\delta\mathcal{P} = \varinjlim{}_n F_n$. As $\mathbb{Z}$-modules, both the F_n and the quotients $F_{n+1}/j(F_n)$ are torsion free.
◇

For more details see the analogous argument given in [19] Thm. 19 p. 62-63.

5.1 The effect of field extensions

Let $L \supset K$ be a finite extension, with branching index $e_{L/K}$ and with $f = [L : K]/e_{L/K}$. Then there is an embedding of the set of vertices $\Delta_K^{(0)} \subset \Delta_L^{(0)}$. In between every two vertices of $\Delta_K^{(0)} \subset \Delta_L^{(0)}$ there are $e_{L/K} - 1$ new vertices of Δ_L. If in Δ_K every vertex has valence $q + 1$, then every vertex of Δ_L has valence $q^f + 1$.

In particular, the image of the tree $\Delta_\Gamma \subset \Delta_K$ in Δ_L is the tree $\Delta_\Gamma \subset \Delta_L$, whereas, when considering the tree Δ'_Γ, we are inserting $e_{L/K} - 1$ new vertices in between each two vertices of $\Delta'_\Gamma \subset \Delta_K$. Notice that the algebras $C^*(\Delta'_\Gamma)$ and $C^*(\Delta_\Gamma)$ are strongly Morita equivalent, since they both are equivalent to the commutative C^*-algebra $C(\Lambda_\Gamma)$. Thus we have:

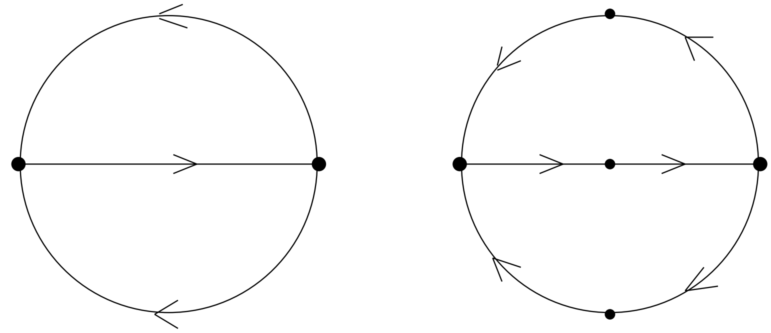

Figure 4: Effect of a field extension with $e_{L/K} = 2$.

Lemma 5.4 The strong Morita equivalence class of the graph C^*-algebras $C^*(\Delta'_\Gamma)$ and $C^*(\Delta'_\Gamma/\Gamma)$ is independent of finite field extensions $L \supset K$.

The following example illustrates how the algebra $\mathcal{O}_{A_+}$ changes under field extensions, by showing the change in the edge matrix A_+. If the first directed graph shown in Figure 4 arises as dual graphs of the closed fiber for a totally split degenerate curve, then the effect on the graph of a field extension with $e_{L/K} = 2$ is illustrated in the second graph in Figure 4.

The edge matrix of the original graph was of the form

$$A_+ = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

while the new edge matrix becomes:

$$A_+ = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

We are interested in understanding the effect of field extensions on the construction considered in the previous paragraph. We have the following result.

Proposition 5.5 *A finite field extension $L \supset K$ determines a morphism*

$$H^1(\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_T) \to H^1(\mathcal{W}(\bar{\Delta}_{K,n}/\Gamma)_T),$$

which is compatible with the filtrations.

Proof. In [14] it is shown that there is a canonical choice of the fundamental domains $\bar{V}$ and V for the action of Γ on the Bruhat-Tits tree Δ_K that is functorial under a finite extensions $L \supset K$. This determines corresponding functorial choices of fundamental domains for the graphs $\Delta_{K,n}$. We obtain this way a corresponding embedding of walk spaces

$$J_{L,K,n} : \mathcal{W}(\bar{\Delta}_{K,n}/\Gamma) \hookrightarrow \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma),$$

which replaces each edge in a sequence $\omega \in \mathcal{W}(\bar{\Delta}_{K,n}/\Gamma)$ (an edge in $\bar{V}$ for $\Delta_{K,n}$) with the corresponding $e_{L/K}$ consecutive edges in the fundamental domain $\bar{V}$ for $\Delta_{L,n}$, thus obtaining an element in $\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)$. This map satisfies $J_{L,K,n} \circ T = T^{e_{L/K}} \circ J_{L,K,n}$. Thus, we obtain an induced map

$$J_{L,K,n,T} : \mathcal{W}(\bar{\Delta}_{K,n}/\Gamma)_T \to \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_{T^{e_{L/K}}},$$

where, for $\ell \geq 1$, $\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_{T^\ell}$ denotes the mapping torus

$$\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_{T^\ell} \simeq \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma) \times [0,\ell]/(x,0) \sim (T^\ell x, \ell),$$

with a covering map $\pi_\ell : \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_{T^\ell} \to \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_T$. Thus, we obtain a map

$$\pi_{e_{L/K}} \circ J_{L,K,n,T} : \mathcal{W}(\bar{\Delta}_{K,n}/\Gamma)_T \to \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_T.$$

This induces a corresponding map in cohomology,

$$(\pi_{e_{L/K}} \circ J_{L,K,n,T})^* : H^1(\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_T) \to H^1(\mathcal{W}(\bar{\Delta}_{K,n}/\Gamma)_T).$$

To see this at the level of the filtrations of Proposition 5.3, notice that the map $J_{L,K,n}$ also induces a restriction map

$$r_{L,K,n} : \mathrm{C}(\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma), \mathbb{Z}) \to \mathrm{C}(\mathcal{W}(\bar{\Delta}_{K,n}/\Gamma), \mathbb{Z}).$$

If we denote by $\mathcal{P}_N^{(L)}$ and $\mathcal{P}_j^{(K)}$ the respective filtrations, then we have restriction maps $r_{L,K,n} : \mathcal{P}^{(L)}_{je_{L/K}} \to \mathcal{P}_j^{(K)}$. If we denote by δ_ℓ the map $\delta_\ell(f) = f - f \circ T^\ell$, then the restriction also satisfies $r_{L,K,n} \circ \delta_{e_{L/K}} = \delta \circ r_{L,K,n}$, hence there is an induced map $\bar{r}_{L,K,n} : F_j^{(L,e_{L/K})} \to F_j^{(K)}$, where we have set $F_j^{(L,e_{L/K})} := \mathcal{P}^{(L)}_{je_{L/K}}/\delta_{e_{L/K}}\mathcal{P}^{(L)}_{(j-1)e_{L/K}}$. An argument analogous to the one used in the proof of Proposition 5.3 shows that the $F_j^{(L,e_{L/K})}$ give a filtration of

$$H^1(\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_{T^{e_{L/K}}}, \mathbb{Z}) = \varinjlim_j F_j^{(L,e_{L/K})}.$$

There is a corresponding map

$$\mathrm{C}(\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma), \mathbb{Z})_T \to \mathrm{C}(\mathcal{W}(\bar{\Delta}_{L,n}/\Gamma), \mathbb{Z})_{T^{e_{L/K}}},$$

induced by the covering $\pi_{e_{L/K}} : \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_{T^{e_{L/K}}} \to \mathcal{W}(\bar{\Delta}_{L,n}/\Gamma)_T$. On the level of filtrations this has the following description. The module $\mathcal{P}^{(L)}_{je_{L/K}}$ can be identified

with the span of functions in $\mathcal{P}^{(L)}_{j,s}$, $s = 0, \ldots, e_{L/K} - 1$, where $\mathcal{P}^{(L)}_{j,s} = T^j(\mathcal{P}^{(L)}_j)$. The inclusion $\iota : \mathcal{P}^{(L)}_j \hookrightarrow \mathcal{P}^{(L)}_{je_{L/K}}$ as $\mathcal{P}^{(L)}_{j,0}$ then satisfies $\delta \circ \iota = \iota \circ \delta_{e_{L/K}}$. We obtain an induced map $F^{(L)}_j \to F^{(L,e_{L/K})}_j$. Thus, the map induced by the covering $\pi_{e_{L/K}}$ also preserves the filtrations, and we obtain maps $F^{(L)}_j \to F^{(K)}_j$ that induce $(\pi_{e_{L/K}} \circ J_{L,K,T})^*$ on the direct limits.
◇

5.2 Dynamical cohomology

Let Δ be a directed tree on which the Schottky group Γ acts, with the same assumptions as in the previous paragraph.

On the set of ends $\partial\Delta$ we consider a measure $d\mu$ defined by first introducing the distance function $d(v) := \mathrm{dist}(v, x_0)$, for $v \in \Delta^0$ and x_0 the base point in Δ^0 with respect to which the structure of directed graph on Δ is determined. Then the measure on $\partial\Delta$ is defined by assigning its value on the clopen set $V(v)$, given by the ends of all paths in Δ starting at a vertex v, to be

$$\mu(V(v)) = q^{-d(v)-1},$$

with $q = \mathrm{card}(\mathcal{O}/m)$.

Proposition 5.6 *This induces a measure on $\mathcal{W}(\Delta/\Gamma)$, with respect to which the shift map T is measure preserving.*

Proof. Notice that, if we identify the points of $V(v)$ with infinite paths starting at v,

$$V(v) = \{w_0 w_1 \ldots w_n \ldots : s(w_0) = v, w_k \in (\Delta)^1_+\},$$

then the image $T(V(v))$ will have measure $\mu(T(V(v))) = \mu(V(v))/q$. In the case of walks starting at a vertex v, the map T scales the measure of the set of walks starting with an edge $w \in (\Delta)^1_+$ by a factor q^{-1} and the measure of the set of walks starting with an edge w with $\bar{w} \in (\Delta)^1_+$ by a factor q.

We can define a map from $\mathcal{W}(\Delta/\Gamma)$ to $V \times V$, by splitting each doubly infinite sequence

$$\ldots w_{-n} w_{-n+1} \ldots w_{-1} w_0 w_1 \ldots w_\ell w_{\ell+1} \ldots$$

into the two sequences

$$(w_0 w_1 \ldots w_\ell w_{\ell+1} \ldots, \; \bar{w}_{-1} \bar{w}_{-2} \ldots \bar{w}_{-n+1} \bar{w}_{-n} \ldots), \tag{5.2}$$

each of which defines a point in the fundamental domain V, if we identify admissible sequences of edges in the fundamental domain Δ/Γ with admissible sequences of edges in Δ with the condition that the first edge w_0 (or $\bar{w}_{-1}$) lies in a chosen fundamental domain $\bar{V}$ of the action of Γ which contains the base point x_0. Then the action of T maps (5.2) to

$$(w_1 \dots w_\ell w_{\ell+1} \dots, \bar{w}_0 \bar{w}_{-1} \bar{w}_{-2} \dots \bar{w}_{-n+1} \bar{w}_{-n} \dots), \tag{5.3}$$

hence it scales the measure on one factor by q and on the other factor by q^{-1}, so that the measure induced on $\mathcal{W}(\Delta/\Gamma)$, by restricting to $V \times V$ the product measure on $\partial\Delta \times \partial\Delta$, is preserved by T.
$\diamond$

Consider the free $\mathbb{Z}$-modules $\mathcal{P}_n$, introduced in the proof of Proposition 5.3, of functions of at most $n+1$ future coordinates. We can realize $\mathcal{P}_n \otimes_{\mathbb{Z}} \mathbb{C}$ as a vector subspace of $L^2(\partial\Delta, d\mu)$. The operator δ is bounded in norm, hence the $F_n = \mathcal{P}_n/\delta\mathcal{P}_{n-1}$ have induced norms and bounded inclusions $F_n \otimes \mathbb{C} \hookrightarrow F_{n+1} \otimes \mathbb{C}$, where the F_n are the torsion free $\mathbb{Z}$-modules of Proposition 5.3.

Using the inner product induced from $L^2(\partial\Delta, d\mu)$, we can split $F_n \cong F_{n-1} \oplus (F_{n-1}^{\perp} \cap F_n)$.

Definition 5.7 *The dynamical cohomology $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$ is the norm completion of the graded complex vector space*

$$H^1_{dyn}(\Delta/\Gamma) := \oplus_{k \geq 0} Gr_k$$

with $Gr_n := (F_n \cap F_{n-1}^{\perp})$.

We now construct a representation, by linear bounded operators, of a graph C*-algebra on this Hilbert space.

Proposition 5.8 *There is a representation of the algebra $C^*(\Delta/\Gamma)$ by bounded linear operators on the Hilbert space $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$.*

Proof. For $w \in (\Delta/\Gamma)^1_+$, we define linear operators T_w of the form

$$(T_w f)(w_0 w_1 w_2 \dots) = \begin{cases} f(w w_0 w_1 w_2 \dots) & r(w) = s(w_0) \\ 0 & r(w) \neq s(w_0). \end{cases}$$

For $v \in \Delta/\Gamma^0$, let P_v denote the projection

$$(P_v f)(w_0 w_1 w_2 \dots) = \begin{cases} f(w_0 w_1 w_2 \dots) & s(w_0) = v \\ 0 & s(w_0) \neq v, \end{cases}$$

We also define projections Π_w of the form

$$(\Pi_w f)(w_0 w_1 w_2 \dots) = \begin{cases} f(w_0 w_1 w_2 \dots) & w_0 = w \\ 0 & w_0 \neq w. \end{cases}$$

Finally, we define linear operators s_w and S_w

$$s_w := \sum_{w'} A(w,w') T_w \Pi_{w'} \quad \text{and} \quad S_w := \sqrt{q}\, s_w. \tag{5.4}$$

These define bounded linear operators on $L^2(\partial\Delta, d\mu)$. The S_w are partial isometries, satisfying $S_w S_w^* = \Pi_w$. Thus, the operators $\{P_v, S_w\}$ form a Cuntz–Krieger family, since we obtain $P_v = \sum_{s(w)=v} \Pi_w = \sum_{s(w)=v} S_w S_w^*$ and $S_w^* S_w = \sum A(w,w') S_{w'} S_{w'}^* = \sum A(w,w') P_{w'} = P_{r(w)}$.

The same argument given in [6], Proposition 4.19, shows that $S_w \delta = \delta S_w$, so that they descend to bounded operators on $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$.

◇

5.3 Embedding of cohomologies

Let Δ be any of the trees $\bar{\Delta}_{K,n}$, $n \geq 0$, or Δ_Γ. We construct a family of embeddings of the cohomology of the dual graph Δ_Γ/Γ in the dynamical cohomology $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$. Let $|\Delta_\Gamma|$ denote the geometric realization of the dual graph Δ_Γ/Γ.

Theorem 5.9 *For each $N \geq 0$ there are embeddings ϕ_N of the cohomology of the dual graph $H^1(|\Delta_\Gamma/\Gamma|, \mathbb{C})$ into the dynamical cohomology $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$.*

Proof. We define maps ϕ_N from the cohomology $H^1(|\Delta_\Gamma/\Gamma|, \mathbb{C})$ to $\mathcal{H}^1_{dyn}(\Delta_K/\Gamma)$ in the following way: let $\{\gamma_i\}_{i=1}^g$ be a chosen set of generators of the Schottky group Γ, $[\gamma_i]$ the corresponding homology classes in $H_1(|\Delta_\Gamma/\Gamma|, \mathbb{Z}) = \Gamma/[\Gamma,\Gamma]$, and η_i the dual generators in cohomology. Let w_i be the finite admissible word in the edges of the directed graph $\bar{V}$ (the fundamental domain for the Γ-action on Δ) that represents the generator γ_i, with length $\ell_i = |w_i|$. Here, in the case of $\Delta = \bar{\Delta}_{K,n}$, we first notice that the first cohomology group of the dual graph is the same as $H^1(|\Delta'_\Gamma/\Gamma|, \mathbb{Z})$, since the insertion of extra vertices does not change the topology of the graph. Since we have $\Delta'_\Gamma \subset \Delta_{K,n}$ we obtain the w_i as above, as edges in the corresponding fundamental domain $\bar{V}$ for $\Delta_{K,n}$.

We then set

$$\phi_N(\eta_i) = P^\perp_{N\ell_i}\ \chi_{i,N}, \tag{5.5}$$

where

$$\chi_{i,N} := \chi_{\mathcal{W}^+(\Delta/\Gamma, \underbrace{w_i \cdots w_i}_{N-times})} \tag{5.6}$$

is the characteristic function of the set $\mathcal{W}^+(\Delta/\Gamma, w_i \cdots w_i)$ of walks in Δ/Γ that begin with the word w_i repeated N-times. The elements $\chi_{i,N}$ lie in $F_{N\ell_i}$. We denote by $P_k^\perp$ the orthogonal projection of F_k onto Gr_k.

The elements $\phi_N(\eta_i)$ are all linearly independent in $H^1_{dyn}(\Delta/\Gamma)$, hence the ϕ_N give linear embeddings of $H^1(|\Delta_\Gamma/\Gamma|, \mathbb{C})$ into $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$.

◇

Corollary 5.10 *For any finite set of distinct* $\{N_k\}$, *the set* $\cup_k \{\phi_{N_k}(\eta_i)\}_{i=1}^g$ *consists of linearly independent vectors in* $H^1_{dyn}(\Delta/\Gamma)$. *Thus, we obtain an embedding*

$$\Phi = \oplus_N \phi_N : \bigoplus_N H^1(|\Delta_\Gamma/\Gamma|, \mathbb{C}) \to \mathcal{H}^1_{dyn}(\Delta/\Gamma). \tag{5.7}$$

Notice that the map Φ of (5.7) does not preserve the graded pieces, due to the rescaling of the degrees by ℓ_i in (5.5). This has to be taken into account if we want to recover arithmetic information such as the local L–factors of [8] from the dynamical cohomology. For this reason, it may be necessary to blow up of some double points on the special fiber.

Lemma 5.11 *It is always possible to reduce to the case where all the* $\ell_i = \ell$, *after blowing up a certain number of double points on the special fiber.*

Proof. Blowing-up a double point on the special fiber $C(k)$, for $k = \mathcal{O}/\mathfrak{m}$, corresponds to introducing one extra vertex in the dual graph Δ_Γ/Γ. This changes by one the lengths ℓ_i for those generators of Γ for which the corresponding chain of edges w_i passes through the newly inserted vertex.
$\diamond$

Thus, possibly after blowing up some double points, we obtain:

$$\phi_N : H^1(|\Delta_\Gamma/\Gamma|, \mathbb{C}) \hookrightarrow Gr_{N\ell} \subset \mathcal{H}^1_{dyn}(\Delta_K/\Gamma),$$

for some $\ell \geq 1$.

5.4 Spectral triples

In this paragraph we show that we can associate to a given Mumford curve a family of spectral triples $(\mathcal{A}_n, \mathcal{H}_n, D_n)$, for $n \geq -1$. Each triple in this family corresponds to the choice of a graph $\Delta = \bar{\Delta}_{K,n}$, for $n \geq 0$ and $\Delta = \Delta_\Gamma$ for $n = -1$. The Dirac operator in these spectral triples depends only on the graded structure of the space $H^1_{dyn}(\Delta/\Gamma)$.

Recall that a *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ is a triple of a C*-algebra, with a representation in the algebra of bounded operators on the Hilbert space $\mathcal{H}$, and a Dirac operator D, which is a self-adjoint unbounded operator acting on $\mathcal{H}$, such that $(\lambda - D)^{-1}$ is a compact operator for all $\lambda \notin \mathbb{R}$ and the commutators $[a, D]$ are bounded operators for all $a \in \mathcal{A}_0$, with $\mathcal{A}_0$ a dense involutive subalgebra of $\mathcal{A}$.

Recall also that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has an associated family of zeta functions of the form

$$\zeta_{a,|D|}(z) = Tr(a|D|^{-z}) = \sum_{\lambda \in \mathrm{Spec}(|D|) \setminus \{0\}} Tr(a\Pi_\lambda)\lambda^{-z}, \tag{5.8}$$

with $a \in \mathcal{A}_0 \cup [D, \mathcal{A}_0]$, and with Π_λ the spectral projection on $\lambda \in \mathrm{Spec}(|D|)$. The properties of these zeta functions are related to the notion of dimension spectrum for the spectral triple (the set of poles of the $\zeta_{a,|D|}$) and to the local index formula of Connes and Moscovici. For the purpose of this paper, we will extend the zeta functions (5.8) to the case where the element a is a weak limit of certain sequences of elements in $\mathcal{A}$, since, in our construction, this is the type of zeta functions that recovers the arithmetic invariant given by the local Euler factor.

There are corresponding two-variable zeta functions

$$\zeta_{a,|D|}(s,z) = \sum_{\lambda \in \mathrm{Spec}(|D|)} Tr(a\Pi_\lambda)(s+\lambda)^{-z}, \tag{5.9}$$

and associated regularized determinants

$$\det_{\infty,a,|D|}(s) = \exp\left(-\frac{d}{dz}\zeta_{a,|D|}(s,z)|_{z=0}\right). \tag{5.10}$$

We consider the Hilbert space

$$\mathcal{H} = \mathcal{H}^1_{dyn}(\Delta/\Gamma) \oplus \mathcal{H}^1_{dyn}(\Delta/\Gamma). \tag{5.11}$$

On this space we consider the diagonal action of $\mathrm{C}^*(\Delta/\Gamma)$. We also introduce the notation $Gr_{n,-} := Gr_n \oplus 0$ and $Gr_{n,+} := 0 \oplus Gr_n$.

We define the Dirac operator D acting on $\mathcal{H}$ by setting

$$D|_{Gr_{+,n}} = n \qquad D|_{Gr_{-,n}} = -n-1. \tag{5.12}$$

Proposition 5.12 *The data* $(\mathrm{C}^*(\Delta/\Gamma), \mathcal{H}^1_{dyn}(\Delta/\Gamma) \oplus \mathcal{H}^1_{dyn}(\Delta/\Gamma), D)$ *determine a spectral triple in the sense of Connes [4].*

Proof. In order to obtain a spectral triple we need to check the compatibility requirement between the Dirac operator D and the action of the algebra $\mathrm{C}^*(\Delta/\Gamma)$. It is sufficient to check that the commutators $[D, S_w]$ are bounded operators. This follows easily since $S_w : Gr_{\pm,n} \to Gr_{\pm,n-1}$, so that $[D, S_w]f = \mp S_w f$. The remaining properties are easily verified.
◇

We can modify slightly the above construction, in order to take into account the scaling factor ℓ in the grading between $\oplus_N H^1(|\Delta_\Gamma/\Gamma|, \mathbb{C})$ and its image under Φ in $\mathcal{H}^1_{dyn}(\Delta/\Gamma)$.

Corollary 5.13 *We modify the operator D of* (5.12)*, by setting*

$$D|_{Gr_{+,n}} = \frac{n}{\ell}\frac{2\pi}{\log q} \qquad D|_{Gr_{-,n}} = \frac{-(n+1)}{\ell}\frac{2\pi}{\log q}. \tag{5.13}$$

Here q is the cardinality of the residue field $k(v)$ and ℓ is the length of all the words representing the generators of Γ, possibly after blowing up some points on the special fiber. With this modified operator D, we still obtain a spectral triple

$$(\mathrm{C}^*(\Delta/\Gamma), \mathcal{H}^1_{dyn}(\Delta/\Gamma) \oplus \mathcal{H}^1_{dyn}(\Delta/\Gamma), D).$$

5.5 Local L-factor

Let X be a curve over a global field $\mathbb{K}$. We assume semi-stability at all places of bad reduction. The local Euler factor at a place v has the following description ([20]):

$$L_v(H^1(X), s) = \det\left(1 - Fr_v^* N(v)^{-s} | H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v}\right)^{-1}. \tag{5.14}$$

Here Fr_v^* is the geometric Frobenius acting on ℓ-adic cohomology of $\bar{X} = X \otimes \mathrm{Spec}(\bar{\mathbb{K}})$, with $\bar{\mathbb{K}}$ the algebraic closure and ℓ a prime with $(\ell, q) = 1$, where q is the cardinality of the residue field $k(v)$ at v. We denote by N the norm map. The determinant is evaluated on the inertia invariants $H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v}$ at v (all of $H^1(\bar{X}, \mathbb{Q}_\ell)$ when v is a place of good reduction).

Suppose v is a place of $k(v)$-split degenerate reduction. Then the completion of X at v is a Mumford curve X_Γ. In this case, the Euler factor (5.14) takes the following form:

$$L_v(H^1(X_\Gamma), s) = \prod_\lambda (1 - \lambda q^{-s})^{-\dim H^1(X_\Gamma)^{I_v}_\lambda} = (1 - q^{-s})^{-g}, \tag{5.15}$$

since the eigenvalues $\{\lambda\}$ of the Frobenius, in this case, are all $\lambda = 1$. Here we denote by $H^1(X_\Gamma)^{I_v}_\lambda$ the eigenspaces of the Frobenius.

Deninger in [8] and [9] obtained the local factor (5.14) as a regularized determinant over an infinite dimensional cohomological theory.

In the case of Mumford curves, Deninger's calculation can be recast in terms of the data of the spectral triples of Proposition 5.12 and of the embeddings $\Phi^\pm$ of cohomologies.

We consider the operator iD (an imaginary rotation of the Dirac operator) and the zeta functions

$$\begin{aligned} \zeta_{a,iD,+}(s,z) &:= \textstyle\sum_{\lambda \in \mathrm{Spec}(iD) \cap i[0,\infty)} \mathrm{Tr}(a\Pi_\lambda)(s+\lambda)^{-z} \\ \zeta_{a,iD,-}(s,z) &:= \textstyle\sum_{\lambda \in \mathrm{Spec}(iD) \cap i(-\infty,0)} \mathrm{Tr}(a\Pi_\lambda)(s+\lambda)^{-z}. \end{aligned} \tag{5.16}$$

Then we have a regularized determinant

$$\det_{\infty,a,iD}(s) := \exp\left(-\zeta'_{a,iD,+}(s,0)\right) \exp\left(-\zeta'_{a,iD,-}(s,0)\right) \tag{5.17}$$

Theorem 5.14 *Let $\pi(\mathcal{V})$ be the orthogonal projection of $\mathcal{H}^1_{dyn}(\Delta/\Gamma) \oplus \mathcal{H}^1_{dyn}(\Delta/\Gamma)$ onto the graded subspace $\mathcal{V} = \mathrm{Im}(\Phi^-) \oplus \mathrm{Im}(\Phi^+)$, where we denote by $\Phi^\pm$ the maps $\Phi \oplus 0$ and $0 \oplus \Phi$. Then the regularized determinant (5.17), with $a = \pi(\mathcal{V})$ and D the Dirac operator of Corollary 5.13 satisfies*

$$\det_{\infty,\pi(\mathcal{V}),iD}(s) = L_v(H^1(X_\Gamma), s)^{-1}. \tag{5.18}$$

Proof. When we compute the zeta functions (5.16) for the Dirac operator of Corollary 5.13, and $a = \pi(\mathcal{V})$, we obtain

$$\begin{aligned} \zeta_{\pi(\mathcal{V}),iD,+}(s,z) &= \sum_{n=0}^{\infty} \mathrm{Tr}(\pi(\mathcal{V})\Pi_{+,n\ell})(\gamma(\tau+n))^{-z} \\ \zeta_{\pi(\mathcal{V}),iD,-}(s,z) &= \sum_{n=0}^{\infty} \mathrm{Tr}(\pi(\mathcal{V})\Pi_{-,n\ell})(\gamma(\tau-n))^{-z} - \mathrm{Tr}(\pi(\mathcal{V})\Pi_{-,0})(\tau\gamma)^{-z}, \end{aligned}$$

for $\gamma = \frac{2\pi i}{\log q}$ and $\tau = \frac{\log q}{2\pi i} s$, with choice of arguments $-\pi < \arg \gamma(\tau+n) < \pi$, as in [9].

Furthermore, we have $Tr(\pi(\mathcal{V})\Pi_{\pm,n\ell}) = \dim(Gr_{\pm,n\ell} \cap \mathcal{V}) = g$. In fact, the space $Gr_{+,n\ell} \cap \mathcal{V} = \mathrm{Im}(\Phi^+)$ is generated by $0 \oplus \chi_{i,n}$ for $i = 1, \dots, g$ and $Gr_{-,n\ell} \cap \mathcal{V} = \mathrm{Im}(\Phi^-)$ is generated by the element $\chi_{i,n} \oplus 0$ for $i = 1, \dots, g$.

The result then follows the calculation of the regularized determinant given in [9]. Namely, we obtain

$$(1-q^{-s})^{-g} = (\tau\gamma)^{-g} \exp(-g\zeta'_{\gamma}(\tau,0)) \exp(-g\zeta'_{-\gamma}(-\tau,0)),$$

which is exactly the regularized determinant $\det_\infty(s - \Theta_q)$ computed in [9] for the spectrum (with multiplicity g)

$$\mathrm{Spec}(s - \Theta_q) = \left\{ \frac{2\pi i}{\log q} \left(\frac{s \log q}{2\pi i} + n \right) : n \in \mathbb{Z} \right\}.$$

◇

It is very interesting to notice an important difference between the archimedean and non-archimedean cases. At the archimedean prime (*cf.* [6], [7]) the local factor is described in terms of zeta functions for a Dirac operator D. On the other hand, at the non-archimedean places, in order to get the correct normalization as in [9], we need to introduce a rotation of the Dirac operator by the imaginary unit, $D \mapsto iD$. This rotation corresponds to the Wick rotation that moves poles on the real line to poles on the imaginary line (zeroes for the local factor) and appears to be a manifestation of a rotation from Minkowskian to Euclidean signature $it \mapsto t$, as already remarked by Manin ([12] p.135), who wrote that *"imaginary time motion" may be held responsible for the fact that zeroes of* $\Gamma(s)^{-1}$ *are purely real whereas the zeroes of all non-archimedean Euler factors are purely imaginary.* This seems to hint to the existence of a more refined construction involving Minkowskian geometry, where the rotation $D \mapsto iD$ could be interpreted as a rotation $it \mapsto t$ of an infinitesimal length element $D^{-1} \sim ic\,dt$. A more precise treatment would require adapting the structure of spectral triple to the case of Minkowskian signature. Another piece of supporting evidence for the idea that a more refined construction should involve Minkowskian geometry comes from the cohomological construction of [5]. In fact, in [6], we only used part of the full symmetry group determined by the Lefschetz module structure, namely the part corresponding to real hyperbolic geometry, so as to match the results of [13]. The full symmetry group is $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$, which is the isometry group not of 3-dimensional real hyperbolic geometry, but of its Minkowskian version, anti de Sitter space AdS_{2+1}. On the relation between the results of [13] and AdS_{2+1} see also [16].

5.6 AF core

In order to understand the zeta functions $\zeta_{\pi(\mathcal{V}),D,\pm}(\tau,z)$ of Theorem 5.14 in terms of the spectral triple, we still need to express the operator $\pi(\mathcal{V})$ in terms of elements in the algebra $C^*(\Delta/\Gamma)$.

The graph C*-algebra $C^*(\Delta/\Gamma)$ contains an AF core given by the AF algebra obtained as the norm closure of $\cup_n \mathcal{F}_n$, where the finite dimensional algebras $\mathcal{F}_n$ are given by

$$\mathcal{F}_n = \text{span}\{S_\mu S_\nu^* : \mu,\nu \in \mathcal{P}^n(\Delta/\Gamma), t(\mu)=t(\nu)\}. \tag{5.19}$$

Here we used the notation $S_\mu = S_{w_1}\cdots S_{w_k}$, for $\mu = w_1\cdots w_k$. The AF core can be identified with the fixed point algebra of the gauge action (*cf.* [1]),

$$\overline{\bigcup_n \mathcal{F}_n} \cong C^*(\Delta/\Gamma)^{U(1)}. \tag{5.20}$$

Let $\{w_i\}_{i=1}^g$ be the words corresponding to the generators of Γ, which we assume all of equal length ℓ, possibly after some blow ups. Let S_{w_i} be the corresponding operators in $C^*(\Delta/\Gamma)$. The operators $S_{w_i}^n S_{w_i}^{*\,n}$ belong to the subalgebra $\mathcal{F}_{n\ell}$ in the AF core of $C^*(\Delta/\Gamma)$.

Each $Q_{i,n} = S_{w_i}^n S_{w_i}^{*\,n}$ acts on $L^2(\partial\Delta, d\mu)$ as multiplication by the characteristic function $\chi_{i,n}$ of (5.6), hence $Q_{i,n}$ maps $Gr_{\pm,n\ell}$ to itself, with range the one dimensional subspace of $Gr_{\pm,n\ell}$ spanned by $\chi_{i,n}$. Thus, the operator $Q_n = \sum_i Q_{i,n}$ projects $Gr_{\pm,n\ell}$ onto the g-dimensional subspace $Gr_{\pm,n\ell}\cap\mathcal{V}$.

For the Dirac operator of (5.13), we write $D = \sum_{n\geq 0}\Pi_n\lambda_n$, where with $\lambda_n = 2\pi n/\ell\log q$ and $\Pi_n = \Pi_{+,n}\oplus\Pi_{-,n-1}$. We then have the following result.

Proposition 5.15 *the zeta function*

$$\zeta_{\pi(\mathcal{V}),|D|}(z) = Tr(\pi(\mathcal{V})|D|^{-z})$$

can be written in the form

$$\zeta_{\pi(\mathcal{V}),|D|}(z) = \text{Tr}\left(\sum_{n>0} Q_{i,n}\Pi_{n\ell}\lambda_n^{-z}\right) \tag{5.21}$$

with the $Q_{i,n}$ in the AF core of $C^*(\Delta/\Gamma)$.

6 Foam spaces

We now consider the local factor (5.14) in the more general case, where we drop the assumption that v is a place of $k(v)$-split degenerate reduction. In this case, we no longer have a p–adic uniformization of the completion of X at v as a Mumford curve, and correspondently, the local factor is no longer determined solely in terms of the combinatorics of the dual graph, but it depends essentially on extra geometric information on the nature of the degeneration. In particular, the inertia invariants $H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v}$ are described only partly by the cohomology of the dual graph, with the extra information provided by the cohomology of the single components of the dual fiber, which in the general case will no longer be just rational curves.

More precisely, if we denote by $H^1(\bar{X})^{I_v}_\lambda$ the eigenspace of the geometric Frobenius, with eigenvalue λ, we can write the Euler factor in the form

$$L_v(H^1(X), s) = \prod_\lambda (1 - \lambda q^{-s})^{-d_\lambda}, \tag{6.1}$$

with $d_\lambda = \dim H^1(\bar{X})^{I_v}_\lambda$. Deninger's description of the local factor as regularized determinant holds in this more general case, in the form $\det_\infty(s - \Theta)$ where Θ has spectrum $\{\alpha_\lambda + \frac{2\pi i n}{\log q}\}$, with $n \in \mathbb{Z}$, $\lambda \in \mathrm{Spec}(Fr_v^*)$, and $q^{\alpha_\lambda} = \lambda$.

We want to modify the graphs Δ/Γ considered in the previous sections, in such a way that the corresponding dynamical cohomology will contain a linear subspace isomorphic to an infinite direct sum $\oplus_N H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v}$, and such that the construction of the spectral triple and the derivation of the regularized determinant described in the case of Mumford curves will extend to this case to recover (6.1).

Using the exact sequence of [17] p.110-111, we obtain an identification

$$H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v} \cong H^1(|\Delta_\Gamma/\Gamma|) \otimes_{\sigma_\ell} \mathbb{Q}_\ell \oplus H^1(X_v^{[0]}) \otimes_{\sigma_\ell} \mathbb{Q}_\ell, \tag{6.2}$$

where $\sigma_\ell : \mathbb{Q}_\ell \to \mathbb{C}$ is a fixed embedding of $\mathbb{Q}_\ell$ in $\mathbb{C}$, for a prime ℓ with $(\ell, q) = 1$, and we denote by $X_v^{[0]}$ the disjoint union of the components of the special fiber. In the case of $k(p)$-split reduction, where all components are $\mathbb{P}^1$'s, (6.2) is simply identified with $H^1(|\Delta_\Gamma/\Gamma|) \otimes_{\sigma_\ell} \mathbb{Q}_\ell$ as in the previous sections. The finite decomposition $H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v} = \oplus_\lambda H^1(\bar{X}, \mathbb{Q}_\ell)^{I_v}_\lambda$ in eigenvalues of the geometric Frobenius provides corresponding spaces $H^1(|\Delta_\Gamma/\Gamma|)_\lambda$ and $H^1(X_v^{[0]})_\lambda$ of dimensions d_λ^Γ and d_λ^0, respectively, with $d_\lambda^\Gamma + d_\lambda^0 = d_\lambda$.

We choose vertices x_λ (not necessarily distinct) of Δ_Γ/Γ, and attach to the vertex x_λ new outgoing edges $w^0_{i,\lambda}$, with $i = 1, \ldots d^0_\lambda$. We denote by E_v the oriented graph obtained via this construction, after appending tails to all sinks.

Remark 6.1 For certain classes of examples, our graph E_v can be embedded as a subgraph of the "foam space" defined in [3]. The foam space is a graph F_v associated to the fiber X_v of an arithmetic surface $\mathfrak{X}$ over $\mathrm{Spec}(\mathcal{O}_\mathbb{K})$, obtained by replacing the special fiber X_v with an infinite series of blowups of its $\mathbb{F}_q$-points. The graph F_v is the limit of the dual graphs associated to this series of blow-ups (*cf.* [15] §35). For this reason, we think of the graphs E_v as a generalization of "foam spaces".

To our foam space E_v, we associate the corresponding dynamical cohomology $\mathcal{H}^1(E_v)$ as in the previous sections, and the graph C*-algebra $C^*(E_v)$. The argument of Proposition 5.12 extends to this case and gives a spectral triple

$$(C^*(E_v), \mathcal{H}^1(E_v) \oplus \mathcal{H}^1(E_v), D).$$

We now define embeddings of cohomology groups as follows. Let $\omega_{i,\lambda}$, for $i = 1,\ldots,d^\Gamma_\lambda$ be loops of edges in Δ_Γ/Γ, with $|\omega_{i,\lambda}| = \ell_{i,\lambda}$, representing homology classes dual to a basis $\{\eta^\Gamma_{i,\lambda}\}$ of $H^1(|\Delta_\Gamma/\Gamma|)_\lambda$. Up to adding vertices to the graph Δ_Γ/Γ by blowing up double points in the closed fiber, we can assume that all the $\ell_{i,\lambda} = \ell$. Adding vertices in this way does not change $H^1(X_v^{[0]})$, since the components of the closed fiber that correspond to the new vertices all have trivial H^1.

We consider then the linear embedding

$$\Phi^\Gamma_{N,\lambda} : H^1(|\Delta_\Gamma/\Gamma|)_\lambda \hookrightarrow Gr_{N\ell} \subset \mathcal{H}^1(E_v),$$

given by

$$\Phi^\Gamma_{N,\lambda}(\eta^\Gamma_{i,\lambda}) = P^\perp_{N\ell_\lambda}\chi_{\mathcal{W}^+(E_v,\underbrace{\omega_{i,\lambda}\cdots\omega_{i,\lambda}}_{N-times})}.$$

We also consider the embeddings

$$\Phi^0_{N,\lambda} : H^1(X_v^{[0]})_\lambda \hookrightarrow Gr_{N\ell} \subset \mathcal{H}^1(E_v),$$

$$\Phi^0_{N,\lambda}(\eta^0_{i,\lambda}) = P^\perp_N\chi_{\mathcal{W}^+(E_v,\underbrace{w^0_{i,\lambda}\cdots w^0_{i,\lambda}}_{N\ell-times})},$$

where the $\eta^0_{i,\lambda}$ form a basis of $H^1(X_v^{[0]})_\lambda$ and the $w^0_{i,\lambda}$ are the corresponding oriented edges of E_v. Let $\Phi^\Gamma_\lambda = \oplus_N \Phi^\Gamma_{N,\lambda}$ and $\Phi^0_\lambda = \oplus_N \Phi^0_{N,\lambda}$, and let $\Phi_\lambda = \Phi^\Gamma_\lambda \oplus \Phi^0_\lambda$. With $\Phi^\pm_\lambda$ defined as the $\Phi^\pm$ in Theorem 5.14, we denote by $\mathcal{V}_\lambda = \mathrm{Im}(\Phi^-_\lambda)\oplus\mathrm{Im}(\Phi^+_\lambda)$, and by $\pi(\mathcal{V}_\lambda)$ the corresponding orthogonal projection.

We then extend the result of Theorem 5.14 to this more general setting.

Theorem 6.2 *Consider the regularized determinants* (5.17)*, with* $a_\lambda = \pi(\mathcal{V}_\lambda)$ *and* D *the Dirac operator of Corollary 5.13 for the spectral triple* $(\mathrm{C}^*(E_v), \mathcal{H}^1(E_v)\oplus\mathcal{H}^1(E_v), D)$*. We obtain*

$$\prod_\lambda \det_{\infty,\pi(\mathcal{V}_\lambda),iD}(s) = L_v(H^1(X), s)^{-1}. \tag{6.3}$$

The operators $\pi(\mathcal{V}_\lambda)$ *are related to the AF core of the* C^**-algebra* $\mathrm{C}^*(E_v)$ *as in* (5.21)*.*

Proof. We compute the zeta functions (5.16) for the Dirac operator of the spectral triple $(\mathrm{C}^*(E_v), \mathcal{H}^1(E_v)\oplus\mathcal{H}^1(E_v), D)$, with $a = \pi(\mathcal{V})$. We obtain

$$\zeta_{\pi(\mathcal{V}_\lambda),iD,+}(s,z) = \sum_{n=0}^\infty \mathrm{Tr}(\pi(\mathcal{V}_\lambda)\Pi_{+,n\ell})(\gamma(\tau_\lambda+n))^{-z}$$

$$\zeta_{\pi(\mathcal{V}_\lambda),iD,-}(s,z) = \sum_{n=0}^\infty \mathrm{Tr}(\pi(\mathcal{V}_\lambda)\Pi_{-,n\ell})(\gamma(\tau_\lambda-n))^{-z} - \mathrm{Tr}(\pi(\mathcal{V}_\lambda)\Pi_{-,0})(\tau_\lambda\gamma)^{-z},$$

for $\gamma = \frac{2\pi i}{\log q}$, $\tau_\lambda = \frac{\log q}{2\pi i}(s-\alpha_\lambda)$, and $q^{\alpha_\lambda} = \lambda$, and with choice of arguments as in [9].

By construction, we have $Tr(\pi(\mathcal{V}_\lambda)\Pi_{\pm,n\ell}) = \dim(Gr_{\pm,n\ell} \cap \mathcal{V}_\lambda) = d_\lambda$, hence the left hand side of (6.3) is the regularized determinant $det_\infty(s - \Theta_q)$ computed in [9], with spectrum (with multiplicities d_λ)

$$\mathrm{Spec}(s-\Theta_q) = \left\{ \frac{2\pi i}{\log q} \left(\frac{\log q}{2\pi i}(s-\alpha_\lambda) + n \right) : n \in \mathbb{Z}, \lambda \in \mathrm{Spec}(Fr_v^*) \right\}.$$

The expression of the operators $\pi(\mathcal{V}_\lambda)$ in terms of operators in the AF core of the C*-algebra $C^*(E_v)$ is analogous to the case of Mumford curves.
◇

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