<u>Def</u>: Contragradient bimodule $\mathcal{M}^o$ of a bimodule $\mathcal{M}$ over $\mathcal{A}_{LR}$

$$\mathcal{M}^o = \{ \bar{\xi} : \xi \in \mathcal{M} \} \quad \text{with action}$$

$$a \bar{\xi} b := \overline{b^* \xi a^*} \qquad a, b \in \mathcal{A}_{LR}$$

Then consider <u>all</u> possible inequivalent irreducible odd $\mathcal{A}_{LR}$ - bimodules

$\mathcal{M}_F$ = direct sum of all these

<u>Prop</u>: $\dim_{\mathbb{C}} \mathcal{M}_F = 32$

$$\mathcal{M}_F = \mathcal{E} \oplus \mathcal{E}^o$$

$$\mathcal{E} = \mathbf{2}_L \otimes \mathbf{1}^o \oplus \mathbf{2}_R \otimes \mathbf{1}^o \oplus \mathbf{2}_L \otimes \mathbf{3}^o \oplus \mathbf{2}_R \otimes \mathbf{3}^o$$

Isomorphism (anti-linear)

$$J_F : \mathcal{M}_F \longrightarrow \mathcal{M}_F^o$$

$$J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi}) \qquad \forall \xi, \eta \in \mathcal{M}_F$$

Satisfies $\boxed{J_F^2 = 1}$ and $\xi b = J_F b^* J_F \xi \qquad \forall \xi \in \mathcal{M}_F$, $\forall b \in \mathcal{A}_{LR}$

Sign: γ_F $\mathbb{Z}/2\mathbb{Z}$-grading on $\mathcal{H}_F$

given by

$\gamma_F = c - J_F c J_F$ where $c = (0, +1, -1, 0) \in \mathcal{A}_{LF}$

chirality

Notice: γ_F and J_F satisfy relations

$J_F^2 = 1$ $\qquad J_F \gamma_F = -\gamma_F J_F$

$\Rightarrow \varepsilon = 1 \quad \varepsilon'' = -1 \quad \Rightarrow \boxed{n = 6 \bmod 8}$

This finite dim. alg. (i.e. metrically zero dimensional space)

is 6-dimensional, from the point of view of KO-dimension!

Generations input N

choose $N = 3$ have models for other choices of N

this is not deduced from previous input

but assigned as additional input

Comment: the model does not predict the number of generations but there are reasons (see later) why $N = 3$ is an especially nice choice in this type of models

$N = 3$ generations

$$\mathcal{H}_f = \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{E} \qquad \mathcal{H}_{\bar{f}} = \mathcal{E}^\circ \oplus \mathcal{E}^\circ \oplus \mathcal{E}^\circ$$

$$\mathcal{H}_F = \underbrace{\mathcal{H}_f}_{\text{matter}} \oplus \underbrace{\mathcal{H}_{\bar{f}}}_{\text{antimatter}} = \mathcal{M}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F$$

Consider the left action of $\mathcal{A}_{LR}$ on $\mathcal{H}_F$

$$\rho(a) = \underbrace{\pi(a)}_{\text{acting on } \mathcal{H}_f} \oplus \underbrace{\pi'(a)}_{\text{acting on } \mathcal{H}_{\bar{f}}}$$

(action of the algebra does not mix matter and antimatter)

π, π' "disjoint" i.e. no equivalent subrepresentations

can see directly from

$$\mathcal{E} = \mathbf{2}_L \otimes \mathbf{1}^\circ \oplus \mathbf{2}_R \otimes \mathbf{1}^\circ \oplus \mathbf{2}_L \otimes \mathbf{3}^\circ \oplus \mathbf{2}_R \otimes \mathbf{3}^\circ$$

More explicit description of $\mathcal{H}_F$ and $\mathcal{A}_{LR}$

Basis for $\mathcal{H}_F$ and physical meaning in terms of particles:

$\mathbf{2}$ repres of $\mathbb{H}$ by $q = \begin{pmatrix}\alpha & \beta \\ -\bar\beta & \bar\alpha\end{pmatrix}$

basis $|\uparrow\rangle$ and $|\downarrow\rangle$:

$\lambda \in \mathbb{C} \subset \mathbb{H}$

$q(\lambda) = \begin{pmatrix}\lambda & 0 \\ 0 & \bar\lambda\end{pmatrix}$ acts on $|\uparrow\rangle$ by

$$q(\lambda)|\uparrow\rangle = \lambda|\uparrow\rangle$$

and on $|\downarrow\rangle$ by

$$q(\lambda)|\downarrow\rangle = \bar\lambda|\downarrow\rangle$$

Use notation

$u, \bar u$ for $u \in |\uparrow\rangle \otimes \mathbf{3}^\circ \subset \mathbf{2}\otimes\mathbf{3}^\circ$

$\bar u \in \mathbf{3}\otimes|\uparrow\rangle \subset \mathbf{3}\otimes\mathbf{2}^\circ$

$u = u_i$ $i = 1,2,3$ index in $\mathbf{3}^\circ$ (color index)

$\bar u = \bar u_j$ $j = 1,2,3$

Since have two copies $\mathbf{2}_L$ and $\mathbf{2}_R$

use notation

$|\uparrow\rangle_L$ $|\uparrow\rangle_R$ and u_L u_R $\bar u_L$ $\bar u_R$

$|\downarrow\rangle_L$ $|\downarrow\rangle_R$ d_L d_R $\bar d_L$ $\bar d_R$

where similarly define

$d, \bar d$ for $d \in |\downarrow\rangle \otimes \mathbf{3}^\circ \subset \mathbf{2}\otimes\mathbf{3}^\circ$

$\bar d \in \mathbf{3}^\circ \otimes |\downarrow\rangle \subset \mathbf{3}^\circ\otimes\mathbf{2}^\circ$

Also use notation

ν for $|\uparrow\rangle \otimes \mathbb{1}^\circ \subset 2 \otimes \mathbb{1}^\circ$

$\bar{\nu}$ for $\mathbb{1} \otimes |\uparrow\rangle \subset \mathbb{1} \otimes 2^\circ$

and

e for $|\downarrow\rangle \otimes \mathbb{1}^\circ \subset 2 \otimes \mathbb{1}^\circ$

$\bar{e}$ for $\mathbb{1} \otimes |\downarrow\rangle \subset \mathbb{1} \otimes 2^\circ$

and ν_L $\bar{\nu}_L$ ν_R $\bar{\nu}_R$

e_L $\bar{e}_L$ e_R $\bar{e}_R$

$u_{L,R}$ $\bar{u}_{L,R}$ up quarks (with extra generation index $\lambda = 1, \dots, N$: $\vec{u}_{L,R}$)

$d_{L,R}$ $\bar{d}_{L,R}$ down quarks

$e_{L,R}$ $\bar{e}_{L,R}$ charged leptons (electron, muon, tau)

$\nu_{L,R}$ $\bar{\nu}_{L,R}$ neutrinos (with right handed neutrinos!)

Then action of $\mathcal{A}_{LR}$: $a = (\lambda, q_L, q_R, m)$

on $\mathcal{H}_f$:

$$a \begin{pmatrix} u_L \\ d_L \end{pmatrix} = q_L^t \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \begin{pmatrix} \alpha_L u_L - \bar{\beta}_L d_L \\ \beta_L u_L + \bar{\alpha}_L d_L \end{pmatrix}$$

$$a \begin{pmatrix} u_R \\ d_R \end{pmatrix} = q_R^t \begin{pmatrix} u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \alpha_R u_R - \bar{\beta}_R d_R \\ \beta_R u_R + \bar{\alpha}_R d_R \end{pmatrix}$$

$$a \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = q_L^t \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \qquad a \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} = q_R^t \begin{pmatrix} \nu_R \\ e_R \end{pmatrix}$$

on $\mathcal{H}_{\bar{f}}$ $\qquad a = (\lambda, q_L, q_R, m)$

$a\bar{f} = \lambda \bar{f}$ for f lepton (i.e. span of $\bar{e}_L, \bar{\nu}_L, \bar{e}_R, \bar{\nu}_R$)

$a\bar{f} = m\bar{f}$ for f quark (i.e. span of $\bar{u}_L, \bar{u}_R, \bar{d}_L, \bar{d}_R$)

$\gamma_F f_L = f_L \quad \gamma_F f_R = -f_R \quad \gamma_F \bar{f}_L = -\bar{f}_L \quad \gamma_F \bar{f}_R = \bar{f}_R$

Breaking of the Left-Right symmetry of the algebra

Introducing the Dirac operator D

Finite dimensional $(\mathcal{A}, \mathcal{H})$ so

$D^* = D$ and $[[D, a], b^\circ] = 0$

conditions

together with $DJ_F = J_F D \qquad (n = 6 \bmod 8)$

Notice that action of $\mathcal{A}_{LR}$ and γ_F and J_F ~~preserve~~ don't mix $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}}$

Need to look for D that mixes $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}}$ (otherwise have completely separate matter/antimatter worlds without interaction)

$\mathcal{A}_{LR}$ with a D mixing $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}}$ does not satisfy $[[D,a],b^\circ]=0$
(no order one condition)

Look for solutions $(\mathcal{A}, D)$ with

$\mathcal{A} \subset \mathcal{A}_{LR}$ subalgebra same $\mathcal{H}_F, J_F, \gamma_F$
and D with $[[D,a],b^\circ]=0 \quad \forall a,b \in \mathcal{A}$
mixing $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}}$

Step 1: $\mathcal{A}(T) := \{ b \in \mathcal{A}_{LR} : \pi'(b)\,T = T\,\pi(b), \; \pi'(b^*)\,T = T\,\pi(b^*) \}$
for a given linear map
$$T : \mathcal{H}_f \longrightarrow \mathcal{H}_{\bar{f}}$$

<u>Lemma</u>: $\mathcal{A} \subset \mathcal{A}_{LR}$ involutive subalgebra, unital

(1) If $\pi|_{\mathcal{A}}$ and $\pi'|_{\mathcal{A}}$ disjoint (no equiv. subrepresentations)
$\Rightarrow \nexists$ off-diagonal Dirac
only: $\begin{pmatrix} D_f & 0 \\ 0 & D_{\bar{f}} \end{pmatrix}$

(2) If $\exists\, D = \begin{pmatrix} D_f & D_{off} \\ D_{off}^* & D_{\bar{f}} \end{pmatrix}$ with $D_{off} \neq 0$ for $\mathcal{A} \subset \mathcal{A}_{LR}$
$\Rightarrow \exists$ pair e, e' min projections in commutants of $\pi(\mathcal{A}_{LR})$ and $\pi'(\mathcal{A}_{LR})$ and T s.t. $e'Te = T \neq 0$ and $\mathcal{A} \subset \mathcal{A}(T)$.

<u>Proof</u>:

1) First notice that $[D, a^\circ]$ commutes w/ all $a \in \mathcal{A}_{LR}$ (all $a \in \mathcal{A}$) by order one condition

i.e. $[D, a^\circ] \in \mathcal{A}'$ = commutant of $\mathcal{A}$

$\Rightarrow$ $[D, a]$ also in $\mathcal{A}'$ (conjugating by J)

$\Rightarrow$ $[D, a]$ cannot have an off diagonal term mixing $\mathcal{H}_f$ and $\mathcal{H}_{\bar f}$ since action of $\mathcal{A}$ diagonal and without equivalent subrepresentations and $[D, a] \in \mathcal{A}'$

$\Rightarrow$ if D has off-diag. terms D_{off} then $[D_{off}, a] = 0$ $\forall a \in \mathcal{A}$ but this again means $D_{off} = 0$

2) if π, π' not disjoint $\exists$ $T : \mathcal{H}_f \to \mathcal{H}_{\bar f}$ s.t. $\mathcal{A} \subset \mathcal{A}(T)$

if x, x' in $\pi(\mathcal{A}_{LR})'$ and $\pi'(\mathcal{A}_{LR})'$ (commutants)

then $\mathcal{A}(T) \subset \mathcal{A}(xTx')$

in fact $\pi'(b) T = T \pi(b) \Rightarrow \pi'(b) x' T x = x' T x \pi(b)$ for x, x' commuting resp. with $\pi'(b)$ and $\pi(b)$

$\exists$ partition of unity by projections

$\exists$ e, e' projections in $\pi(\mathcal{A}_{LR})'$ and $\pi'(\mathcal{A}_{LR})'$ s.t. $e' T e \neq 0$

So can assume T is of this form