Adeles and ideles

Def: X_α topological spaces $U_\alpha \subset X_\alpha$ subspaces $\alpha \in I$

Restricted product of the X_α w/ respect to U_α

$$\prod_\alpha' X_\alpha \subset \prod_\alpha X_\alpha \quad \text{subspace of product}$$

given by $x = (x_\alpha)$ s.t. $x_\alpha \in U_\alpha$ for all but finitely many α

i.e. $\prod_\alpha' X_\alpha = \bigcup_{S \text{ fin. subset of } I} Y_S$ with $Y_S = \prod_{\alpha \in S} X_\alpha \times \prod_{\alpha \notin S} U_\alpha$

If X_α loc. comp. & U_α comp. (for all but fin. many α) then $\prod_\alpha' X_\alpha$ loc. comp.

$\mathbb{Q}$ = rational numbers

Valuations $\Sigma_{\mathbb{Q}}$ $\begin{cases} \text{Archimedean } |\cdot| \text{ (also noted } |\cdot|_\infty) \\ \text{non-archimedean } |\cdot|_v \quad v = p \in \mathrm{Spec}(\mathbb{Z}) \end{cases}$

a prime number

p-adic valuations

Completions of $\mathbb{Q}$ in valuations

Archimedean $\rightsquigarrow \mathbb{R}$ $(= \mathbb{Q}_\infty)$

p-adic $\rightsquigarrow \mathbb{Q}_p$

(non-archimedean)

topological (metric) spaces

adèles $\mathbb{A}_{\mathbb{Q}} = \prod'_{v \in \Sigma_{\mathbb{Q}}} \mathbb{Q}_v$ restricted product with respect to

$\Sigma_{\mathbb{Q}} = \mathrm{Spec}(\mathbb{Z}) \cup \{\infty\}$ $\qquad$ $\mathbb{Z}_p \subset \mathbb{Q}_p$

(p-adic integers)

Another description:

$$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$$ maps: $m|n$

$$\mathbb{A}_{\mathbb{Q},f} = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$$ ← finite adèles

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f} \times \mathbb{R}$$

component at ∞

$$\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p \subset \mathbb{A}_{\mathbb{Q},f}$$ maximal compact subring

idèles = invertible adèles

$$\mathbb{A}^*_{\mathbb{Q}} = GL_1(\mathbb{A}_{\mathbb{Q}})$$

$$\mathbb{A}^*_{\mathbb{Q},f} = GL_1(\mathbb{A}_{\mathbb{Q},f})$$

idèles class group:

$$C_{\mathbb{Q}} = GL_1(\mathbb{A}_{\mathbb{Q}})/GL_1(\mathbb{Q})$$

$$C_{\mathbb{Q}}/D_{\mathbb{Q}} = \mathbb{A}^*_{\mathbb{Q},f}/\mathbb{Q}^*_+ = \hat{\mathbb{Z}}^*$$

mod connected component of id.

$\mathbb{Q}$-lattices

$\mathbb{Q}$-lattice in $\mathbb{R}^n$ $\quad (\Lambda, \phi)$ $\quad \Lambda \subset \mathbb{R}^n$ lattice (cocompact subgroup $\Lambda \simeq \mathbb{Z}^n$)

$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda$ group homomorphism

invertible $\mathbb{Q}$-lattice if ϕ = isomorphism

Commensurability: $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ iff

$\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ (lattices are commensurable)

and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$

lattice because Λ_1 & Λ_2 commens.

Equivalence relation: transitivity

$L_1 = (\Lambda_1, \phi_2)$ $\quad L_2 = (\Lambda_2, \phi_2)$ $\quad L_3 = (\Lambda_3, \phi_3)$

$\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2 = \mathbb{Q}\Lambda_3$ $\quad \Rightarrow \Lambda_i$ finite index in $\Lambda_1 + \Lambda_2 + \Lambda_3$
also $\Lambda_i + \Lambda_j$

$\phi_1 - \phi_2 = 0 \mod \Lambda_1 + \Lambda_2$ $\quad \phi_2 - \phi_3 = 0 \mod \Lambda_2 + \Lambda_3$

so both also $= 0 \mod \Lambda_1 + \Lambda_2 + \Lambda_3$

$\Rightarrow \phi_1 - \phi_3 = 0 \mod \Lambda_1 + \Lambda_2 + \Lambda_3$ $\quad$ want $\mod \Lambda_1 + \Lambda_3$

$\Lambda_1 + \Lambda_3 \subset \Lambda_1 + \Lambda_2 + \Lambda_3$ fin. index $\quad \phi_1 - \phi_3 : \mathbb{Q}^n/\mathbb{Z}^n \to \dfrac{\Lambda_1 + \Lambda_2 + \Lambda_3}{\Lambda_1 + \Lambda_3}$ fin. group

but such homom. $\mathbb{Q}^n/\mathbb{Z}^n \to$ finite group trivial
(infinitely divisible)

$$\mathcal{L}_n = \left\{ \begin{array}{c} \text{~~set of~~ } \mathbb{Q}\text{-lattices in } \mathbb{R}^n \\ L=(\Lambda,\phi) \end{array} \right\} \Big/ \text{commensurability}$$

Claim: better regard $\mathcal{L}_n$ as a noncommutative space

($\mathcal{L}_n$ uncountable set but $\nexists$ countable collection of functions separating points)

Note: Commensurable $L_1 \sim L_2$: if L_1 invertible, L_2 not (at most one invertible representative; usually none)

Noncommutative spaces from equivalence relations

$$Y = X/\sim \qquad \sim \; = \text{equivalence relation}$$
$$= R \subset X \times X$$
$$R = \{(x,y) : x \sim y\}$$

Usual description of quotient:

$$C(Y) = \{ f \in C(X) \text{ s.t. } f(x) = f(y) \quad \forall x \sim y \}$$

Problem: very often $C(Y) = \mathbb{C}$ only constant functions remain

Noncommutative space description of quotients:

$$\text{"}C(Y)\text{"} = \mathcal{A}_Y = C(R)$$

functions on the graph of the equivalence relation $R \subset X \times X$

non-commutative algebra

$C_c(R)$

with product $\displaystyle (f_1 * f_2)(x,y) = \sum_{x \sim z \sim y} f_1(x,z) f_2(z,y)$

First defined on fin. support: finite sum; then completion

1-dimensional $\mathbb{Q}$-lattices :

$$\mathcal{L}_1 \xleftarrow[\ell]{\sim} \hat{\mathbb{Z}} \times \mathbb{R}_+^*$$

$$L = (\Lambda, \phi) = (\lambda^{-1}\mathbb{Z}, \lambda^{-1}\rho) = \ell(\rho, \lambda)$$

Equivalence relation of commensurability
$\Rightarrow$ groupoid

$$\mathcal{G}_1 = \left\{ (r, \rho, \lambda) : r \in \mathbb{Q}_+^*, \rho \in \hat{\mathbb{Z}}, \lambda \in \mathbb{R}_+^* \text{ such that } r\rho \in \hat{\mathbb{Z}} \right\}$$

Source and range maps

$$s(r, \rho, \lambda) = (\rho, \lambda) \qquad t(r, \rho, \lambda) = (r\rho, r\lambda)$$

Composition of arrows defined iff $\begin{cases} r_2\rho_2 = \rho_1 \\ r_2\lambda_2 = \lambda_1 \end{cases}$

$$(r_1, \rho_1, \lambda_1) \circ (r_2, \rho_2, \lambda_2) = (r_1 r_2, \rho_2, \lambda_2)$$

1-dimensional $\mathbb{Q}$-lattices up to scaling:

$$\mathcal{L}_1 / \mathbb{R}_+^* \simeq \hat{\mathbb{Z}}$$

<u>Notice</u>: After identifying $\Lambda \subset \mathbb{R}$ with $\Lambda = \lambda^{-1}\mathbb{Z}$ for some $\lambda \in \mathbb{R}_+^*$

$\phi : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}\Lambda/\Lambda$ is of the form

$$\phi = \lambda^{-1}\rho \qquad \rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}$$

Also commensurability:

$(r, \rho, \lambda) \rightsquigarrow ((r^{-1}\lambda^{-1}\mathbb{Z}, \lambda^{-1}\rho), (\lambda^{-1}\mathbb{Z}, \lambda^{-1}\rho))$ pair of commensurable $\mathbb{Q}$-lattices

Notice also $\mathcal{G}_1/\mathbb{R}_+^*$ is still a groupoid and it corresponds to commensurability relation on $\mathbb{Q}$-lattices up to scaling

$$\mathcal{U}_1 = \mathcal{G}_1/\mathbb{R}_+^* = \{ (r,\rho) \in \mathbb{Q}_+^* \times \hat{\mathbb{Z}} : r\rho \in \hat{\mathbb{Z}} \}$$

$s(r,\rho) = \rho \quad t(r,\rho) = r\rho$ and composition

$(r_1,\rho_1) \circ (r_2, \rho_2) = (r_1 r_2, \rho_2)$ when $r_2\rho_2 = \rho_1$

Algebra: groupoid algebra $C_c(\mathcal{G})$

$$f_1 * f_2 (g) = \sum_{g = g_1 \circ g_2} f_1(g_1) f_2(g_2)$$

$f^*(g) = \overline{f(g^{-1})}$ groupoid: all elements (arrows) have inverses

involution

Units of the groupoid: $\rho \in \hat{\mathbb{Z}}$ i.e. (r,ρ) with $r = 1 \in \mathbb{Q}_+^*$

Each unit $y \in \mathcal{G}^{(0)}$ $\rightsquigarrow$ representation of the algebra

$\pi_y(f)$ on vector space $\mathcal{H}_y = \ell^2(\mathcal{G}_y)$

$\mathcal{G}_y = \{ g \in \mathcal{G}$ s.t. $s(g) = y \}$

$$(\pi_y(f))(\xi)(g) = \sum_{g = g_1 g_2} f(g_1)\xi(g_2)$$

If $\mathcal{U}(\mathcal{G}) = \mathcal{G}^{(0)}$ set of units of $\mathcal{G}$ compact $\Rightarrow$ $\|f\| = \sup_{y \in \mathcal{G}^{(0)}} \|\pi_y(f)\|_{\mathcal{H}_y}$

Complete $C_c(\mathcal{G})$ to $C^*(\mathcal{G})$

Prop: $-$ $C^*(\mathcal{G}_1/\mathbb{R}_+^*) = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$

$-$ Presentation μ_n, $e(r)$ generators $n \in \mathbb{N}$ $r \in \mathbb{Q}/\mathbb{Z}$

relations:

$$\mu_n^* \mu_n = 1 \qquad \mu_k \mu_n = \mu_{kn} \qquad \forall\, k,n \in \mathbb{N}$$

$$e(0) = 1 \qquad e(r)^* = e(-r) \qquad e(r)e(s) = e(r+s)$$
$$\forall\, r,s \in \mathbb{Q}/\mathbb{Z}$$

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)$$

$-$ Morita equiv. to $C_0(\mathbb{A}_{\mathbb{Q},f}) \rtimes \mathbb{Q}_+^*$

Pf: $n \in \mathbb{N}$ $\quad n\hat{\mathbb{Z}} \subset \hat{\mathbb{Z}}$ range of multipl.
open/closed subset $\quad \pi_n = \chi_{n\hat{\mathbb{Z}}}$ characteristic function

$$= \begin{cases} 1 & x \in n\hat{\mathbb{Z}} \\ 0 & x \notin n\hat{\mathbb{Z}} \end{cases}$$

$$\pi_n \pi_m = \pi_{l.c.m(n,m)}$$ lowest common multiple

$\pi_n =$ char. function of set of $\mathbb{Q}$-lattices "divisible by n"

(Λ, ϕ) : $\phi_N : (\mathbb{Z}/N\mathbb{Z})^n \to \Lambda/N\Lambda$
N-torsion

(Λ, ϕ) divisible by N if $\phi_N = 0$

Semigroup action of $\mathbb{N}$ on $C(\hat{\mathbb{Z}})$:

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho) & \rho \in n\hat{\mathbb{Z}} \\ 0 & \rho \notin n\hat{\mathbb{Z}} \end{cases}$$

isom. of $C(\hat{\mathbb{Z}})$ w/ reduced algebra $C(\hat{\mathbb{Z}})_{\pi_n}$ (compressed w/ proj. π_n)

In terms of $\mathbb{Q}$-lattices:

$$\alpha_n(f)(\Lambda,\phi) = \begin{cases} f(n\Lambda,\phi) & (\Lambda,\phi)\in \mathrm{Support}(\pi_n) \\ 0 & \text{otherwise} \end{cases}$$

$$\mu_n(r,\rho) = \begin{cases} 0 & r\neq n \\ 1 & r=n \end{cases} \qquad \mu_n^*\mu_n = 1 \quad \forall n$$

$$\mu_k\mu_n = \mu_{kn} \qquad \pi_n = \mu_n\mu_n^*$$

Semigroup action α_n implemented by the μ_n:

$$\alpha_n(f) = \underbrace{\mu_n f \mu_n^*}_{\text{product = convol. prod in alg.}}$$

The other generators $e(r)$ from identification (Pontrjagin dual)

$$C(\hat{\mathbb{Z}}) = C^*(\mathbb{Q}/\mathbb{Z})$$

where $C^*_{(r)}(G)$ = completion of group ring $\mathbb{C}[G]$ in left reg. rep on $\ell^2(G)$

$e(r)$ $r\in\mathbb{Q}/\mathbb{Z}$ basis of $\mathbb{C}[\mathbb{Q}/\mathbb{Z}]$

then $\mu_n e(r)\mu_n^* = \alpha_n(e(r))$ gives remaining relation

char. function of $\hat{\mathbb{Z}}\subset\mathbb{A}_{\mathbb{Q},f}$

Morita equiv.: $C(\hat{\mathbb{Z}})\rtimes\mathbb{N} = \left(C_0(\mathbb{A}_{\mathbb{Q},f})\rtimes\mathbb{Q}_+^*\right)_\pi$

Time evolution on $C^*(\mathcal{G}_1/\mathbb{R}_+^*)$

$$f((\Lambda,\phi),(\Lambda',\phi'))$$ with $L=(\Lambda,\phi)\sim(\Lambda',\phi')=L'$

f homogeneous of deg. zero w/respect to the scaling action:

$$f(\lambda(\Lambda,\phi),\lambda'(\Lambda',\phi'))=f((\Lambda,\phi),(\Lambda',\phi'))$$

$$\mathrm{Covol}(\Lambda)=\mathrm{Vol}(\mathbb{R}/\Lambda)$$

$$|L/L'|:=\frac{\mathrm{Covol}(\Lambda')}{\mathrm{Covol}(\Lambda)}=\frac{\mathrm{Covol}(\lambda\Lambda')}{\mathrm{Covol}(\lambda\Lambda)}$$

$$\sigma_t(f)(L,L'):=|L/L'|^{it}f(L,L')$$

In coordinates: $f(r,\rho)$

$$\sigma_t(f)(r,\rho)=r^{it}f(r,\rho)$$

in terms of generators

$\sigma_t(f)$ for μ_n and $e(r)$: $\begin{cases}\sigma_t(e(r))=e(r)\\ \sigma_t(\mu_n)=n^{it}\mu_n\end{cases}$

in fact: if $((\Lambda,\phi),(\Lambda',\phi'))=((r^{-1}\mathbb{Z},\rho),(\mathbb{Z},\rho))$

then $\dfrac{\mathrm{Covol}(\Lambda')}{\mathrm{Covol}(\Lambda)}=\dfrac{\mathrm{Covol}(\mathbb{Z})}{\mathrm{Covol}(r^{-1}\mathbb{Z})}=r$

Symmetries:

$g \in \hat{\mathbb{Z}}^*$

$$g(f)(r,\rho) = f(r, g\rho)$$

in fact if $r\rho \in \hat{\mathbb{Z}}$ also $rg\rho \in \hat{\mathbb{Z}}$ for $g \in \hat{\mathbb{Z}}^*$

$$g\sigma_t = \sigma_t g \qquad g \in \mathrm{Aut}(\mathcal{A}, \sigma_t)$$

$$g(\Lambda, \phi) = (\Lambda, \phi \circ g)$$

$$\phi : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}\Lambda/\Lambda$$

$$\mathbb{Q}/\mathbb{Z} \xrightarrow{g} \mathbb{Q}/\mathbb{Z} \xrightarrow{\phi} \mathbb{Q}\Lambda/\Lambda$$

$g \in \hat{\mathbb{Z}}^* = \mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ since $\hat{\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$

In coordinates (generators)

$$g\mu_n = \mu_n \qquad g\, e(r) = e(g(r))$$

again thinking of $g \in \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}^*$

Arithmetic subalgebra

$\mathcal{A}_{\mathbb{Q}}$ = gen. algebraically over $\mathbb{Q}$ by the $e(r)$ and μ_n

Nicer presentation in terms of function of lattices and trigonometric analog of Eisenstein series