How Noncommutative Geometry looks at Arithmetic

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Based on:

- Alain Connes, Caterina Consani, Matilde Marcolli, *Noncommutative geometry and motives: the thermodynamics of endomotives*, Advances in Mathematics, Vol.214 (2007) N.2, 761-831.

- Caterina Consani, Matilde Marcolli, *Quantum statistical mechanics over function fields* Journal of Number Theory 123 (2007) 487-528.

- Alain Connes, Matilde Marcolli, Niranjan Ramachandran, *KMS states and complex multiplication*, Selecta Mathematica, Vol.11 (2005) N.3-4, 325–347.

- Alain Connes, Matilde Marcolli, *Quantum statistical mechanics of $\mathbb{Q}$-lattices* in "Frontiers in number theory, physics, and geometry. I", 269–347, Springer, Berlin, 2006.

- Alain Connes, Caterina Consani, Matilde Marcolli, *The Weil proof and the geometry of the adeles class space*, to appear in Manin's Festschrift.

Some Goals:

Moduli spaces in arithmetic geometry:

Enrich the boundary structure with "invisible" degenerations:

• Modular curves
(elliptic curves and noncommutative tori)
• Shimura varieties
• Moduli spaces of Drinfeld modules, *etc.*

L-functions:

• Modular forms and modular symbols extensions on the boundary
• Spectral realization of L-functions and cohomological interpretation
• L-functions of motives
(geometry of archimedean factors)
• L-functions for real quadratic fields
(Stark's numbers)

Noncommutative geometry as a tool

Equivalence relation $\mathcal{R}$ on X:
quotient $Y = X/\mathcal{R}$.

Even for very good $X \Rightarrow X/\mathcal{R}$ pathological!

Classical: functions on the quotient
$\mathcal{A}(Y) := \{f \in \mathcal{A}(X) \,|\, f \text{ is } \mathcal{R}-\text{invariant}\}$

$\Rightarrow$ often too few functions
$\mathcal{A}(Y) = \mathbb{C}$ only constants

NCG: $\mathcal{A}(Y)$ noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation
(compact support or rapid decay)

Convolution product

$$(f_1 * f_2)(x,y) = \sum_{x \sim u \sim y} f_1(x,u) f_2(u,y)$$

involution $f^*(x,y) = \overline{f(y,x)}$.

$\mathcal{A}(\Gamma_{\mathcal{R}})$ noncommutative algebra $\Rightarrow Y = X/\mathcal{R}$ *noncommutative space*

Recall: $C_0(X) \Leftrightarrow X$ Gelfand–Naimark equiv of categories abelian C^*-algebras, loc comp Hausdorff spaces

Result of NCG:

$Y = X/\mathcal{R}$ *noncommutative space* with NC algebra of functions $\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$ is

- as good as X to do geometry
  (deRham forms, cohomology, vector bundles, connections, curvatures, integration, points and subvarieties)

- but with *new* phenomena
  (time evolution, thermodynamics, quantum phenomena)

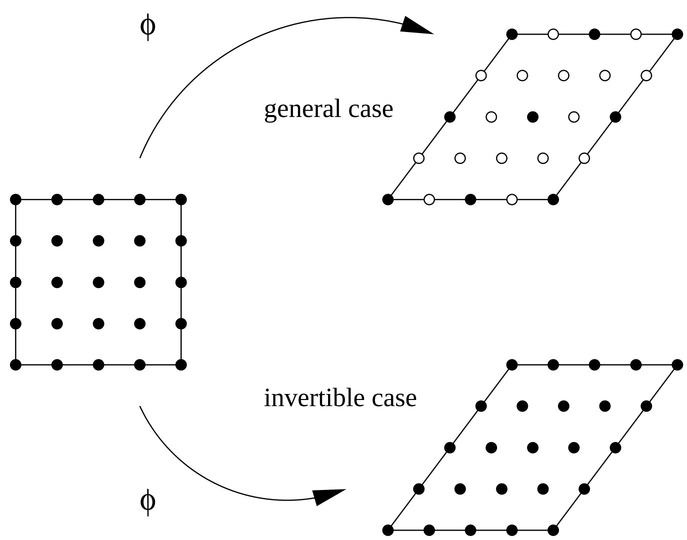
An example: $\mathbb{Q}$-lattices

(from joint work with Alain Connes)

Definition: (Λ, ϕ) $\mathbb{Q}$-lattice in $\mathbb{R}^n$
lattice $\Lambda \subset \mathbb{R}^n$ + labels of torsion points

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

group *homomorphism* (invertible $\mathbb{Q}$-lat if isom)



Commensurability $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$
iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

$\mathbb{Q}$-lattices / Commensurability $\Rightarrow$ NC space

More concretely: 1-dimension

$$(\Lambda, \phi) = (\lambda\, \mathbb{Z}, \lambda\, \rho) \quad \lambda > 0$$

$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$

Up to scaling λ: algebra $C(\hat{\mathbb{Z}})$

Commensurability Action of $\mathbb{N} = \mathbb{Z}_{>0}$

$$\alpha_n(f)(\rho) = f(n^{-1}\rho)$$

(partially defined action of $\mathbb{Q}_+^*$)

Invertible:

$$\mathbb{A}_f^*/\mathbb{Q}_+^* = \mathrm{GL}_1(\mathbb{Q})\backslash(\mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\}) = Sh(\mathrm{GL}_1, \pm 1)$$

simplest (zero-dim) example of Shimura variety

Non-invertible:

$$C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^* \quad \Leftrightarrow \quad Sh^{nc}(\mathrm{GL}_1, \pm 1)$$

"non-commutative" Shimura variety

1-dimensional $\mathbb{Q}$-lattices up to scale / Commens.

$$\Rightarrow \text{ NC space } \quad C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$$

Crossed product algebra

$$f_1 * f_2(r,\rho) = \sum_{s\in\mathbb{Q}^*_+, s\rho\in\hat{\mathbb{Z}}} f_1(rs^{-1}, s\rho) f_2(s,\rho)$$

$$f^*(r,\rho) = \overline{f(r^{-1}, r\rho)}$$

Representations: $R_\rho = \{r \in \mathbb{Q}^*_+ : r\rho \in \hat{\mathbb{Z}}\}$

$$(\pi_\rho(f)\xi)(r) = \sum_{s\in R_\rho} f(rs^{-1}, s\rho)\xi(s)$$

on $\ell^2(R_\rho)$. Completion: $\|f\| = \sup_\rho \|\pi_\rho(f)\|$

Time evolution
(ratio of covolumes of commensurable pairs)

$$(\sigma_t f)((\Lambda,\phi),(\Lambda',\phi')) = \left(\frac{\mathrm{covol}(\Lambda')}{\mathrm{covol}(\Lambda)}\right)^{it} f((\Lambda,\phi),(\Lambda',\phi'))$$

$$(\sigma_t f)(r,\rho) = r^{it} f(r,\rho)$$

Quantum statistical mechanics

$(\mathcal{A}, \sigma_t)$ C^*-algebra and time evolution

State: $\varphi : \mathcal{A} \to \mathbb{C}$ linear $\varphi(1) = 1$, $\varphi(a^* a) \geq 0$

Representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$: Hamiltonian

$$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}$$

Symmetries:

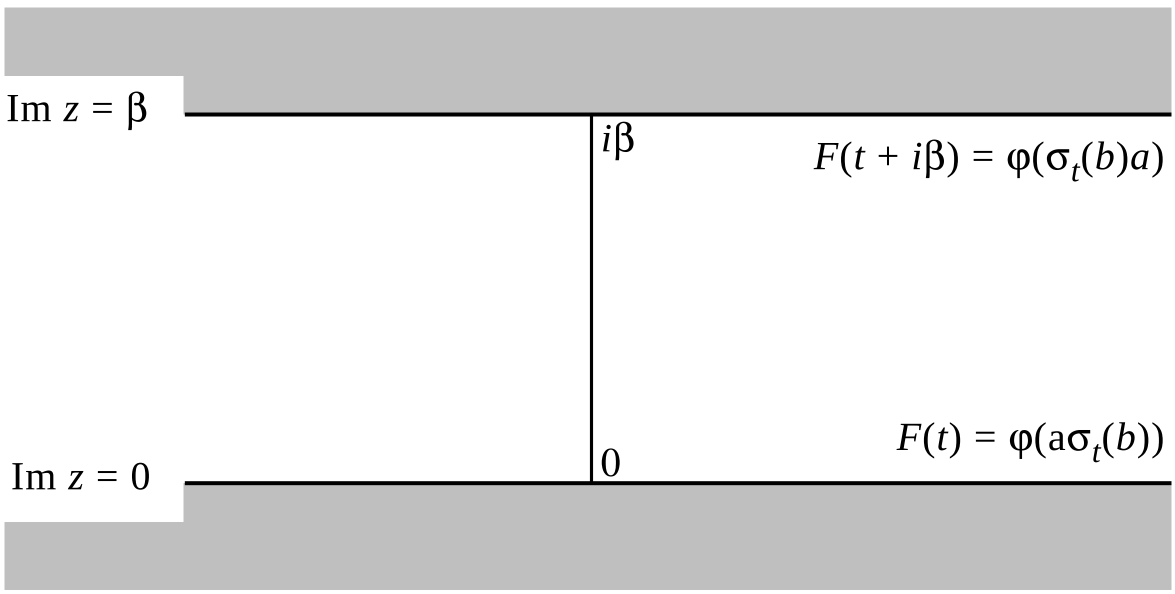
- *Automorphisms:* $G \subset \mathrm{Aut}(\mathcal{A})$, $g\sigma_t = \sigma_t g$

Inner: $u =$ unitary, $\sigma_t(u) = u$, $a \mapsto uau^*$

- *Endomorphisms:* $\rho\sigma_t = \sigma_t\rho$, $e = \rho(1)$

Inner: $u =$ isometry $\sigma_t(u) = \lambda^{it} u$

KMS states $\varphi \in \mathsf{KMS}_\beta$



$\forall a, b \in \mathcal{A}$ $\exists$ holom function $F_{a,b}(z)$ on strip:

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b)a)$$

$\forall t \in \mathbb{R}$

Example: Gibbs states inverse temperature $0 < \beta < \infty$

$$\frac{1}{Z(\beta)} \,\mathrm{Tr}\left(a\, e^{-\beta H}\right) \qquad Z(\beta) = \mathrm{Tr}\left(e^{-\beta H}\right)$$

At $T > 0$ simplex $\mathrm{KMS}_\beta \quad \rightsquigarrow$ extremal $\mathcal{E}_\beta$
At $T = 0$ (ground states) weak limits

$$\varphi_\infty(a) = \lim_{\beta\to\infty} \varphi_\beta(a)$$

Classical points of NC space

Action of symmetries (mod inner): endomorphisms

$$\rho^*(\varphi) = \frac{1}{\varphi(e)}\,\varphi \circ \rho, \quad \text{for } \varphi(e) \neq 0$$

On $\mathcal{E}_\infty$ warming up/cooling down

$$W_\beta(\varphi)(a) = \frac{\mathrm{Tr}(\pi_\varphi(a)\, e^{-\beta\, H})}{\mathrm{Tr}(\, e^{-\beta\, H})}$$

Main idea: Recover classical moduli spaces as set of low temperature KMS states.

Extra structure: high temperature regime, phase transitions, algebra of quantum mechanical observables

Bost–Connes: For $(C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}, \sigma_t)$

Classification of KMS states:

- $\beta \leq 1 \Rightarrow$ unique KMS$_\beta$ state
- $\beta > 1 \Rightarrow \mathcal{E}_\beta = Sh(\mathrm{GL}_1, \pm 1)$

$$\varphi_{\beta,\alpha}(x) = \frac{1}{\zeta(\beta)} \mathrm{Tr}\left(\pi_\alpha(x)\, e^{-\beta H}\right)$$

- $\beta = \infty$ Galois action θ : $\mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \stackrel{\cong}{\to} \hat{\mathbb{Z}}^*$

$$\gamma\, \varphi(x) = \varphi(\theta(\gamma)\, x)$$

Generalizations:

System	GL_1	GL_2	$\mathbb{K} = \mathbb{Q}(\sqrt{-d})$
$Z(\beta)$	$\zeta(\beta)$	$\zeta(\beta)\zeta(\beta-1)$	$\zeta_{\mathbb{K}}(\beta)$
Symm	$\mathbb{A}^*_{\mathbb{Q},f}/\mathbb{Q}^*$	$GL_2(\mathbb{A}_{\mathbb{Q},f})/\mathbb{Q}^*$	$\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$
Aut	$\hat{\mathbb{Z}}^*$	$GL_2(\hat{\mathbb{Z}})$	$\hat{\mathcal{O}}^*/\mathcal{O}^*$
End		$GL_2^+(\mathbb{Q})$	$Cl(\mathcal{O})$
Gal	$Gal(\mathbb{Q}^{ab}/\mathbb{Q})$	$Aut(F)$	$Gal(\mathbb{K}^{ab}/\mathbb{K})$
$\mathcal{E}_\infty$	$Sh(GL_1, \pm 1)$	$Sh(GL_2, \mathbb{H}^\pm)$	$\mathbb{A}^*_{\mathbb{K},f}/\mathbb{K}^*$

In 2-dimensions GL_2 (Connes-Marcolli)
$\mathbb{Q}(\sqrt{-d})$ (Connes-Marcolli-Ramachandran)
Shimura varieties (Ha-Paugam)
Function fields (Consani-Marcolli)

Function fields case $\mathbb{K} = \mathbb{F}_q(C)$ requires char p valued functions; time evolution $\sigma : \mathbb{Z}_p \to Aut(\mathcal{A})$ and Goss L-function

$$Z(s) = \sum_I I^{-s} \quad s \in S_\infty = \mathbb{C}^*_\infty \times \mathbb{Z}_p$$

The function field case (Consani-M.)

2-dim $\mathbb{Q}$-lattices $\Leftrightarrow$ pointed Tate module [CMR]

$$TE = H^1(E, \hat{\mathbb{Z}}), \quad \xi_1, \xi_2 \in TE.$$

Commensurability $\Leftrightarrow$ isogeny

Function fields $\mathbb{K} = \mathbb{F}_q(C)$:
Elliptic curves/lattices $\Rightarrow$ Drinfeld modules

$\mathcal{L}_{\mathbb{K},1}$ 1-dim $\mathbb{K}$-lattices mod commens. $\mathbb{A}_{\mathbb{K},f}/\mathbb{K}^*$
(choice of $v = \infty$ on curve C) $\Rightarrow$ convolution algebra

$\mathbb{K}_\infty$ = completion at $v = \infty$; $\bar{\mathbb{K}}_\infty$ = algebraic closure; $\mathbf{C}_\infty$ = completion; Goss L-function

$$Z(s) = \sum_I I^{-s} \quad s \in S_\infty = \mathbb{C}_\infty^* \times \mathbb{Z}_p$$

Quantum statistical mechanics:
p-adic time evolution $\sigma : \mathbb{Z}_p \to \mathrm{Aut}(\mathcal{A})$
$\Rightarrow$ Partition function $Z(s)$, KMS states $\mathbb{A}_{\mathbb{K}}^*/\mathbb{K}^*$

General procedure: *Endomotives*

(from joint work with Connes and Consani)

Algebraic category of endomotives:

Objects: $\mathcal{A}_{\mathbb{K}} = A \rtimes S$

$$A = \varinjlim_{\alpha} A_\alpha \quad X_\alpha = \mathrm{Spec}(A_\alpha)$$

X_α = Artin motives over $\mathbb{K}$

S = unital abelian semigroup of endomorphisms with $\rho : A \xrightarrow{\simeq} eAe$ with $e = \rho(1)$

Morphisms: étale correspondences $\mathcal{G}(X_\alpha, S) - \mathcal{G}(X'_{\alpha'}, S')$ spaces Z such that the right action of $\mathcal{G}(X'_{\alpha'}, S')$ is étale.

(i.e. $Z = \mathrm{Spec}(M)$ right $\mathcal{A}_{\mathbb{K}}$-module: M finite projective)

$\mathbb{Q}$-linear space $M((X_\alpha, S), (X'_{\alpha'}, S'))$ formal linear combinations $U = \sum_i a_i Z_i$

$$\text{Composition:} \qquad Z \circ W = Z \times_{\mathcal{G}'} W$$

fibered product over groupoid of the action of S' on X'

<u>Analytic category of endomotives:</u> $X(\bar{\mathbb{K}})/S$

<u>Objects:</u> C^*-algebras $C(\mathcal{X}) \rtimes S$

$$C^* - \text{algebra} \quad \mathcal{A} = C(X(\bar{\mathbb{K}})) \rtimes S = C^*(\mathcal{G})$$

Uniform condition: $\mu = \varprojlim \mu_\alpha$ counting on X_α

$$\frac{d\rho^*\mu}{d\mu} \quad \text{loc constant on} \quad \mathcal{X} = X(\bar{\mathbb{K}})$$

$\Rightarrow$ state φ on $\mathcal{A}$

<u>Morphisms:</u> étale correspondences $\mathcal{Z}$

$g : \mathcal{Z} \to \mathcal{X}$ discrete fiber and 1 =comp operator in $\mathcal{M}_\mathcal{Z}$
right module over $C^*(\mathcal{X})$ from $C_c(\mathcal{G})$-valued inn prod

$$\langle \xi, \eta \rangle (x, s) := \sum_{z \in g^{-1}(x)} \bar{\xi}(z) \eta(z \circ s)$$

For $\mathcal{G}$–$\mathcal{G}'$ spaces $Z \mapsto Z(\bar{\mathbb{K}}) = \mathcal{Z}$

$C_c(\mathcal{Z})$ right mod over $C_c(\mathcal{G})$

Morphisms $\Rightarrow$ in KK or cyclic category

Galois action: $G = \text{Gal}(\bar{\mathbb{K}}/\mathbb{K})$
on characters $X(\bar{\mathbb{K}}) = \text{Hom}(A, \bar{\mathbb{K}})$

$$A_n \xrightarrow{\chi} \bar{\mathbb{K}} \quad \mapsto \quad A_n \xrightarrow{\chi} \bar{\mathbb{K}} \xrightarrow{g} \bar{\mathbb{K}}$$

automorphisms of $\mathcal{A} = C(\mathcal{X}) \rtimes S$

Revisit the example: $C(\hat{\mathbb{Z}}) \cong C^*(\mathbb{Q}/\mathbb{Z})$
(Fourier transform)

$$X_n = \text{Spec}(A_n), \qquad A_n = \mathbb{Q}[\mathbb{Z}/n\mathbb{Z}]$$

$$A = \varinjlim_n A_n = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$$

$S = \mathbb{N}$ action on canonical basis e_r, $r \in \mathbb{Q}/\mathbb{Z}$

$$\rho_n(e_r) = \frac{1}{n} \sum_{ns=r} e_s$$

Galois action $\zeta_n = \chi(e_{1/n}) \Rightarrow$ cyclotomic action of
$G^{ab} = \text{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q})$

Examples from self-maps of algebraic varieties

Data: (X, S) endomotive $/\mathbb{K}$:

- C^*-algebra $\mathcal{A} = C(\mathcal{X}) \rtimes S$
- arithmetic subalgebra $\mathcal{A}_{\mathbb{K}} = A \rtimes S$
- state $\varphi : \mathcal{A} \to \mathbb{C}$ (from uniform μ)
- Galois action $G \subset \mathrm{Aut}(\mathcal{A})$

Enters Thermodynamics:

$(\mathcal{A}, \varphi) \Rightarrow \sigma_t$ with φ KMS$_1$ (Tomita–Takesaki)

GNS $\mathcal{H}_\varphi$ with cyclic separating vector ξ
$\mathcal{M}\xi$ and $\mathcal{M}'\xi$ dense in $\mathcal{H}_\varphi$ ($\mathcal{M}$ =von Neumann alg)

$$S_\varphi : \mathcal{M}\xi \to \mathcal{M}\xi \quad a\xi \mapsto S_\varphi(a\xi) = a^*\xi$$

$$S_\varphi^* : \mathcal{M}'\xi \to \mathcal{M}'\xi \quad a'\xi \mapsto S_\varphi^*(a'\xi) = a'^*\xi$$

closable $\Rightarrow$ polar decomposition $S_\varphi = J_\varphi \Delta_\varphi^{1/2}$

J_φ conjugate-linear involution $J_\varphi = J_\varphi^* = J_\varphi^{-1}$

$\Delta_\varphi = S_\varphi^* S_\varphi$ self-adjoint positive $J_\varphi \Delta_\varphi J_\varphi = S_\varphi S_\varphi^* = \Delta_\varphi^{-1}$

- $J_\varphi \mathcal{M} J_\varphi = \mathcal{M}'$ and $\Delta_\varphi^{it} \mathcal{M} \Delta_\varphi^{-it} = \mathcal{M}$

$$\alpha_t(a) = \Delta_\varphi^{it} a \Delta_\varphi^{-it} \qquad a \in \mathcal{M}$$

- The state φ is a KMS$_1$ state for the modular automorphism group $\sigma_t^\varphi = \alpha_{-t}$

<u>Classical points:</u> Ω_β

if σ_t^φ preserves $\mathcal{A}_{\mathbb{K}} \rtimes \mathbb{C} \Rightarrow$ KMS$_1$ state on $\mathcal{A}$

$\Omega_\beta \subset \mathcal{E}_\beta$ regular extremal KMS$_\beta$ states

(low temparature: type I_∞, i.e. factor $\mathcal{M}$ type I_∞)

$\epsilon \in \Omega_\beta \Rightarrow$ irreducible representation

$$\pi_\epsilon : \mathcal{A} \to \mathcal{B}(\mathcal{H}(\epsilon))$$

$$\mathcal{H}_\epsilon = \mathcal{H}(\epsilon) \otimes \mathcal{H}' \quad \mathcal{M} = \{T \otimes 1 : T \in \mathcal{B}(\mathcal{H}(\epsilon))\}$$

Gibbs states: $\sigma_t^\varphi(\pi_\epsilon(a)) = e^{itH}\pi_\epsilon(a)e^{-itH}$

with $\mathrm{Tr}(e^{-\beta H}) < \infty$

$$\epsilon(a) = \frac{\mathrm{Tr}(\pi_\epsilon(a)e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

Notice: H not uniquely determined $H \leftrightarrow H + c$

Real line bundle $\tilde{\Omega}_\beta = \{(\varepsilon, H)\}$

$$\lambda\,(\varepsilon, H) = (\varepsilon, H + \log\lambda) \qquad \forall \lambda \in \mathbb{R}_+^*$$

$\mathbb{R}_+^* \to \tilde{\Omega}_\beta \to \Omega_\beta$ with section $\mathrm{Tr}(e^{-\beta H}) = 1$

$$\tilde{\Omega}_\beta \simeq \Omega_\beta \times \mathbb{R}_+^*$$

Injections $c_{\beta',\beta} : \Omega_\beta \to \Omega_{\beta'}$ for $\beta' > \beta$

Dual system: $(\hat{\mathcal{A}}, \theta)$ algebra: $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$

$(x \star y)(s) = \int_{\mathbb{R}} x(t)\, \sigma_t(y(s-t))\, dt, \quad x, y \in \mathcal{S}(\mathbb{R}, \mathcal{A}_{\mathbb{C}})$

$\int x(t)\, U_t\, dt \in \hat{\mathcal{A}}_{\mathbb{C}}$

Scaling action θ of $\lambda \in \mathbb{R}_+^*$ on $\hat{\mathcal{A}}$

$$\theta_\lambda(\int x(t)\, U_t\, dt) = \int \lambda^{it}\, x(t)\, U_t\, dt$$

$(\varepsilon, H) \in \tilde{\Omega}_\beta \Rightarrow$ irred reps of $\hat{\mathcal{A}}$

$$\pi_{\varepsilon,H}(\int x(t)\, U_t\, dt) = \int \pi_\varepsilon(x(t))\, e^{itH}\, dt$$

Scaling action: $\pi_{\varepsilon,H} \circ \theta_\lambda = \pi_{\lambda(\varepsilon,H)}$

Function field case: (Consani-M.):
Scaling action on dual system
= Frobenius action (up to Wick rotation)
$\Rightarrow$ "Scaling = Frobenius in characteristic zero"

Function fields: scaling and Frobenius

$\mathbb{K}_\infty^* = \mathbb{F}_{q^{d_\infty}}^* \times u_\infty^{\mathbb{Z}} \times U \quad x = \mathrm{sign}(x) u_\infty^{v_\infty(x)} \langle x \rangle$ like $z = re^{i\theta}$

$s \in S_\infty = \mathbb{C}_\infty^* \times \mathbb{Z}_p$, $\lambda \in \mathbb{K}_\infty^*$

$\lambda^s = x^{\deg(\lambda)} \langle \lambda \rangle^y \Rightarrow I^s = x^{\deg(I)} \langle I \rangle^y$

$$\text{time evolution} \quad \sigma_y(f)(L, L') = \frac{\langle I \rangle^y}{\langle J \rangle^y} f(L, L')$$

Dual system and scaling by $\lambda \in \mathbb{K}_\infty^*$

$H = G \times \mathbb{Z}_p$, $G \subset \mathbb{C}_\infty^*$, $\ell \in C(H, \mathcal{A})$

$$X = \int_H \ell(s) U_s d\mu(s)$$

scaling $\theta_\lambda(X) = \int_H \ell(s) \lambda^s U_s d\mu(s)$

$$\theta_\lambda|_G(X) = \int_H \ell(s) x^{\deg(\lambda)} U_s d\mu(s)$$

$$\theta_\lambda|_{\mathbb{Z}_p}(X) = \int_H \ell(s) \langle \lambda \rangle^y U_s d\mu(s)$$

Algebra $\hat{\mathcal{A}}$ maps to $\mathcal{A}(\mathbb{A}_{\mathbb{K}}/\mathbb{K}^*)$

injective Artin homomorphism $\Theta : K \to \mathrm{Gal}(K^{ab}/K)$ (local class field theory $K = \mathbb{K}_\infty$)

- $\theta_\lambda|_G \mapsto Fr^{\mathbb{Z}}$ Frobenius
- $\theta_\lambda|_{\mathbb{Z}_p} \mapsto \mathrm{Gal}(K^{ab}/K^{un})$ inertia

Trace class property:

$$\pi_{\varepsilon,H}(\int x(t)\, U_t\, dt) \in \mathcal{L}^1(\mathcal{H}(\varepsilon))$$

for $x \in \hat{\mathcal{A}}_\beta$ (analytic continuation to strip of KMS_β with rapid decay along boundary)

Restriction map:

$$\hat{\mathcal{A}}_\beta \xrightarrow{\pi} C(\tilde{\Omega}_\beta, \mathcal{L}^1) \xrightarrow{\mathrm{Tr}} C(\tilde{\Omega}_\beta)$$

$$\pi(x)(\varepsilon, H) = \pi_{\varepsilon,H}(x) \qquad \forall (\varepsilon, H) \in \tilde{\Omega}_\beta$$

(no obstruction hypothesis for Tr)

Morphism of cyclic modules

$$\hat{\mathcal{A}}_\beta^\natural \xrightarrow{\pi} C(\tilde{\Omega}_\beta, \mathcal{L}^1)^\natural$$

$\hat{\mathcal{A}}_\beta^\natural \xrightarrow{(\mathrm{Tr}\circ\pi)^\natural} C(\tilde{\Omega}_\beta)^\natural$ equivariant for scaling action of $\mathbb{R}_+^*$

Abelian category: can take cokernels

$$D(\mathcal{A}, \varphi) = \mathrm{Coker}(\delta)$$

Cyclic cohomology: $HC_0(D(\mathcal{A}, \varphi))$ with

- Scaling action: induced $\mathbb{R}_+^*$ representation
- If $(\mathcal{A}, \varphi)$ from an endomotive: Galois representation

Quick excursus: cyclic category and NC spaces

(Connes)

Cyclic category: $[n] \in Obj(\Lambda)$
$\delta_i : [n-1] \to [n]$, $\sigma_j : [n+1] \to [n]$

$$\delta_j \delta_i = \delta_i \delta_{j-1} \quad \text{for } i < j, \qquad \sigma_j \sigma_i = \sigma_i \sigma_{j+1}, \quad i \leq j$$

$$\sigma_j \delta_i = \begin{cases} \delta_i \sigma_{j-1} & i < j \\ 1_n & \text{if } i = j \text{ or } i = j+1 \\ \delta_{i-1} \sigma_j & i > j+1. \end{cases}$$

$\tau_n : [n] \to [n]$

$$\tau_n \delta_i = \delta_{i-1} \tau_{n-1} \quad 1 \leq i \leq n, \quad \tau_n \delta_0 = \delta_n$$

$$\tau_n \sigma_i = \sigma_{i-1} \tau_{n+1} \quad 1 \leq i \leq n, \quad \tau_n \sigma_0 = \sigma_n \tau_{n+1}^2$$

$$\tau_n^{n+1} = 1_n.$$

Category $\mathcal{C}$ cyclic objects: covariant functors $\Lambda \to \mathcal{C}$

Unital algebra $\mathcal{A}$ over a field $\mathbb{K}$: $\mathbb{K}(\Lambda)$-module
$\mathcal{A}^\natural =$ covariant functor $\Lambda \to Vect_{\mathbb{K}}$

$$[n] \Rightarrow \mathcal{A}^{\otimes^{(n+1)}} = \mathcal{A} \otimes \mathcal{A} \cdots \otimes \mathcal{A}$$

$$\delta_i \Rightarrow (a^0 \otimes \cdots \otimes a^n) \mapsto (a^0 \otimes \cdots \otimes a^i a^{i+1} \otimes \cdots \otimes a^n)$$

$$\sigma_j \Rightarrow (a^0 \otimes \cdots \otimes a^n) \mapsto (a^0 \otimes \cdots \otimes a^i \otimes 1 \otimes a^{i+1} \otimes \cdots \otimes a^n)$$

$$\tau_n \Rightarrow (a^0 \otimes \cdots \otimes a^n) \mapsto (a^n \otimes a^0 \otimes \cdots \otimes a^{n-1})$$

More morphisms:

- Morphism of algebras $\phi : \mathcal{A} \to \mathcal{B} \Rightarrow \phi^\natural : \mathcal{A}^\natural \to \mathcal{B}^\natural$
- Traces

$$\tau : \mathcal{A} \to \mathbb{K} \Rightarrow \tau^\natural : \mathcal{A}^\natural \to \mathbb{K}^\natural$$

$$\tau^\natural(x^0 \otimes \cdots \otimes x^n) = \tau(x^0 \cdots x^n)$$

- $\mathcal{A} - \mathcal{B}$ bimodules $\mathcal{E} \Rightarrow \mathcal{E}^\natural = \tau^\natural \circ \rho^\natural$

$\rho : \mathcal{A} \to \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ and $\tau : \mathrm{End}_{\mathcal{B}}(\mathcal{E}) \to \mathcal{B}$

Abelian category: $HC^n(\mathcal{A}) = \mathrm{Ext}^n(\mathcal{A}^\natural, \mathbb{K}^\natural)$

For non-unital $\mathcal{A}$: $\mathcal{A} \subset \mathcal{A}^{comp}$ essential ideal
(e.g. $\mathcal{A}^{comp} = \tilde{\mathcal{A}} =$ 1-point compactification)
Λ-module $(\mathcal{A}, \mathcal{A}^{comp})^\natural$

$$\sum a_0 \otimes \cdots \otimes a_n \qquad a_j \in \mathcal{A}^{comp}$$

at least one a_j belongs to $\mathcal{A}$
Trace class operators $\mathcal{L}^1$, algebra $\mathcal{B}$

$$(\mathcal{B} \otimes \tilde{\mathcal{L}}^1)^\natural \to \mathcal{B}^\natural$$

$$\mathrm{Tr}((x_0 \otimes t_0) \otimes \ldots \otimes (x_n \otimes t_n)) = x_0 \otimes \ldots \otimes x_n \,\mathrm{Tr}(t_0 \cdots t_n)$$

Back to our chosen example: $\mathcal{A} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$

$$\varphi(f) = \int_{\hat{\mathbb{Z}}} f(1,\rho) d\mu(\rho) \quad \Rightarrow \sigma_t(f)(r,\rho) = r^{it} f(r,\rho)$$

$\tilde{\Omega}_\beta = \hat{\mathbb{Z}}^* \times \mathbb{R}_+^* = C_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$ (for $\beta > 1$)

Dual system $\quad \hat{\mathcal{A}} = C^*(\tilde{\mathcal{G}})$

$$h(r,\rho,\lambda) = \int f_t(r,\rho) \lambda^{it} U_t dt$$

where commensurability of $\mathbb{Q}$-lattices (not up to scale): groupoid

$$\tilde{\mathcal{G}} = \{(r,\rho,\lambda) \in \mathbb{Q}_+^* \times \hat{\mathbb{Z}} \times \mathbb{R}_+^* : \, r\rho \in \hat{\mathbb{Z}}\}$$

$\mathcal{A} = C^*(\mathcal{G})$ with $\mathcal{G} = \tilde{\mathcal{G}}/\mathbb{R}_+^*$

Scaling+Galois $\Rightarrow \hat{\mathbb{Z}}^* \times \mathbb{R}_+^* = C_{\mathbb{Q}}$ action

$\chi =$ character of $\hat{\mathbb{Z}}^*$

$$p_\chi = \int_{\hat{\mathbb{Z}}^*} g\chi(g)\, dg$$

$p_\chi =$ idempotent in cat of endomotives and in $\mathrm{End}_\Lambda D(\mathcal{A},\varphi)$

$$HC_0(p_\chi \, D(\mathcal{A},\varphi))$$

Adeles class space

Morita equivalence $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N} = (C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}^*_+)_\pi$
($\pi =$ char function of $\hat{\mathbb{Z}}$)

$$\mathbb{A}_{\mathbb{Q},f}/\mathbb{Q}^*_+ \quad \text{dual system} \quad \mathbb{A}^{\cdot}_{\mathbb{Q}}/\mathbb{Q}^*$$

$\mathbb{A}^{\cdot}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f} \times \mathbb{R}^*$

Adeles class space $X_{\mathbb{Q}} := \mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*$
(added point $0 \in \mathbb{R}$)

The adeles class space and Riemann's zeta (Connes)

$$0 \to L^2_\delta(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*)_0 \to L^2_\delta(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*) \to \mathbb{C}^2 \to 0$$

$f(0) = 0$ and $\hat{f}(0) = 0$

$$0 \to L^2_\delta(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*)_0 \stackrel{\mathfrak{E}}{\to} L^2_\delta(C_{\mathbb{Q}}) \to \mathcal{H} \to 0$$

$$\mathfrak{E}(f)(g) = |g|^{1/2} \sum_{q \in \mathbb{Q}^*} f(qg), \quad \forall g \in C_{\mathbb{Q}}$$

compatible with $C_{\mathbb{Q}}$ actions

$$U(h) = \int_{C_{\mathbb{Q}}} h(g)\, U_g\, d^*g \quad h \in \mathcal{S}(C_{\mathbb{Q}})$$

(comp support) acts on $\mathcal{H}$

$$\mathcal{H} = \oplus_\chi \mathcal{H}_\chi \quad \chi \in \text{characters of } \hat{\mathbb{Z}}^*$$

$\mathcal{H}_\chi = \{\xi \in \mathcal{H} : U_g \xi = \chi(g)\xi\}$ w/ $\mathbb{R}^*_+$-action gen by D_χ

Connes' spectral realization and trace formula

$$\mathrm{Spec}(D_\chi) = \left\{ s \in i\mathbb{R} \,|\, L_\chi\left(\frac{1}{2} + s\right) = 0 \right\}$$

$L_\chi =$ L-function with Grössencharakter χ

$\chi = 1 \Rightarrow \zeta(s)$ Riemann zeta

Trace formula (semi-local): $S =$ fin many places

$$\mathrm{Tr}(R_\Lambda U(h)) = 2h(1)\log\Lambda + \sum_{v\in S} \int'_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1-u|} d^*u + o(1)$$

$R_\Lambda =$ cutoff regularization, $\int' =$ principal value

Weil's explicit formula (distributional form):

$$\widehat{h}(0) + \widehat{h}(1) - \sum_\rho \widehat{h}(\rho) = \sum_v \int'_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1-u|} d^*u$$

Geometric idea: periodic orbits of the action of $C_{\mathbb{Q}}$ on $X_{\mathbb{Q}} \smallsetminus C_{\mathbb{Q}}$

Guillemin–Sternberg distributional trace formula

Flow on manifold $F_t = \exp(tv)$

$$(U_t f)(x) = f(F_t(x)) \qquad f \in C^\infty(M)$$

$$(F_t)_* : T_x/\mathbb{R}v_x \to T_x/\mathbb{R}v_x = N_x$$

transversality: $1 - (F_t)_*$ invertible

$$\mathrm{Tr}_{distr}\left(\int h(t) U_t \, dt\right) = \sum_\gamma \int_{I_\gamma} \frac{h(u)}{|1 - (F_u)_*|} \, d^* u$$

$\gamma =$ periodic orbits and $I_\gamma =$ isotropy group

Schwartz kernel $(Tf)(x) = \int k(x,y) f(y) \, dy$

$$\mathrm{Tr}_{distr}(T) = \int k(x,x) \, dx$$

For $(Tf)(x) = f(F(x))$ kernel

$$(Tf)(x) = \int \delta(y - F(x)) f(y) dy$$

Cohomological interpretation:

(from joint work w/ Connes and Consani)

The restriction morphism $\delta = (\mathrm{Tr} \circ \pi)^\natural$ for the BC system $(C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}, \sigma_t)$:

$$\delta(f) = \sum_{n \in \mathbb{N}} f(1, n\rho, n\lambda) = \sum_{q \in \mathbb{Q}^*} \tilde{f}(q(\rho, \lambda)) = \mathfrak{E}(\tilde{f})$$

$\tilde{f}$ = ext by zero outside $\hat{\mathbb{Z}} \times \mathbb{R}^+ \subset \mathbb{A}_\mathbb{Q}$

Hilbert space $L^2_\delta(\mathbb{A}_\mathbb{Q}/\mathbb{Q}^*)_0$ replaced by Λ-module $\hat{\mathcal{A}}^\natural_{\beta,0}$

different analytic techniques (as in R.Meyer)

$\Rightarrow$ Cohomological interpretation of $\mathfrak{E}$ via scaling action θ on $HC_0(D(\mathcal{A}, \varphi))$

Action of $C_\mathbb{Q} = \mathbb{A}^*_\mathbb{Q}/\mathbb{Q}^*$ on $\mathcal{H}^1 = HC_0(D(\mathcal{A}, \varphi))$

$$\vartheta(f) = \int_{C_\mathbb{Q}} f(g) \vartheta_g \, d^*g \quad f \in S(C_\mathbb{Q})$$

$\Rightarrow$ Weil's explicit formula

$$\mathrm{Tr}(\vartheta(f)|_{\mathcal{H}^1}) = \hat{f}(0) + \hat{f}(1) - \Delta \bullet \Delta f(1) - \sum_v \int'_{(\mathbb{K}^*_v, e_{K_v})} \frac{h(u^{-1})}{|1-u|} d^*u$$

$f \in S(C_\mathbb{K})$ (strong Schwartz space)

Self inters of diagonal $\Delta \bullet \Delta = \log|a| = -\log|D|$

(D = discriminant for #-field, Euler char $\chi(C)$ for $\mathbb{F}_q(C)$)

Observation:

• Tr($R_\Lambda U(f)$): only zeros on critical line
Trace formula (global) $\Leftrightarrow$ RH

• Tr($\vartheta(f)|_{\mathcal{H}^1}$): all zeros involved
RH $\Leftrightarrow$ positivity

$$\mathrm{Tr}\left(\vartheta(f \star f^\sharp)|_{\mathcal{H}^1}\right) \geq 0 \quad \forall f \in S(C_{\mathbb{Q}})$$

where

$$(f_1 \star f_2)(g) = \int f_1(k) f_2(k^{-1}g) d^*g$$

multiplicative Haar measure d^*g and adjoint

$$f^\sharp(g) = |g|^{-1}\overline{f(g^{-1})}$$

$\Rightarrow$ Better for comparing with Weil's proof for function fields

• Explicit formula
• Positivity: (correspondences, linear equiv, RR, etc.)

Weil's proof in a nutshell

$\mathbb{K} = \mathbb{F}_q(C)$ function field, $\Sigma_{\mathbb{K}}$ = places deg $n_v = \#$ orbit of Fr on fiber $C(\bar{\mathbb{F}}_q) \to \Sigma_{\mathbb{K}}$

$$\zeta_{\mathbb{K}}(s) = \prod_{\Sigma_{\mathbb{K}}} (1 - q^{-n_v s})^{-1} = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})}$$

$P(T) = \prod (1 - \lambda_n T)$ char polynomial of Fr^* on $H^1_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell)$

$$C(\bar{\mathbb{F}}_q) \supset \mathrm{Fix}(\mathrm{Fr}^j) = \sum_k (-1)^k \mathrm{Tr}(\mathrm{Fr}^{*j} | H^k_{\mathrm{et}}(\bar{C}, \mathbb{Q}_\ell))$$

RH $\Leftrightarrow$ eigenvalues λ_n with $|\lambda_j| = q^{1/2}$

Correspondences: divisors $Z \subset C \times C$; degree, codegree, trace:

$$d(Z) = Z \bullet (P \times C) \qquad d'(Z) = Z \bullet (C \times P)$$

$$\mathrm{Tr}(Z) = d(Z) + d'(Z) - Z \bullet \Delta$$

RH $\Leftrightarrow$ Weil positivity $\mathrm{Tr}(Z \star Z') > 0$

Steps: Frobenius correspondence

• Adjust degree mod trivial correspondences $C \times P$ and $P \times C$

• Riemann–Roch: $P \mapsto Z(P)$ of deg $= g$ lin equiv to effective

• Using $d(Z \star Z') = d(Z)d'(Z) = gd'(Z) = d'(Z \star Z')$

$$\mathrm{Tr}(Z \star Z') = 2gd'(Z) + (2g-2)d'(Z) - Y \bullet \Delta$$
$$\geq (4g-2)d'(Z) - (4g-4)d'(Z) = 2d'(Z) \geq 0$$

Building a dictionary

Alg Geom/NT	**NCG**
$C(\bar{\mathbb{F}}_q)$ alg points	$\Xi_{\mathbb{K}}$ classical points
Weil explicit formula	$\mathrm{Tr}(\vartheta(f)\|_{\mathcal{H}^1})$
Frobenius correspondence	$Z(f) = \int_{C_{\mathbb{K}}} f(g)\, Z_g\, d^*g$
Trivial correspondences	$\mathcal{V} = \mathrm{Range}(\mathrm{Tr} \circ \pi)$
Adjusting the degree by trivial correspondences	Fubini step on test functions in $\mathcal{V}$
Principal divisors	???
Riemann–Roch	Index theorem

Classical points of the periodic orbits

$C_{\mathbb{K}}$ action on $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

$P = \{(x,u) \in X_{\mathbb{K}} \times C_{\mathbb{K}} \,|\, u\,x = x\}$ $\quad u \neq 1 \Rightarrow \exists v \in \Sigma_{\mathbb{K}}$:

$$X_{\mathbb{K},v} = \{x \in X_{\mathbb{K}} \,|\, x_v = 0\}$$

Isotropy $\supset$ cocompact $\mathbb{K}_v^* = \{(k_w) \,|\, k_w = 1 \; \forall w \neq v\} \subset C_{\mathbb{K}}$

$X_{\mathbb{K},v}$ NC spaces $\mathbb{A}_{\mathbb{K},v}/\mathbb{K}^*$ $(\mathbb{A}_{\mathbb{K},v} = \{a \in \mathbb{A}_{\mathbb{K}} | a_v = 0\})$

$$\mathcal{A}_{\mathbb{K},v} = C^*(\mathcal{G}_{\mathbb{K},v}), \quad \mathcal{G}_{\mathbb{K},v} = \{(k,x) \in \mathcal{G}_{\mathbb{K}} \,|\, x_v = 0\}$$

groupoid $\mathcal{G}_{\mathbb{K}} = \mathbb{K}^* \ltimes \mathbb{A}_{\mathbb{K}}$ with $C^*(\mathcal{G}_{\mathbb{K}})$ alg of $X_{\mathbb{K}} = \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

smooth subalgebra $\mathcal{S}(\mathcal{G}_{\mathbb{K},v})$

restricted groupoid for $X_{\mathbb{K},v}$:

$$\mathcal{G}^{(1)}_{\mathbb{K},v} := \mathbb{K}^* \ltimes \mathbb{A}^{(1)}_{\mathbb{K},v} = \{(g,a) \in \mathcal{G}_{\mathbb{K},v} \,|\, a, ga \in \mathbb{A}^{(1)}_{\mathbb{K},v}\}$$

$\mathbb{A}^{(1)}_{\mathbb{K},v} = \prod_w \mathbb{K}^{(1)}_w$ with $\mathbb{K}^{(1)}_w$ = interior of $\{x \in \mathbb{K}_w \,|\, |x| \leq 1\}$

Quantum Statistical Mechanics on $X_{\mathbb{K},v}$

state $\varphi(f) = \int_{\mathbb{A}^{(1)}_{\mathbb{K},v}} f(1,a)da \ \Rightarrow$ time evolution:

$$\sigma_t^v(f)(k,x) = |k|_v^{it} f(k,x)$$

additive Haar measure scales $d(ka_v) = |k|_v da_v \ \Rightarrow$ KMS$_1$

$$\Xi_{\mathbb{K}} := \bigcup_{v \in \Sigma_{\mathbb{K}}} C_{\mathbb{K}} \cdot a^{(v)}$$

with $a_w^{(v)} = 1$ for $w \neq v$ and $a_v^{(v)} = 0$

$y \in \Xi_{\mathbb{K}}$ positive energy representation of $\mathcal{A}_{\mathbb{K},v}$

Hamiltonian

$$(H_y\, \xi)(k,y) = \log |k|_v \; \xi(k,y)$$

low temperature KMS states: classical points of $X_{\mathbb{K},v}$

Function field: $\quad \Xi_{\mathbb{K}} = C(\bar{\mathbb{F}}_q)$

equiv Fr action: $N/N_v = q^{\mathbb{Z}}/q^{n_v\mathbb{Z}} = \mathbb{Z}/n_v\mathbb{Z}$

$\Rightarrow \Xi_{\mathbb{Q}} = C(\bar{\mathbb{F}}_1)$ over "field with one element"?

Correspondences:

Graph of scaling action by $g \in C_{\mathbb{K}}$

$$Z_g = \{(x, g^{-1}x)\} \subset \mathbb{A}_{\mathbb{K}}/\mathbb{K}^* \times \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$$

$Z(f) = \int_{C_{\mathbb{K}}} f(g)\, Z_g\, d^*g$ with $f \in S(C_{\mathbb{K}})$

degree and codegree

$$d(Z(f)) = \hat{f}(1) = \int f(u)|u|\, d^*u$$

with $d(Z_g) = |g|$

$$d'(Z(f)) = d(Z(\overline{f}^{\sharp})) = \int f(u)\, d^*u = \hat{f}(0)$$

Adjusting degree $d(Z(f)) = \hat{f}(1)$ adding $h \in \mathcal{V}$

$$h(u, \lambda) = \sum_{n \in \mathbb{Z}^\times} \eta(n\lambda)$$

$\lambda \in \mathbb{R}^*_+$, $u \in \hat{\mathbb{Z}}^*$, $C_{\mathbb{Q}} = \hat{\mathbb{Z}}^* \times \mathbb{R}^*_+$

Notice: can find $h \in \mathcal{V}$ with $\hat{h}(1) \neq 0$ since

$$\int_{\mathbb{R}} \sum_n \eta(n\lambda)\, d\lambda \neq \sum_n \int_{\mathbb{R}} \eta(n\lambda) d\lambda = 0$$

Fubini thm does not apply

Back to the dictionary:

Virtual correspondences	bivariant class Γ
Modulo torsion	$KK(A, B \otimes \mathrm{II}_1)$
Effective correspondences	Epimorphism of C^*-modules
Degree of correspondence	Pointwise index $d(\Gamma)$
deg $D(P) \geq g \Rightarrow \sim$ effective	$d(\Gamma) > 0 \Rightarrow \exists K, \Gamma + K$ onto
Lefschetz formula	bivariant Chern of $Z(h)$ (localization on graph $Z(h)$)