dim 2 mod 8 ①

Product geometry $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$

$$\begin{cases} J^2 = -1 \\ JD = DJ \\ J\gamma = -\gamma J \end{cases}$$

Real part:

$$F = (\mathcal{A}_F, \mathcal{H}_F, D_F, J_F, \gamma_F)$$

$$(\mathcal{A}_F)_{J_F} = \mathbb{R} = \{ (\lambda, \lambda, \lambda) \mid \lambda \in \mathbb{R} \} \subset \mathcal{A}_F$$

and real part $\mathcal{A}_J = C^\infty(M, \mathbb{R})$

real part of algebra does not see the NC space

Pf: $a = (\lambda, q, m) \in \mathcal{A}_F$

$[a, J_F] = 0 \Rightarrow$ $\pi(a)$ action on $\mathcal{H}_f$

$a = J_F a J_F$ $= \pi'(a^*)$ action on $\mathcal{H}_f$

$$\pi(a) = \begin{pmatrix} \lambda & & & \\ & \bar\lambda & & \\ & & \alpha & \beta \\ & & -\bar\beta & \bar\alpha \end{pmatrix} \qquad \pi'(a^*) = \begin{cases} \bar\lambda \\ m^* \end{cases}$$

on leptons $\lambda = \bar\lambda$ and $q = \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} = \bar\lambda$

$\Rightarrow \lambda \in \mathbb{R}$ and $q = \begin{pmatrix} \lambda & \\ & \lambda \end{pmatrix}$

on quarks $m^* = \lambda\, id_3$ as well $\Rightarrow (\lambda, \lambda, \lambda)_{\lambda \in \mathbb{R}}$

then prod. geom. $a \in C^\infty(M; \mathcal{A}_F)$

$[a, J_M \otimes J_F] = 0$ $\qquad [f, J_M] = fJ_M - J_M f = 0$

$f \otimes a_F = J_M f J_M^{-1} \otimes J_F a_F J_F$ $\qquad a_F = J_F a_F J_F \quad a_F = \lambda$

$f \in C^\infty(M, \mathbb{R})$ $\qquad fJ_M - J_M f = 0$

$a_F \in \mathbb{R}$ $\qquad f = \bar f$

Bosons and inner fluctuations of Dirac $D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$

Local gauge transformations $C^\infty(M, SU(\mathcal{A}_F))$

$(\mathcal{A}, \mathcal{H}, D)$ prod. geom.

Lemma:

- U unitary operator on $\mathcal{H}$ $\quad [U, \gamma] = 0 = [U, J]$

  and $U \mathcal{A} U^* = \mathcal{A}$

  $\Rightarrow \exists\, \varphi \in \mathrm{Diff}(M)$ s.t.

  $$U f U^* = f \circ \varphi \quad \forall f \in \mathcal{A}_J = C^\infty(M, \mathbb{R})$$

- U as above w/ $\varphi = id$

  $\Rightarrow \exists\, u \in C^\infty(M, SU(\mathcal{A}_F))$

  s.t. $U\, Ad(u)^* \in$ Commutant of alg. gen. by $\mathcal{A}$ and $J \mathcal{A} J^{-1}$

Inner fluctuations of Dirac

$$D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$$

$$D_A := D + A = D + A^{(1,0)} + A^{(0,1)}$$

where $A^{(1,0)}$ fluctuation of $\partial\!\!\!/_M$

"continuous part" change along M-direction $\rightarrow$ $\partial\!\!\!/_M + A^{(1,0)}$ and $A^{(0,1)}$ fluct. of $\gamma_5 \otimes D_F$

$\gamma_5 \otimes (D_F + A^{(0,1)})$

"discrete part" change along F nc-direction

Discrete part $\Rightarrow$ Higgs $\quad q_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\bar{\beta}_i & \bar{\alpha}_i \end{pmatrix}$

$$a_i(x) = (\lambda_i, q_i, m_i) \qquad a_i'(x) = (\lambda_i', q_i', m_i')$$

Set:
$$\begin{cases} \varphi_1 = \sum \lambda_i (\alpha_i' - \lambda_i') \\ \varphi_2 = \sum \lambda_i \beta_i' \\ \varphi_1' = \sum \alpha_i (\lambda_i' - \alpha_i') + \beta_i \bar{\beta}_i' \\ \varphi_2' = \sum -\alpha_i \beta_i' + \beta_i (\bar{\lambda}_i' - \bar{\alpha}_i') \end{cases}$$

Get Variation $\displaystyle \sum_i a_i [\gamma_5 \otimes D_F, a_i'](x) \Big|_{\mathcal{H}_f} = \gamma_5 \otimes \left(A_q^{(0,1)} + A_\ell^{(0,1)} \right)$

Where $A_q^{(0,1)}$ acts on quark sector of $\mathcal{H}_f$ by

$$A_q^{(0,1)} = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix} \otimes id_3 \quad \text{with}$$

$$X = \begin{pmatrix} Y_u^* \varphi_1 & Y_u^* \varphi_2 \\ -Y_d^* \bar{\varphi}_2 & Y_d^* \varphi_1 \end{pmatrix} \qquad X' = \begin{pmatrix} Y_u \varphi_1' & Y_d \varphi_2' \\ -Y_u \bar{\varphi}_2' & Y_d \bar{\varphi}_1' \end{pmatrix}$$

Similarly $A_\ell^{(0,1)}$

$$A_\ell^{(0,1)} = \begin{pmatrix} 0 & Y \\ Y' & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} Y_\nu^* \varphi_1 & Y_\nu^* \varphi_2 \\ -Y_e^* \bar{\varphi}_2 & Y_e^* \bar{\varphi}_1 \end{pmatrix}$$

with $\quad Y' = \begin{pmatrix} Y_\nu \varphi_1' & Y_e \varphi_2' \\ -Y_\nu \bar{\varphi}_2' & Y_e \bar{\varphi}_1' \end{pmatrix}$

$\Rightarrow$ inner fluctuations $A^{(0,1)}$

in terms of a quaternion valued function

$$H \in C^\infty(M, \mathbb{H}) \qquad H = \varphi_1 + \varphi_2 j \qquad \varphi_1, \varphi_2 \in C^\infty(M, \mathbb{C})$$

(Notice that X matrix above
second line like first w/
Y_u instead of Y_d and
jH instead of H)

$$\left\{ \sum_i a_i [D_F, a_i'] \right\} = \Omega_D^{(0,1)} = \text{Span}\{(q(x), q'(x))\}$$

pairs of quaternion valued functions

$$\varphi_1 + \varphi_2 j \,,\quad \varphi_1' + \varphi_2' j$$

Self adjoint

$$A^* = A \quad \Rightarrow \quad q' = q^*$$

$\Rightarrow$ only one quaternion function
$\varphi_1 + \varphi_2 j$

$$\left(\begin{array}{c} (\varphi_1, \varphi_2) \mapsto (-\bar{\varphi}_2, \bar{\varphi}_1) \\ H \mapsto jH \end{array} \right)$$

change the hypercharge of the Higgs doublet
to its opposite to couple it to the up quark
(Glashow-Weinberg-Salam model
extension from leptons to quarks)

Continuous part of inner fluctuation:
gauge bosons

$$A^{(1,0)} = \sum_i a_i [\not{\partial}_M \otimes 1, a_i']$$

$a_i = (\lambda_i, q_i, m_i)$

$a_i' = (\lambda_i', q_i', m_i')$

- A $U(1)$ gauge field

$$\Lambda = \sum_i \lambda_i d\lambda_i'$$

- An $SU(2)$ gauge field

$$Q = \sum_i q_i dq_i'$$

- A $U(3)$ gauge field

$$V' = \sum m_i dm_i'$$

Pf:

$$\sum_i \lambda_i [\not{\partial}_M \otimes 1, \lambda_i']$$

Notice λ can act by λ or by $\bar{\lambda}$
So will have two actions of these

$$\Lambda = \sqrt{-1} \sum_i \lambda_i \partial_\mu \lambda_i' \gamma^\mu = \Lambda_\mu \gamma^\mu$$

$$\Lambda_\mu = \sqrt{-1} \sum_i \lambda_i \partial_\mu \lambda_i'$$ real valued from $A^* = A$ condition

So in case of $\bar{\lambda}$ action

$$\sum \bar{\lambda}_i [\not{\partial}_M \otimes 1, \bar{\lambda}_i'] = \sqrt{-1} \sum \bar{\lambda}_i \partial_\mu \bar{\lambda}_i' \gamma^\mu = -\Lambda_\mu \gamma^\mu$$

both actions <u>same</u> gauge potential Λ

$q = f_0 + \sum_\alpha i f_\alpha \sigma^\alpha$ on basis of quaternions

w/ $f_0, f_\alpha \in C^\infty(M, \mathbb{R})$

$f_0 [\not{\partial}_M \otimes 1, i f'_\alpha \sigma^\alpha]$ self adjoint since the σ^α self-adjoint

σ^α commute w/ $\not{\partial}_M$

$$\Rightarrow \sum q_i [\not{\partial}_M \otimes 1, q'_i] = f_0 [\not{\partial}_M \otimes 1, f'_0] + \sum_\alpha f_\alpha [\not{\partial}_M \otimes 1, i f'_\alpha \sigma^\alpha]$$

Self adjoint part only:

$$Q = \sum_\alpha f_\alpha [\not{\partial}_M \otimes 1, i f'_\alpha \sigma^\alpha]$$

$Q = Q_\mu \gamma^\mu$

1-form values in Lie alg. of $SU(2)$

$\sum_i q_i \, dq'_i$

all crossed terms

$i f_\alpha \sigma^\alpha [\not{\partial}_M, i f_\beta \sigma^\beta]$ have

$-f_\alpha f_\beta (\sigma^\alpha \sigma^\beta + \sigma^\beta \sigma^\alpha) = 0$

Generally

$A = \sum_i a_i [\not{\partial}_M \otimes 1, a'_i]$ $\quad a_i, a'_i \in C^\infty(M, M_N(\mathbb{C}))$

matrix valued 1-form $a_i \, da'_i$ condition $A = A^* \Rightarrow$ in $Lie(U(N))$

From $U(3)$ to $SU(3)$ if (as in def of $SU(A_F)$) take unimodular A to satisfy $Tr(A) = 0$ ($\det(u) = 1$ for A_F)

Notation:

$$\Lambda_\mu = \frac{g_1}{2} B_\mu$$

$$Q_\mu = \frac{g_2}{2} W_\mu^\alpha \sigma^\alpha$$

$$V' = -V - \frac{1}{3} \Lambda \, id_3$$

V': $U(3)$ gauge field

V: $SU(3)$ gauge field

$A + JAJ^{-1}$ acts on quarks and leptons

(in basis $|\uparrow\rangle_R \quad |\downarrow\rangle_R \quad |\uparrow\rangle_L \quad |\downarrow\rangle_L$) as

$$\cancel{A^q_\mu = \begin{pmatrix} -\frac{2i}{3} g_1 B_\mu \otimes id_3 & 0 \\ 0 & \frac{i}{3} g_1 B_\mu \otimes id_3 \end{pmatrix}}$$

Gell-Mann matrices
basis of matrices

$$A^q_\mu = id_3 \otimes \left(\begin{array}{cc|cc} -\frac{2i}{3} g_1 B_1 & 0 & 0 & 0 \\ 0 & \frac{i}{3} B_\mu & 0 & 0 \\ \hline 0 & 0 & & \\ 0 & 0 & & \end{array}\right) + -\frac{i}{2} g_3 V^i_\mu \lambda^i \otimes id_4$$

~~quaternion~~ SU(2) gauge field $\quad -\frac{i}{2} g_2 W^\alpha_\mu \sigma^\alpha - \frac{i}{6} B_\mu \otimes id_2$

2x2 Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\lambda^i = \underset{\lambda^1}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}, \quad \underset{\lambda^2}{\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\lambda^3 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \lambda^4$$

$$\lambda^5 \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \lambda^6$$

$$\lambda^7 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \lambda^8$$

$$\mathrm{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$$

$$A^\ell_\mu = \begin{pmatrix} 0 & 0 & 0 \quad 0 \\ 0 & ig_1 B_\mu & 0 \quad 0 \\ 0 & 0 & -\frac{i}{2} g_2 W^\alpha_\mu \sigma^\alpha + \frac{i}{2} g_1 B_\mu \otimes id_2 \end{pmatrix}$$

<u>Pf</u>: from fact that action of A on $\mathcal{H}_f$

$$\begin{pmatrix} \Lambda & 0 & 0 & 0 \\ 0 & -\Lambda & 0 & 0 \\ 0 & 0 & Q_{11} & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} \end{pmatrix} \quad \text{and} \quad A + JAJ^{-1}$$

(minus sign)

then gives

$$\begin{pmatrix} \Lambda - V' & 0 & 0 & 0 \\ 0 & -\Lambda - V' & 0 & 0 \\ 0 & 0 & Q_{11} - V' & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} - V' \end{pmatrix}$$

($-V'$ from JAJ^{-1})

and

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\Lambda & 0 & 0 \\ 0 & 0 & Q_{11} - \Lambda & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} - \Lambda \end{pmatrix}$$

($-\Lambda$ from JAJ^{-1})

and then using $V' = -V - \frac{1}{3} \Lambda \otimes id_3$

The spectral action on spectral triples

Action functional that always defined for spectral triples with

- additivity property
- invariance under "changes of coordinates"

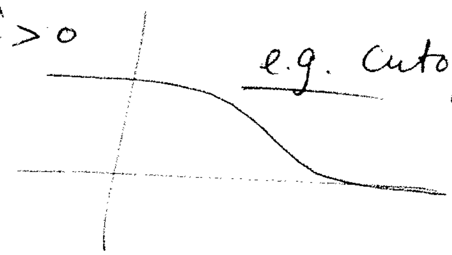
Chamseddine-Connes $\Rightarrow \exists$ functional

$$S(A, \mathcal{H}, D) = Tr\left(f\left(\frac{D}{\Lambda}\right)\right)$$

Λ = energy scale to make $\frac{D}{\Lambda}$ dimensionless

f = function of D

$f > 0$ e.g. cutoff function



Spectral action

Thm (C.C.) Asymptotic expansion for the spectral action functional

Assuming that e^{-tD^2} has

$$Tr(e^{-tD^2}) \sim \sum a_\alpha t^\alpha \quad \text{for } t \to 0$$

with no $\log(t)$ terms

(Simple dimension spectrum)

finitely summable triple:

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \mathrm{Tr}(\Delta^{-s/2}) \qquad \Delta = D^2 \text{ Laplacian}$$

($\mathrm{Ker}\, D = 0$ or replace D by $D + \varepsilon P$ $P =$ proj onto $\mathrm{Ker}\, D$)

- $a_\alpha \neq 0$ with $\alpha < 0$ in $\sum_\alpha a_\alpha t^\alpha$ corresponds to a pole of ζ_D at $s = -2\alpha$ with Residue

$$\mathrm{Res}_{s=-2\alpha}\, \zeta_D(s) = \frac{2 a_\alpha}{\Gamma(-\alpha)}$$

- $a_0 = \zeta_D(0) + \dim \mathrm{Ker}\, D$

In fact $$|D|^{-s} = \Delta^{-s/2} = \frac{1}{\Gamma(\frac{s}{2})} \int_0^\infty e^{-t\Delta}\, t^{s/2 - 1}\, dt$$

then using $\mathrm{Tr}(e^{-t\Delta}) \sim \sum_\alpha a_\alpha t^\alpha \qquad t \to 0$

and $$\int_0^1 t^{\alpha + \frac{s}{2} - 1}\, dt = (\alpha + \tfrac{s}{2})^{-1}$$

get relation between a_α and residues

from $\frac{1}{\Gamma(\frac{s}{2})} \sim \frac{s}{2}$ for $s \to 0$ get that at $s = 0$

value $\zeta_D(0)$ from pole part of $\int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^{s/2 - 1}\, dt$

i.e. $a_0 \int_0^1 t^{s/2 - 1}\, dt = a_0 \frac{2}{s}$

Thm (Chamseddine-Connes)

Asymptotic formula for the spectral action

$$\mathrm{Tr}\left(f\left(\frac{D}{\Lambda}\right)\right) \sim \sum_{\beta \in \mathrm{DimSp}} f_\beta \Lambda^\beta ⨍ |D|^{-\beta} + f(0)\,\zeta_D(0)$$

$f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, dv$ momenta of f

where $⨍ |D|^{-\beta} = \mathrm{Res}_{s=\beta} \mathrm{Tr}(|D|^{-s}) = \mathrm{Res}_{s=\beta}\, \zeta_D(s)$

In general: $⨍ P := \mathrm{Res}_{z=0} \mathrm{Tr}(P|D|^{-z})$

In the case of $(\mathcal{A}, H, D) = (C^\infty(M), L^2(M,S), \partial\!\!\!/_M)$

$M = 4$-mfld

$ds = |D|^{-1}$

- A term in Λ^4 proportional to $⨍ ds^4$
  $\rightsquigarrow$ cosmological term
- A term in Λ^2 proportional to $⨍ ds^2$
  $\rightsquigarrow$ Einstein-Hilbert action (Euclidean)

$$S_{EH}(g) = \frac{1}{16\pi G}\left(\int_M R\, dv + 2\Lambda_c \int_M dv\right)$$

- A term $\Lambda^0 \rightsquigarrow$ YM terms from inner fluctuations of metric

$\partial\!\!\!/_M + A \rightsquigarrow YM(A)$

e.g. for $C^\infty(M, M_N(\mathbb{C}))$

YM

$R = R_{\mu\nu} g^{\mu\nu}$

$R_{\mu\nu} = R^\lambda_{\mu\lambda\kappa}$

$R^\lambda_{\mu\nu\kappa} = \partial_\kappa \Gamma^\lambda_{\mu\nu} - \partial_\nu \Gamma^\lambda_{\mu\kappa} + \Gamma^\rho_{\mu\nu}\Gamma^\lambda_{\kappa\rho} - \Gamma^\rho_{\mu\kappa}\Gamma^\lambda_{\nu\rho}$

$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$

$\mathrm{Tr}(f(\frac{D}{\Lambda}))$ produces

Bosonic part of the SM Lagrangian

for $(A, H, D) = (C^{\infty}(M), L^2(M,S), \gamma_M) \otimes (A_F, H_F, D_F)$

$$S = \frac{1}{\pi^2}\left(48 f_4 \Lambda^4 - f_2 \Lambda^2 c + \frac{f_0}{4} d\right) \int \sqrt{g}\, d^4x$$

$$+ \frac{96 f_2 \Lambda^2 - f_0 c}{24\pi^2} \int R \sqrt{g}\, d^4x$$

$$+ \frac{f_0}{10\pi^2} \int \frac{11}{6} \left(R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}\right) \sqrt{g}\, d^4x$$

$$+ \frac{(-2a f_2 \Lambda^2 - f_0 c)}{\pi^2} \int |\varphi|^2 \sqrt{g}\, d^4x$$

$$+ \frac{f_0}{2\pi^2} \int a |D_\mu \varphi|^2 \sqrt{g}\, d^4x$$

$$- \frac{f_0}{12\pi^2} \int a R |\varphi|^2 \sqrt{g}\, d^4x$$

$$+ \frac{f_0}{2\pi^2} \int \left(g_3^2 G_{\mu\nu}^i \bar{G}^{\mu\nu i} + g_2^2 F_{\mu\nu}^\alpha \bar{F}^{\mu\nu\alpha} + \frac{5}{3} g_1^2 B_{\mu\nu} \bar{B}^{\mu\nu}\right) \sqrt{g}\, d^4x$$

$$+ \frac{f_0}{2\pi^2} \int b |\varphi|^4 \sqrt{g}\, d^4x$$

$a = \mathrm{Tr}(Y_\nu^* Y_\nu + Y_e^* Y_e + 3 Y_u^* Y_u + Y_d^* Y_d)$ $\quad b = \mathrm{Tr}((Y_\nu^* Y_\nu)^2 + (Y_e^* Y_e)^2 + 3(Y_u^* Y_u)^2 + 3(Y_d^* Y_d)^2)$

$c = \mathrm{Tr}(Y_R^* Y_R)$ $\quad d = \mathrm{Tr}((Y_R^* Y_R)^2)$ $\quad e = \mathrm{Tr}(Y_R^* Y_R Y_\nu^* Y_\nu)$