From Noncommutative to Arithmetic Geometry

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Credits:

- C.Consani and M.Marcolli, *Noncommutative geometry, dynamics and ∞-adic Arakelov geometry*, preprint

- C.Consani and M.Marcolli, *Spectral triples from Mumford curves*, preprint.

- C.Consani and M.Marcolli, *Triplets spectraux en geometrie d'Arakelov*, C.R.Acad.Sci. Paris, Ser. I 335 (2002) 779-784.

- Yu.Manin and M.Marcolli, *Continued fractions, modular symbols and noncommutative geometry*, Selecta Math. 8(3) (2002) 475-520.

- Yu.Manin and M.Marcolli, *Holography principle and arithmetic of algebraic curves*, Adv.Theor.Math.Phys. Vol.5, N.3 (2001) 617-650.

- M.Marcolli, *Limiting modular symbols and the Lyapunov spectrum*, J. Number Theory, Vol.98 N.2 (2003) 348-376.

What is a Noncommutative space?

measure theory	von Neumann algebras
topology	C^*–algebras
smooth structures	smooth subalgebras
Riemannian geometry	spectral triples

Warning: More rigid structure in the noncommutative case!

C^*-algebras: Gel'fand–Naimark correspondence (loc. comp. Hausdorff space $\Leftrightarrow$ Commutative C^*-algebra)

$$X \Leftrightarrow C_0(X) \quad \text{Topology}$$

but for NC tori C^*-algebra T_θ "like" $\mathbb{C}$ structure, NC elliptic curves

Noncommutativity from quotients

Algebra of functions for a quotient space $X = Y/\sim$:

- Functions on Y with $f(a) = f(b)$ for $a \sim b$. **Poor!**

- Functions f_{ab} on the graph of the equivalence relation. **Good!**

For sufficiently *nice* quotients: Morita equivalent.

Simplest example: $Y = [0,1] \times \{0,1\}$; equivalence $(x,0) \sim (x,1)$ for $x \in (0,1)$. First method: constant functions $\mathbb{C}$; second method:

$$\{f \in C([0,1]) \otimes M_2(\mathbb{C}) : f(0) \text{ and } f(1) \text{ diagonal } \}$$

Warning: Different NC spaces from same quotient (Quantization is an art not a functor)

Geometry on NC spaces

Connes, "Geometry from the spectral point of view", Lett.Math.Phys. 34 (1995) 3

Infinitesimal distance element ds on a compact Riemannian spin manifold in terms of Dirac operator D. $\Rightarrow$ Data $(C^\infty(X), L^2(S), D)$ completely determine the Riemannian geometry

Spectral Triples

$\mathcal{A} \rightsquigarrow C^*$-algebra (or dense sub-algebra)
$\mathcal{H} \rightsquigarrow$ Hilbert space H
$D \rightsquigarrow$ (unbounded) "Dirac operator" on H

With the following conditions:

1. D self-adjoint

2. $\forall \lambda \notin \mathbb{R}$: $(D-\lambda)^{-1}$ compact

3. $\forall\ a \in \mathcal{A}$: $[D, a]$ bounded on H

Spectral triples and zeta functions

$(\mathcal{A}, \mathcal{H}, D)$ spectral triple

$$s, z \in \mathbb{C} \quad \zeta_{a,D} := \sum_\lambda \mathrm{Tr}(a, \prod(\lambda, |D|))(s-\lambda)^{-z}$$

$\prod(\lambda, |D|) =$ orthog. proj. on the eigsp. $E(\lambda, |D|)$
$a \in \tilde{\mathcal{A}} =$ algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$

Have: Mellin transform

$$\zeta_{a,D}(s,z) = \frac{1}{\Gamma(z)} \int_0^{+\infty} \theta_{a,D,s}(t) t^{z-1} dt$$

$$(*) \quad \theta_{a,D,s}(t) := \sum_\lambda \mathrm{Tr}(a, \prod(\lambda, |D|)) e^{(s-\lambda)t}$$

Under suitable hypothesis on asymptotic expansion of $(*)$:
$\zeta_{a,D}(s,z)$ admits a unique analytic continuation and there is an associated
Regularized determinant (Ray-Singer)

$$\det_{\infty,a,D}(s) := \exp(-\frac{d}{dz}\, \zeta_{a,D}(s,z)_{|z=0})$$

Arithmetic (Compactifications)

- Moduli spaces: NC spaces as “boundaries”

- Infinite primes: archimedean fibers as NC spaces

NC Geom and compactifications (philosophy):

(1) NC tori in M-theory (Connes–Douglas–Schwarz)

(2) degenerations of CY manifolds (Kontsevich–Soibelman);

(3) Foliations and boundary of Teichmüller spaces (Thurston);

(4) Instantons (Nekrasov–Schwarz);

Modular curves (Simplest significant example of moduli spaces)

G = finite index subgroup of $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ or $\mathrm{PGL}(2,\mathbb{Z})$; $\mathbb{P} := \Gamma/G$ cosets; $\mathbb{H}^2$ = 2-dim real hyperbolic plane

$$X_G := G\backslash\mathbb{H}^2 \cong \Gamma\backslash(\mathbb{H}^2 \times \mathbb{P})$$

<u>Compactification</u> by cusps

$$G\backslash\mathbb{P}^1(\mathbb{Q}) \cong \Gamma\backslash(\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P})$$

If count irrational points $\mathbb{P}^1(\mathbb{R})$ as "part of boundary" $\Rightarrow$ NC spaces

NC spaces associated to X_G:

- $\Gamma\backslash(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \Rightarrow C^*$-algebra

$$C(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}) \rtimes \Gamma$$

Modular interpretation.

Precise: $\Gamma\backslash\mathbb{P}^1(\mathbb{R})$ = Morita equivalence classes of NC tori T_θ

Heuristic: degeneration of elliptic curves

Jacobi uniformization: $q \in \mathbb{C}^*$, $|q| < 1$ ($q = \exp(2\pi i\tau)$, $Im(\tau) > 0$)

$$E_q = \mathbb{C}^*/q^{\mathbb{Z}}$$

fundamental domain: annulus radii 1 and q, identification via scaling and rotation

As $q \to \exp(2\pi i\theta) \in S^1$, $\theta \in \mathbb{R} \setminus \mathbb{Q}$,

$$E_q \longrightarrow \text{NC torus } T_\theta$$

Same classical quotient!

PGL$(2,\mathbb{Z})$ orbits in $\mathbb{P}^1(\mathbb{R}) \simeq$ orbits of $[0,1]$ under $Tx = 1/x - [1/x]$ of continued fraction expansion

$$x \sim_T y \Leftrightarrow \exists n, m : T^n x = T^m y$$

• Action of shift $T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$

$$T(x,s) = \left(\frac{1}{x} - \left[\frac{1}{x} \right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} \cdot s \right)$$

Groupoid

$\mathcal{G}([0,1] \times \mathbb{P}, T) = \{((x,s), m-n, (y,t)) : T^m(x,s) = T^n(y,t)\}$
$\mathcal{G}^0 = \{((x,s), 0, (x,s))\}$

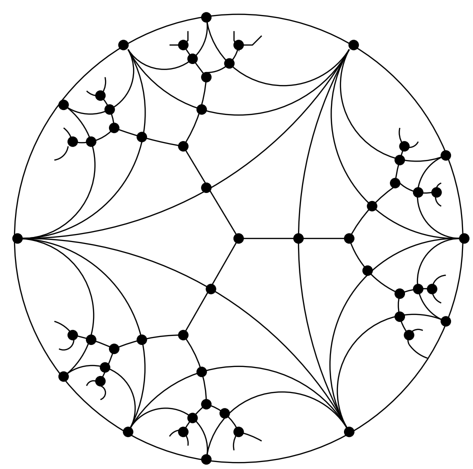
$\Rightarrow$ C^*-algebra $C^*(\mathcal{G}([0,1] \times \mathbb{P}, T))$, encodes properties of the dynamics of T, Cuntz-like algebras (Renault)

$\Rightarrow$ *limiting behavior* (for arithmetic invariants defined on modular curves) when $\tau \to \theta \in \mathbb{R} \setminus \mathbb{Q}$ (e.g. limiting modular symbols)

Recover modular curves from NC spaces

Notation: $\tilde{I} = \Gamma \cdot i$, $\tilde{R} = \Gamma \cdot \rho$, $\rho = e^{\pi i/3}$, $I = G\backslash\tilde{I}$, $R = G\backslash\tilde{R}$; σ,τ gen $\mathrm{PSL}(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$, $\sigma^2 = 1$ and $\tau^3 = 1$; $\mathbb{P}_I = \langle\sigma\rangle\backslash\mathbb{P}$ and $\mathbb{P}_R = \langle\tau\rangle\backslash\mathbb{P}$

Modular complex computing $H_1(X_G)$



Relative homology sequence: $(H_A^B := H_1(\overline{X_G}\backslash A, B; \mathbb{Z}))$

$$0 \to H_{\text{cusps}} \to H_{\text{cusps}}^{R\cup I} \stackrel{(\tilde{\beta}_R,\tilde{\beta}_I)}{\to} H_0(R) \oplus H_0(I) \to \mathbb{Z} \to 0,$$

$H_0(I) \cong \mathbb{Z}[\mathbb{P}_I]$, $H_0(R) \cong \mathbb{Z}[\mathbb{P}_R]$, $H_{\text{cusps}}^{R\cup I} = \mathbb{Z}[\mathbb{P}]$:

$$0 \to H_{\text{cusps}} \to \mathbb{Z}[\mathbb{P}] \to \mathbb{Z}[\mathbb{P}_R] \oplus \mathbb{Z}[\mathbb{P}_I] \to \mathbb{Z} \to 0$$

Pimsner–Voiculescu exact sequence for group Γ acting on a tree ($T^1 \simeq \Gamma$, $T^0 \simeq \Gamma/\langle\sigma\rangle \cup \Gamma/\langle\tau\rangle$)

$X := \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$, $\Gamma_\sigma = \mathbb{Z}/2 = \langle\sigma\rangle$, $\Gamma_\tau = \mathbb{Z}/3 = \langle\tau\rangle$:

$$\begin{array}{ccccc} K_0(C(X)) & \xrightarrow{\alpha} & K_0(C(X) \rtimes \Gamma_\sigma) \oplus K_0(C(X) \rtimes \Gamma_\tau) & \xrightarrow{\tilde{\alpha}} & K_0(C(X) \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(C(X) \rtimes \Gamma) & \xleftarrow[\tilde{\beta}]{} & K_1(C(X) \rtimes \Gamma_\sigma) \oplus K_1(C(X) \rtimes \Gamma_\tau) & \xleftarrow[\beta]{} & K_1(C(X)) \end{array}$$

Can identify $K_i(C(X)) = \mathbb{Z}[\mathbb{P}]$, $K_i(C(X) \rtimes \Gamma_\sigma) = \mathbb{Z}[P_I]$, $K_i(C(X) \rtimes \Gamma_\tau) = \mathbb{Z}[\mathbb{P}_R]$

$$0 \to Ker(\beta) \to \mathbb{Z}[\mathbb{P}] \to \mathbb{Z}[P_I] \oplus \mathbb{Z}[P_R] \to Im(\tilde{\beta}) \to 0$$

canonically isomorphic to

$$0 \to H_{\text{cusps}} \to \mathbb{Z}[\mathbb{P}] \to \mathbb{Z}[\mathbb{P}_R] \oplus \mathbb{Z}[\mathbb{P}_I] \to \mathbb{Z} \to 0$$

(Similar for K_0 with extra torsion)

e.g. Modular symbols $\{g(0), g(i\infty)\}_G$ identified with elements in K-theory

Result from dynamics: shift $T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$.

• Modular symbols at irrational points of the "boundary" (localized on closed geodesics = periodic cont fr)

geodesics between cusps $\Rightarrow$ modular symbols $\{\alpha, \beta\}_G$ homology classes, lin combin of $\varphi(s) = \{g(0), g(i\infty)\}_G$ for $gG = s \in \mathbb{P}$

Limiting modular symbols β irrational

$$\{\{*, \beta\}\}_G := \lim_{\tau \to \infty} \frac{1}{\tau} \{x, y(\tau)\}_G \in H_1(X_G, \mathbf{R}),$$

$$\lim_{n \to \infty} \frac{1}{\lambda(\beta) n} \sum_{k=1}^{n} \varphi \circ T^k(s)$$

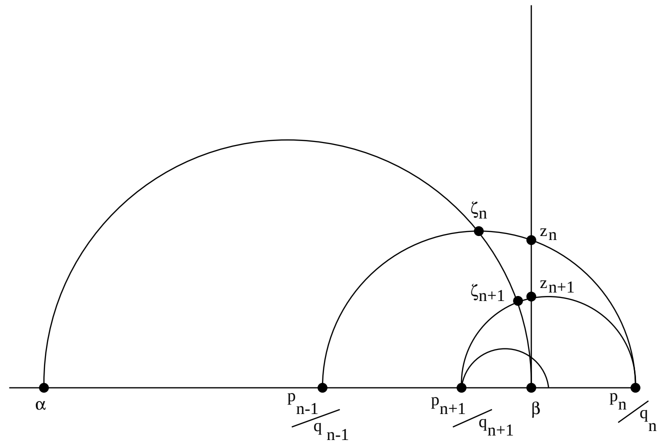
($\lambda(\beta) =$ Lyapunov exponent of T)

Periodic case:

$$\{\{*, \beta\}\}_G = \frac{\{0, g(0)\}_G}{\log \Lambda_g} = \frac{1}{\lambda(\beta) \ell} \sum_{k=1}^{\ell} \varphi \circ T^k(s)$$

Non-periodic case: vanishing result a.e. T-inv measure (depends on spectral theory of Ruelle transfer operator)

• Modular forms obtained as integral averages on the boundary



Application: Mixmaster Universe

Cosmological models with $SO(3)$ symmetry on the space-like hypersurfaces and cosmological singularity at $t \to 0$

$$ds^2 = -dt^2 + a(t)dx^2 + b(t)dy^2 + c(t)dz^2$$

Kasner metric ($\sum p_i = 1 = \sum_i p_i^2$)

$$ds^2 = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$$

Mixmaster universe (p_i depend on a parameter u)

$p_1 = -u/(1+u+u^2)$ $p_2 = (1+u)/(1+u+u^2)$ $p_3 = u(1+u)/(1+u+u^2)$

Discretization: Eras, cycles: $u_{n+1} = \frac{1}{u_n - [u_n]}$ and permutation of space axis

Result: Infinite geodesics on the modular curve $X_{\Gamma_0(2)}$ not ending at cusps $\Leftrightarrow$ solutions of the mix-master universe

NC and Arakelov geometry

Arithmetic surfaces ($\mathbb{K}$ = number field $O_{\mathbb{K}}$ = ring of integers) Smooth algebraic curve X over $\mathbb{K} \Rightarrow$ model $X_{O_{\mathbb{K}}}$, arithmetic surface over Spec($O_{\mathbb{K}}$). Closed fiber of $\mathcal{X}_{O_{\mathbb{K}}}$ over a prime $\wp \in O_{\mathbb{K}}$: $X_{\wp} =$ reduction mod $\wp$

Arithmetic infinity: embeddings $\alpha : \mathbb{K} \hookrightarrow \mathbb{C}$ (real or complex conjugate) = archimedean primes $\Rightarrow$ Riemann surfaces X_{α}: $X_{/\mathbb{R}}$ or $X_{/\mathbb{C}}$

$\overline{\text{Spec}(O_{\mathbb{K}})}$: adding "archimedean places" $\{\alpha\}$

$\overline{\mathcal{X}}$: adding formal real linear combinations of "fibers at ∞":

$$\sum_{\alpha} \lambda_{\alpha} F_{\alpha}, \quad F_{\alpha} = \underline{\text{formal symbols}}$$

Hermitian metric ds^2_{α} on each Riemann surface X_{α}

Arakelov's philosophy: Hermitian geometry on the $X_{\alpha} \Rightarrow$ geometry of the "fibers at infinity" F_{α}

Green function

compact Riemann surface $X_{\mathbb{C}}$, Green function $g_{\mu,A}$: divisor $A = \sum_x m_x(x)$, positive real–analytic 2-form $d\mu$

- *Laplace equation*: g_A satisfies

$$\partial\bar{\partial}\, g_A = \pi i\, (\deg(A)\, d\mu - \delta_A)$$

  with δ_A the δ–current $\varphi \mapsto \sum_x m_x \varphi(x)$.

- *Singularities*: $z =$ loc coord in neighb of x $\Rightarrow g_A - m_x \log|z|$ loc real analytic.

- *Normalization*: g_A satisfies $\int_X g_A d\mu = 0$.

Geometric description of the F_α?

Manin, "3-dimensional hyperbolic geometry as ∞-adic Arakelov geometry", Invent.Math. 104 (1991)

• Enrich metric structure on $X_{/\mathbb{C}}$ with a choice of **Schottky uniformization:**

Schottky group: $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$: discrete, free group of rank $g \geq 1$ purely loxodromic elements

$\mathbb{P}^1(\mathbb{C}) \supset \Lambda_\Gamma =$ limit set of Γ: accumulation pts. of Γ-orbits (Cantor set for $g \geq 2$)
$\Omega_\Gamma := \mathbb{P}^1(\mathbb{C}) \smallsetminus \Lambda_\Gamma$ connected, non-simply connected Γ-invariant domain of discontinuity of Γ

$$X_{\mathbb{C}} = \Omega_\Gamma / \Gamma$$

$\mathfrak{X}_\Gamma = \mathbb{H}^3/\Gamma$ **hyperbolic handlebody** $X_{\mathbb{C}} = \partial \mathfrak{X}_\Gamma$

• Green function on $X_{/\mathbb{C}}$ with Schottky uniformization:

$$g((a)-(b),(c)-(d)) =$$

$$\sum_{h\in\Gamma} \log|\langle a,b,hc,hd\rangle| - \sum_{\ell=1}^{g} X_\ell(a,b) \sum_{h\in S(g_\ell)} \log|\langle z^+(h), z^-(h), c, d\rangle|$$

$S(\gamma)$ conjugacy class of γ in Γ

$\langle a,b,c,d\rangle$ = cross ratio

In terms of geodesics in the handlebody $\mathfrak{X}_\Gamma$:

$$-\sum_{h\in\Gamma} \text{ordist}(a * \{hc,hd\}, b * \{hc,hd\})$$

$$+\sum_{\ell=1}^{g} X_\ell(a,b) \sum_{h\in S(g_\ell)} \text{ordist}(z^+(h) * \{c,d\}, z^-(h) * \{c,d\}).$$

Coefficients $X_\ell(a,b)$ also in terms of geodesics

Geometric idea: Bounded geodesics in $\mathfrak{X}_\Gamma$ give the dual graph of the fiber F_α

Physical interpretation: Holography principle: gravity on bulk space (Euclidean AdS black holes), field theory (boson/fermion) on the boundary

Example: Bañados–Teitelboim–Zanelli black hole

Genus one case: $\mathbb{H}^3/(q^{\mathbb{Z}}) \leftrightsquigarrow X_q(\mathbb{C}) = \mathbb{C}^*/(q^{\mathbb{Z}})$ (Jacobi uniformization) $q : (z, y) \mapsto (qz, |q|y)$

$$q = \exp\left(\frac{2\pi(i|r_-| - r_+)}{\ell}\right) \quad r_\pm^2 = \frac{1}{2}\left(M\ell \pm \sqrt{M^2\ell^2 + J}\right)$$

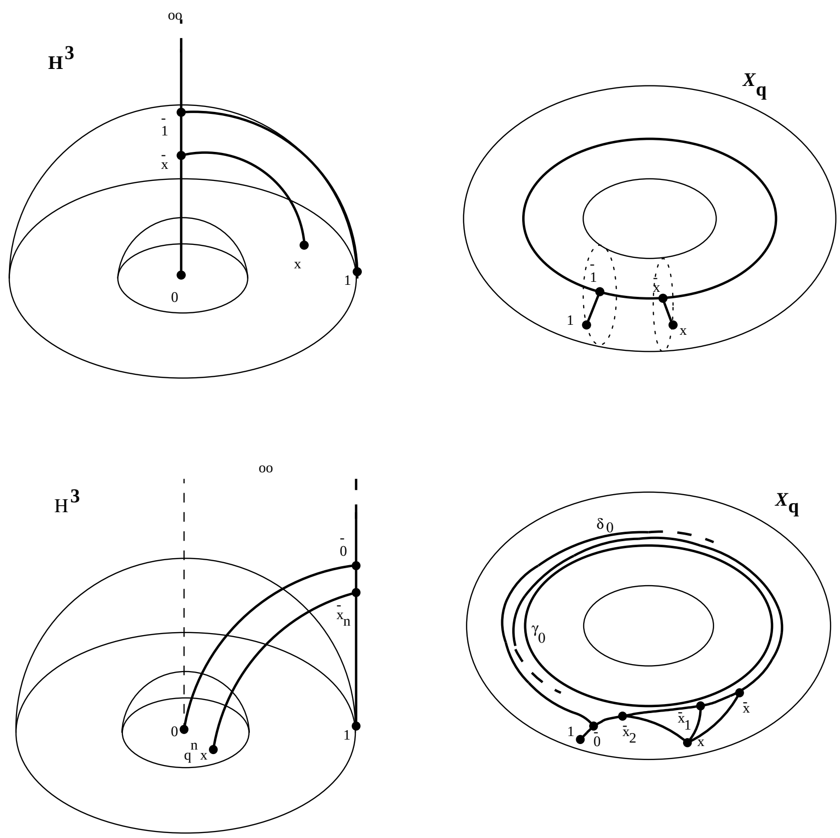
mass and angular momentum of black hole, $-1/\ell^2 =$ cosmological constant.

Operator product expansion of path integral on the elliptic curve $X_q(\mathbb{C})$ = Arakelov Green function (Alvarez-Gaume,Moore,Vafa Comm.Math.Phys. 106 1 (1986))

$$g(z,1) = \log\left(|q|^{B_2(\log|z|/\log|q|)/2}|1-z| \prod_{n=1}^{\infty} |1-q^n z|\ |1-q^n z^{-1}|\right)$$

in terms of geodesics (gravity on bulk space):

$$= -\frac{1}{2}\ell(\gamma_0)\, B_2\left(\frac{\ell_{\gamma_0}(\bar{z}, \bar{1})}{l(\gamma_0)}\right) + \sum_{n\geq 0} \ell_{\gamma_1}(\bar{0}, \bar{z}_n) + \sum_{n\geq 1} \ell_{\gamma_1}(\bar{0}, \tilde{z}_n).$$



$$\bar{x} = x * \{0, \infty\};\ \bar{z}_n = q^n z * \{1, \infty\},\ \tilde{z}_n = q^n z^{-1} * \{1, \infty\}$$

Arithmetic of the fibers F_α (Serre)

$X_{/\kappa}$ = smooth, proper curve over $\kappa = \mathbb{C}, \mathbb{R}$, $H_B^* := H_{Betti}^*(X_{/\kappa}, \mathbb{C})$, Hodge structure

$$H_B^* = \bigoplus_{p,q} H^{p,q}$$

$$s \in \mathbb{C}, \quad L_\kappa(H_B^*, s) :=$$

$$= \begin{cases} \prod_{p,q} \Gamma_{\mathbb{C}}(s - \min(p,q))^{h^{p,q}} & \kappa = \mathbb{C} \\ \prod_{p<q} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \prod_p \Gamma_{\mathbb{R}}(s-p)^{h^{p+}} \Gamma_{\mathbb{R}}(s-p+1)^{h^{p-}} & \kappa = \mathbb{R} \end{cases}$$

$h^{p,q} := \dim_{\mathbb{C}} H^{p,q}$ $h^{p,\pm} := \dim(\pm(-1)^p\text{-eigenspace})$ of de-Rham conj. on $H^{p,p}$

$$\Gamma_{\mathbb{C}}(s) := (2\pi)^{-s}\Gamma(s), \quad \Gamma_{\mathbb{R}}(s) := 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$$

$$\text{(Gamma function) } \Gamma(s) := \int_0^{+\infty} e^{-t} t^s \, \frac{dt}{t}.$$

Conclusion The archimedean factor is constructed using a cohomological theory on $X_{/\kappa}$

Complex with monodromy

Consani, "Double complexes and Euler L-factors", Compositio Math. 111 (1998)

$X_{/\kappa}$ smooth projective algebraic curve over $\kappa = \mathbb{C}, \mathbb{R}$

$\exists$ $(K^{\cdot,\cdot}, d)$ double complex of real differentiable Tate-twisted forms on $X_{/\kappa}$ with N = local monodromy at ∞ (also real Frobenius $\bar{F}_\infty$ for $\kappa = \mathbb{R}$)

For $\kappa = \mathbb{C}$, $\mathbb{H}^*(K^\cdot, d)^{N=0}$ decomposes as ($p \in \mathbb{Z}$):

$$(\mathbb{H}^0)^{N=0} = \bigoplus_{p \leq 0} H^0(X_{/\mathbb{C}}, \mathbb{R}(p))$$
$$(\mathbb{H}^1)^{N=0} = \bigoplus_{p \leq 0} H^1(X_{/\mathbb{C}}, \mathbb{R}(p))$$
$$(\mathbb{H}^2)^{N=0} = \bigoplus_{p \leq 1} H^2(X_{/\mathbb{C}}, \mathbb{R}(p))$$

(For $\kappa = \mathbb{R}$: take $\bar{F}_\infty$-invariants)

Regularized determinant

linear operator $\phi_q : \mathbb{H}^q(K^\cdot, d)^{N=0} \to \mathbb{H}^q(K^\cdot, d)^{N=0}$

$$\phi_q|_{gr_p^{\mathsf{W}}\mathbb{H}^q(K^\cdot,d)^{N=0}} = \text{ multipl. by weight } p$$

Archimedean factor:

$$L_\kappa(H_B^q, s) = \det_\infty\left(\frac{1}{2\pi}(s - \phi_q) \mid (\mathbb{H}^q)^{N=0}\right)^{-1}$$

Recall: T = self-adjoint operator with pure point spectrum (m_λ = multipl. of λ)

$$\det_\infty(s - T) := \exp\left(-\frac{d}{dz}\zeta_T(s,z)_{|z=0}\right)$$

$$\zeta_T(s,z) = \sum_{\lambda \in \mathrm{Spec}(T)} m_\lambda (s-\lambda)^{-z}$$

cf. Deninger, "On the Γ-factors attached to motives", Invent.Math. 104 (1991)

$\mathbb{H}^*(K^\cdot, d)^{N=0} \subset \mathbb{H}^*(\mathrm{Cone}(N)^\cdot, d)$: ϕ restriction of a 'weight' operator $\Phi : \mathbb{H}^*(\mathrm{Cone}(N)^\cdot, d) \to \mathbb{H}^*(\mathrm{Cone}(N)^\cdot, d)$

Conclusion: $\mathbb{H}^*(\mathrm{Cone}(N)^\cdot, d)$ carries arithmetical information on the closed fiber at ∞

Polarized Hodge–Lefschetz modules

$$(\mathbb{H}^*(K^\cdot, d), N, \ell, <\cdot,\cdot>)$$

Loc. monodromy at ∞: $N(\eta) = (2\pi\sqrt{-1})^{-1}\eta$

(Tate twist, index shift)

$$\text{Lefschetz: } \ell(\eta) = (2\pi\sqrt{-1})\eta \wedge \omega$$

(ω = Kähler form)

$$\text{Polarization: } \psi : K^{\cdot,\cdot} \otimes K^{\cdot,\cdot} \to \mathbb{R}(1)$$

Equivalently: representation

$$\sigma : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Aut}(K^{\cdot,\cdot})$$

$$x \in K^{i,j}, \quad \sigma\left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}, \begin{pmatrix} b & 0 \\ 0 & b^{-1}\end{pmatrix}\right\}(x) = a^i b^j x$$

$$d\sigma\left\{\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}, 0\right\} = N, \quad d\sigma\left\{0, \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\right\} = l$$

Symm. pos. def. bilinear form:

$$\langle x, y\rangle = \psi(x, \sigma(w,w)y) \quad w = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$$

Bers simultaneus uniformization

$X_{/\mathbb{C}} = \Omega_\Gamma/\Gamma$ compact, smooth Riemann surface of genus $g \geq 2$ uniformized by
$\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ Schottky group of rank g

- **Quasi-circle** (Bowen) Γ-invariant Jordan curve $C \subset \mathbb{P}^1(\mathbb{C})$
(limit set $\Lambda_\Gamma \subset C$, $\mathbb{P}^1(\mathbb{C}) \setminus C = \Omega_1 \cup \Omega_2$)
$$\hat{C} = \pi_\Gamma(C \cap \Omega_\Gamma) \subset X_{/\mathbb{C}},$$

$$\exists\ \alpha_i : \Omega_i \stackrel{\simeq}{\to} U_i, \quad U_1 \cup U_2 = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$$

α_i = conformal maps, $U_i \simeq \mathbb{H}^2$ = upper half plane

- $G_i := \{\alpha_i \gamma {\alpha_i}^{-1} \ :\ \gamma \in \tilde{\Gamma}\} \simeq \Gamma$ ($G_i \subset \mathrm{PSL}(2,\mathbb{R})$ Fuchsian Schottky grps.)

- $X_{/\mathbb{C}} = X_1 \cup_{\partial X_1 = \hat{C} = \partial X_2} X_2$, $X_i = U_i/G_i$
X_i = Riemann surfaces with boundary $\hat{C}$

For real structure $\hat{C} = X(\mathbb{R})$

Arithmetic spectral triple

From Lefschetz module $\Rightarrow (\mathbb{H}^*(\text{Cone}(N)), \langle\cdot,\cdot\rangle)$ pre-Hilbert space, with action of (right) $\text{SL}(2,\mathbb{R})$ by bounded operators

Input Schottky uniformization: (real) C^*-algebra $A_{\tilde{\Gamma}}$ acting on $\mathcal{H} = (\mathbb{H}^*(\text{Cone}(N)), \langle\cdot,\cdot\rangle)$ completion in $B(\mathcal{H})$ of group ring $\mathbb{R}[\tilde{\Gamma}]$ ($\tilde{\Gamma} \subset \text{SL}(2,\mathbb{R})$; when unitary action $A_{\tilde{\Gamma}} = C^*(\tilde{\Gamma})$)

$$(A_{\tilde{\Gamma}}, \mathbb{H}^*(\text{Cone}(N)), \Phi) \text{ is a spectral triple}$$

In the family of zeta-functions for the spectral triple $\zeta_{a,D}(s,z)$ choose

$$a := \sigma_2(-\text{id}) \in \mathcal{A}_{\tilde{\Gamma}} \quad P_-(\Phi) := \Phi_{|\mathbb{H}^*(K^\cdot,d)^{N=0}}$$

Obtain:

$$\exp\left(-\frac{d}{dz}\zeta_{a,\frac{P_-(\Phi)}{2\pi}}\left(\frac{s}{2\pi},z\right)|_{z=0}\right)^{-1}$$

$$= \frac{L_{\mathbb{C}}(H^1_B,s)}{L_{\mathbb{C}}(H^0_B,s)L_{\mathbb{C}}(H^2_B,s)}$$

Model of the dual graph Geodesics in $\mathfrak{X}_\Gamma$:

• Closed geodesics: $\forall\, \gamma \in \Gamma$, $\exists\, \{z^\pm(\gamma)\} \in \mathbb{P}^1(\mathbb{C})$ fixed points; geodesic in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ connecting $\{z^\pm(\gamma)\}$, for $\gamma \in \Gamma \Rightarrow$ closed geodesic in $\mathfrak{X}_\Gamma$

• Bounded geodesics: images in $\mathfrak{X}_\Gamma$ of geodesics in $\mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ having both ends on Λ_Γ

Coding of geodesics: Generators $\{g_i\}_{i=1}^g$ of Γ $(g_{i+g} = g_i^{-1})$

Subshift of finite type: $(\mathcal{S}, T)$

$$\mathcal{S} = \{\ldots a_{-m} \ldots a_{-1} a_0 a_1 \ldots a_\ell \ldots \mid$$

$$a_i \in \{g_i\}_{i=1}^{2g},\ a_{i+1} \neq a_i, \forall i \in \mathbb{Z}\}.$$

$$T(\ldots a_{-m} \ldots a_{-1} a_0 a_1 \ldots a_\ell \ldots) =$$

$$\ldots a_{-m+1} \ldots a_0 a_1 a_2 \ldots a_{\ell+1} \ldots$$

Mapping Torus (suspension flow)

$$\mathcal{S}_T := \mathcal{S} \times [0,1]/(x,0) \sim (Tx,1)$$

generalized solenoid

$H^1(\mathcal{S}_T)$ has a filtration F_n (dim $F_n = 2g(2g-1)^{n-1}(2g-2)+1$)

$H_1(\mathcal{S}_T)$ has a filtration $\mathcal{K}_n$ (dim $\mathcal{K}_n = (2g-1)^n + 1$, n even, $(2g-1)^n + (2g-1)$, n odd)

- *dynamical cohomology*: $H^1_{dyn} := \oplus_{p\leq 0}\, gr^{\Gamma}_{2p} H^1_{dyn}$

$$gr^{\Gamma}_{2p} H^1_{dyn} := (F_{-p}/F_{-p-1}) \otimes_{\mathbb{R}} \mathbb{R}(p)$$

- graded subspace $\mathcal{V} := \oplus_{p\leq 0}\, gr^{\Gamma}_{2p}\mathcal{V}$

$$gr^{\Gamma}_{2p}\mathcal{V} = \text{span}\{(2\pi\sqrt{-1})^p \chi_{-p+1,k}\}$$

$$\chi_{n,k} := [\chi_{\mathcal{S}^+(w_{n,k})}] \in (F_{n-1}/F_{n-2})$$

- *dynamical homology*: $H_1^{dyn} := \oplus_{p\geq 1}\, gr^{\Gamma}_{2p} H_1^{dyn}$

$$gr^{\Gamma}_{2p} H_1^{dyn} := \mathcal{K}_{p-1} \otimes \mathbb{R}(p)$$

- graded subspace $\mathcal{W} = \oplus_{p\geq 1}\, gr^{\Gamma}_{2p}\mathcal{W}$

$$gr^{\Gamma}_{2p}\mathcal{W} = \text{span}\{(2\pi\sqrt{-1})^p \underbrace{g_k g_k \ldots g_k}_{p-times}\}$$

- Involution $\bar{F}_\infty$ induced by change of orientation

Archimedean vs. dynamical cohomology

There are $\bar{F}_\infty$-equivariant isomorphisms U and $\tilde{U}$: the diagram commutes ($p \leq 0$)

$$\begin{array}{ccc} gr^{\mathsf{W}}_{2p} H^1(\tilde{X}^*)^{N=0} & \xrightarrow{\delta_1} & gr^{\mathsf{W}}_{2(-p+2)} H^2(X^*) \\ \downarrow U & & \downarrow \tilde{U} \\ gr^{\Gamma}_{2p} \mathcal{V} & \xrightarrow{\mathcal{D}} & gr^{\Gamma}_{2(-p+1)} \mathcal{W} \end{array}$$

$\delta_1 =$ arithmetic duality isom (between Ker and $Coker$ of monodromy N)

$\mathcal{D} =$ isom from homology/cohomology pairing on $\mathcal{S}_T$

Note: $Ker(N)$ and $Coker(N)$ of arithmetic construction exchanged with $Coker(1-T)$ and $Ker(1-T)$ of dynamical construction: intrinsic duality in U.

<u>Conclusion:</u> Geometric model for Archimedean cohomology.

A non-commutative space

A = matrix giving the admissibility condition for sequences in $\mathcal{S}$ ($A_{ij} = 1$ for $|i-j| \neq g$; $= 0$ otherwise)

Cuntz-Krieger algebra: $\mathcal{O}_A$ generated by S_i, $i = 1, \ldots, 2g$ partial isometries ($S = SS^*S$) with relations:

$$\sum_j S_j S_j^* = 1 \qquad S_i^* S_i = \sum_j A_{ij} S_j S_j^*$$

Dynamics of action of Γ on Λ_Γ

$$\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma$$

Other description as crossed product (up to stabilization): $\mathcal{O}_A \simeq \mathcal{F}_A \rtimes_T \mathbb{Z}$ with $\mathcal{F}_A$ = AF-algebra

Action: of $\mathcal{O}_A$ on $H^1(\mathcal{S}_T) \Rightarrow$ induced action on H^1_{dyn}

Morita equivalent NC-spaces

• Action of $\mathcal{O}_A \cong C(\Lambda_\Gamma) \rtimes \Gamma$ on H^1_{dyn} extends to action of the algebra

$$(C(\Lambda_\Gamma) \otimes C_0(\mathbb{H}^3)) \rtimes \Gamma$$

• On the homology H_1^{dyn} action of $C(\Lambda_\Gamma)$ that extends to action of

$$C_0(\mathfrak{X}_\Gamma, \mathcal{E})$$

sections of bundle $\mathcal{E} = (C(\Lambda_\Gamma) \times \mathbb{H}^3)/\Gamma \to \mathfrak{X}_\Gamma$

Morita equivalence:

$$(C(\Lambda_\Gamma) \otimes C_0(\mathbb{H}^3)) \rtimes \Gamma \simeq C_0(\mathfrak{X}_\Gamma, \mathcal{E})$$

same non-commutative space

Dynamical spectral triple

Dynamical homology and cohomology fit in a spectral triple $(\mathcal{A}, \mathcal{H}, \tilde{D})$:

- $\mathcal{A} = (\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma$
  ("Reduction mod ∞")

- $\mathcal{H} = \mathcal{H}^1_{dyn} \oplus \mathcal{M} \otimes_{\mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})} \mathcal{H}^{dyn}_1$

- $\mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$ sections of $\mathcal{E} = (\mathrm{C}(\Lambda_\Gamma) \times \mathbb{H}^3)/\Gamma \to \mathfrak{X}_\Gamma$

- $\mathcal{M}$ bimodule implementing the Morita equivalence $(\mathrm{C}(\Lambda_\Gamma) \otimes \mathrm{C}_0(\mathbb{H}^3)) \rtimes \Gamma \simeq \mathrm{C}_0(\mathfrak{X}_\Gamma, \mathcal{E})$

- D multiplication by the weight
$$D|_{gr^\Gamma_{2p} H^1_{dyn}} = p \cdot \qquad D|_{gr^\Gamma_{2p} H^{dyn}_1} = p \cdot$$

- $\tilde{D}|_{\mathcal{H}^1_{dyn}} = D$ and $\tilde{D}|_{\mathcal{M} \otimes \mathcal{H}^{dyn}_1} = 1 \otimes D$

Archimedean factors from dynamics

Consider the zeta function

$$\zeta_{\pi_{\mathcal{V},\bar{F}_\infty=id},D}(s,z) := \sum_{\lambda\in \mathrm{Spec}(D)} \mathrm{Tr}\left(\pi_{\mathcal{V},\bar{F}_\infty=id}\Pi(\lambda,D)\right)(s-\lambda)^{-z}$$

$\pi_{\mathcal{V},\bar{F}_\infty=id}$ = orthogonal projection on $+1$ eigenspace of $\bar{F}_\infty$ in $\mathcal{V}$

The regularized determinant

$$\exp\left(-\frac{d}{dz}\zeta_{\pi_{\mathcal{V},\bar{F}_\infty=id},\frac{D}{2\pi}}\left(\frac{s}{2\pi},z\right)|_{z=0}\right)^{-1}$$

$$= L_{\mathbb{R}}(H^1(X_{/\mathbb{R}},\mathbb{R}),s)$$

algebra role:

$$\Pi_N \pi_{\mathcal{V},\bar{F}_\infty=id}\Pi_N = \sum_{i=1}^{g} S_i^N S_i^{*N}$$

Non-archimedean places

Mumford curves K = finite extension of $\mathbb{Q}_p$; Δ_K Bruhat-Tits building $\partial\Delta_K = \mathbb{P}^1(K)$; $\Gamma \subset PGL(2,K)$ p-adic Schottky group

$X_\Gamma = \Omega_\Gamma/\Gamma$ Schottky–Mumford curve

Δ'_Γ/Γ dual graph of closed fiber (Δ'_Γ = tree of free group Γ extended to subtree of Δ_K; closed fiber of min model over $O_K \subset K$ ring of integers: totally split degenerate)

For any Γ-invariant $\Delta \subset \Delta_K$: $\mathcal{W}(\Delta/\Gamma)$ = doubly infinite walks

subshift of finite type and mapping torus $\mathcal{W}(\Delta/\Gamma)_T$ $\Rightarrow$ dynamical cohomology $H^1_{dyn}(\Delta/\Gamma)$

Spectral triple

Algebra: C*-algebra of graph C$^*(\Delta/\Gamma)$ (generalization of Cuntz-Krieger algebras)

$E =$ graph, C$^*(E)$ generated by $\{P_v\}_{v\in E^0}$ orthogonal projections and $\{S_w\}_{w\in E^1_+}$ partial isometries with

$$S_w^* S_w = P_{r(w)} \quad P_v = \sum_{w:s(w)=v} S_w S_w^*$$

Action of C$^*(\Delta/\Gamma)$ on dynamical cohomology $H^1_{dyn}(\Delta/\Gamma)$ (as in archimedean case)

$$(\mathrm{C}^*(\Delta/\Gamma), H^1_{dyn}(\Delta/\Gamma) \oplus H^1_{dyn}(\Delta/\Gamma), D)$$

spectral triple ($D =$ multipl by weight and $\frac{2\pi}{\log q}$)

Local factor($\lambda =$ eigenvalues of Frobenius)

$$L(H^1(X_\Gamma), s) = \prod_\lambda (1 - \lambda q^{-s})^{-\dim H^1(X)^\lambda}$$

Mumford curves: $L(H^1(X), s) = (1 - q^{-s})^{-g}$

From zetas of spectral triple:

$$\det_{\infty,a,iD}(s) = \exp\left(-\zeta'_{a,iD,+}(s,0)\right) \exp\left(-\zeta'_{a,iD,-}(s,0)\right)$$

$$\zeta_{a,iD,-}(s,z) := \sum_{\lambda \in \mathrm{Spec}(iD) \cap i(-\infty,0)} Tr(a\Pi_\lambda)(s+\lambda)^{-z}$$

$$\zeta_{a,iD,+}(s,z) := \sum_{\lambda \in \mathrm{Spec}(iD) \cap i[0,\infty)} Tr(a\Pi_\lambda)(s+\lambda)^{-z}$$

$$\det_{\infty,\pi(\mathcal{V}),iD}(s) = L(H^1(X_\Gamma), s)^{-1}$$

$$\Pi_N \pi(\mathcal{V}) \Pi_N = \sum_i S^N_{w_i} S^{*\,N}_{w_i}$$

$w_i =$ words repres. generators of Γ

Generalizations for non-split degenerate cases