Thm: (Bost-Connes) classification of KMS states for the system $(\mathcal{A},\sigma)$ $\quad \mathcal{A} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ $\quad \sigma_t(\mu_n) = n^{it}\mu_n$ 1

$\sigma_t(e(r)) = e(r)$

- for all $0 < \beta \leq 1$ $\exists$ unique KMS_β state

  this satisfies

$$\varphi_\beta(e(\tfrac{a}{b})) = \frac{f_{-\beta+1}(b)}{f_1(b)}$$

  where $f_k(b) = \sum_{d|b} \mu(d)(\frac{b}{d})^k$

  μ = Möbius function

- $1 < \beta \leq \infty$ set $\mathcal{E}_\beta$ = extremal KMS_β states

  = invertible $\mathbb{Q}$-lattices (up to scale)

  $= \hat{\mathbb{Z}}^* = GL_1(\hat{\mathbb{Z}})$

  equivalently described as $GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A}_\mathbb{Q}) / \mathbb{R}_+^* = C_\mathbb{Q}/D_\mathbb{Q}$

  for $1<\beta<\infty$

$$\varphi_{\beta,\rho}(e(r)) = \frac{1}{\zeta(\beta)} \mathrm{Li}_\beta(\rho(\zeta_r))$$

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s} \qquad \zeta(\beta) = \sum_{n\geq 1} n^{-s}$$

Riemann zeta function

- $GL_1(\hat{\mathbb{Z}})$ acts on $\mathcal{E}_\beta$ transitively

- $\mathcal{E}_\infty \simeq \hat{\mathbb{Z}}^*$ $\quad \varphi_{\infty,\rho}(\mathcal{A}_\mathbb{Q}) \subset \mathbb{Q}^{ab} = \mathbb{Q}^{cycl}$ (gen. over $\mathbb{Q}$ by all roots of 1)

Intertwining property of zero-temperature states (on arithmetic elements)

$\forall\, \gamma \in \mathrm{Aut}(\mathcal{A}, \sigma_t) = \hat{\mathbb{Z}}^*$
$\exists\, \theta(\gamma) \in \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ s.t.

$$\varphi(\gamma a) = \theta(\gamma)\, \varphi(a) \quad \forall a \in \mathcal{A}_{\mathbb{Q}}$$

The group homomorphism (isomorphism)

$$\theta : \hat{\mathbb{Z}}^* \to \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$$

is known in number theory as the class field theory isomorphism

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Sketch of proof: easy part, to see that these are KMS states, extremal
hard part: to check all of them

At low temperature $1 < \beta < \infty$:

Hamiltonian for BC system

Representations: $\mathcal{H} = \ell^2(\mathbb{N})$

$\rho \in \hat{\mathbb{Z}}^* \Rightarrow$ a representation π_ρ

$$\pi_\rho(f)(\xi)(\alpha) = \sum_{\alpha = \alpha_1 \alpha_2} f(\alpha_1)\, \xi(\alpha_2)$$

$\alpha \in \ell^2(\mathcal{U}_{1,\rho})$

$s(\alpha) = \rho$

$$\begin{cases} \pi_\rho(\mu_n)\, \varepsilon_k = \varepsilon_{kn} \\ \pi_\rho(e(r))\, \varepsilon_k = \rho(\zeta_r^k)\, \varepsilon_k \end{cases}$$

Since $\sigma_t(\mu_n) = n^{it}\mu_n$ $\quad \sigma_t(e(r)) = e(r)$

Hamiltonian $e^{itH}\pi_\rho(f)\, e^{-itH} = \pi_\rho(\sigma_t(f))$

$$\Rightarrow \quad e^{itH}\,\varepsilon_k = k^{it}\,\varepsilon_k \qquad H\varepsilon_k = \log k \;\varepsilon_k$$

positive spectrum $(k \geq 1)$ multiplicities $=1$

$$\zeta(\beta) = Tr(e^{-\beta H}) = \sum_{n\geq 1} n^{-\beta} \quad = \text{Riemann } \zeta\text{-function}$$

Partition function of the QSM system

Then Gibbs-like states

$$\varphi_\beta(a) = \frac{Tr(\pi_\rho(a)\, e^{-\beta H})}{Tr(e^{-\beta H})} \overset{a = e(r)}{=} \zeta(\beta)^{-1} \sum_{k\geq 1} \rho(\zeta_r)^k / k^\beta$$

$$= \zeta(\beta)^{-1}\, Li_\beta(\rho(\zeta_r))$$

$\rho \in \hat{\mathbb{Z}}^* = Aut(\mathbb{Q}/\mathbb{Z})$ can think of them as all different possible embeddings of roots of unit in $\mathbb{C}$
(fix one and obtain all others by precomposing w/ ρ)

The zero temperature $\lim_{\beta\to\infty} \varphi_{\beta,\rho}(a) = \varphi_{\infty,\rho}(a)$
give just projection onto Kernel of H
i.e. ε_1-span in $\ell^2(\mathbb{N})$

$$\varphi_{\infty,\rho}(e(r)) = \rho(\zeta_r) \in \mathbb{Q}^{cycl}$$

Comment: $\mathbb{Q}^{ab} = \mathbb{Q}^{cycl}$ (Kronecker-Weber theorem)

$\Rightarrow$ have <u>explicit</u> <u>generators</u> for $\mathbb{Q}^{ab}$
(a priori only defined as that extension of $\mathbb{Q}$ with $Gal(\mathbb{Q}^{ab}/\mathbb{Q}) = Gal(\bar{\mathbb{Q}}/\mathbb{Q})^{ab}$ abelianization)

and <u>explicit</u> description of <u>Galois</u> <u>action</u> on them.

For fields $\mathbb{K}$ other than $\mathbb{Q}$ (e.g. other number fields)
"Explicit Class Field Theory problem" (Hilbert)

Why the QSM approach may help:

explicit presentation of field $\mathbb{Q}^{ab}$ $\longleftrightarrow$ explicit presentation of an algebra $\mathcal{A}_{\mathbb{Q}}$

Generally the second problem may be easier than first

<u>e.g. here</u>: $H = \mathbb{Q}/\mathbb{Z}$ algebra $\mathbb{Q}[H] \subset C^*(H)$

J ideal generated by idempotents

$$\pi_m = \frac{1}{m} \sum_{r \in H:\, mr = 0} u(r) \leftarrow u(r) \text{ canonical basis of } \mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$$

($u(r)(a) = \langle r, a \rangle = e(ra)$ $\forall a \in \hat{\mathbb{Z}}$ Pontrjagin duality of $\mathbb{Q}/\mathbb{Z}$ and $\hat{\mathbb{Z}}$)

- The quotient $\mathbb{Q}[H] / (\mathbb{Q}[H] \cap J)$ is a field and $\cong \mathbb{Q}^{cycl}$

- All KMS_∞ states vanish on ideal J
(KMS_β states $\varphi(\pi_n) = \varphi(\mu_n \mu_n^*) = n^{-\beta} \to 0$ as $\beta \to \infty$

2-dimensional $\mathbb{Q}$-lattices (Connes-M.)

Set of (Λ, ϕ) $\mathbb{Q}$-lattices in $\mathbb{R}^2$

$$= \Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}))$$

Up to scaling $\lambda \in \mathbb{C}^*$ $\qquad \Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times \mathbb{H})$ ← upper half plane

$GL_2^+(\mathbb{R})$ acts on $\mathbb{C}$ by $\mathbb{R}$-linear transformations when identify $\mathbb{C} \simeq \mathbb{R}^2$ w/ basis $\{1, -i\} = \{e_1, e_2\}$

$$\alpha(xe_1 + ye_2) = (ax+by)e_1 + (cx+dy)e_2$$

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$$

$M_2(\hat{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$ $\qquad \rho \in M_2(\hat{\mathbb{Z}})$

$$\rho : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}\Lambda_0/\Lambda_0 \qquad \Lambda_0 = \mathbb{Z} + i\mathbb{Z} = \mathbb{Z}e_1 + \mathbb{Z}e_2$$

$$\rho(a) = \rho_1(a)e_1 + \rho_2(a)e_2$$

(Λ, ϕ) : $\quad \Lambda = \alpha^{-1}\Lambda_0 \quad \phi = \alpha^{-1}\rho \quad$ for some $\alpha \in GL_2^+(\mathbb{R})$ some $\rho \in M_2(\hat{\mathbb{Z}})$

$(\alpha^{-1}\Lambda_0, \alpha^{-1}\rho) = (\beta^{-1}\Lambda_0, \beta^{-1}\rho')$ iff

$$\underbrace{\beta\alpha^{-1}\Lambda_0 = \Lambda_0}_{\gamma = \beta\alpha^{-1} \in SL_2(\mathbb{Z})} \qquad \underbrace{\beta\alpha^{-1}\rho = \rho'}_{\rho' = \gamma\rho}$$

so $SL_2(\mathbb{Z})$ quotient

Scaling $\lambda \in \mathbb{C}^*$

$$\lambda = a + ib \longleftrightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL_2^+(\mathbb{R})$$

$$\lambda(\Lambda, \phi) = (\lambda\Lambda, \lambda\phi)$$

$\alpha \mapsto \alpha\lambda^{-1}$ so up to scaling get

$$\Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}))/\mathbb{C}^* = \Gamma \backslash (M_2(\hat{\mathbb{Z}}) \times \mathbb{H})$$

$\mathbb{H}$ = upper half plane $\cong GL_2^+(\mathbb{R})/\mathbb{C}^*$

(while $GL_2(\mathbb{R})/\mathbb{C}^* = \mathbb{H}^+ \cup \mathbb{H}^- = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$)

identification $\mathbb{H} \simeq GL_2^+(\mathbb{R})/\mathbb{C}^*$:

$GL_2^+(\mathbb{R})$ acts on $\mathbb{H}$ by fractional linear transformations $\alpha(z) = \dfrac{az+b}{cz+d}$ $\quad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\alpha \in GL_2^+(\mathbb{R}) \longmapsto \alpha(i) = z = \frac{ai+b}{ci+d} \in \mathbb{H}$$

fixed by $\alpha \in \mathbb{C}^* \subset GL_2^+(\mathbb{R})$

$$\Rightarrow GL_2^+(\mathbb{R})/\mathbb{C}^* = \mathbb{H}$$

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}z), \lambda(\rho_1 - z\rho_2))$$

$z \in \mathbb{H}$, $\lambda \in \mathbb{C}^*$, $\rho = \rho_1 e_1 + \rho_2 e_2 \in M_2(\hat{\mathbb{Z}})$

Commensurability relation on 2-dim $\mathbb{Q}$-lattices up to scaling:

Partially defined action of $GL_2^+(\mathbb{Q})$

$$G_\rho = \{ g \in GL_2^+(\mathbb{Q}) \text{ such that } g\rho \in M_2(\hat{\mathbb{Z}}) \}$$

(Note: An equivalent more geometric description

(Λ, ϕ) 2-dim $\mathbb{Q}$-lattice up to scaling

$\Updownarrow$

E elliptic curve together with $\xi_1, \xi_2 \in TE$
two points in the "total Tate module" of E

$$TE = \varprojlim_n E[n] = \Lambda \otimes \hat{\mathbb{Z}} \qquad E = \mathbb{C}/\Lambda$$

$$TE = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, E_{tor})$$

Commensurability $\Longleftrightarrow$ isogeny & induced map on Tate modules)

Groupoids:

$$\tilde{\mathcal{G}}_2 = \{ (g, \rho, \alpha) \in GL_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}) \text{ with } g\rho \in M_2(\hat{\mathbb{Z}}) \}$$

units $\tilde{\mathcal{G}}_2^{(0)} = \{ (\rho, \alpha) \in M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R}) \}$

Source and target: $s(g, \rho, \alpha) = (\rho, \alpha)$
$t(g, \rho, \alpha) = (g\rho, g\alpha)$

Composition: $(g_1, \rho_1, \alpha_1) \circ (g_2, \rho_2, \alpha_2) = (g_1 g_2, \rho_2, \alpha_2)$ when $g_2\rho_2 = \rho_1$

Quotient by $\Gamma = SL_2(\mathbb{Z})$

$$\Gamma \backslash GL_2^+(\mathbb{Q}) \times_\Gamma M_2(\hat{\mathbb{Z}}) \times GL_2^+(\mathbb{R})$$

$$(g, \rho, \alpha) \sim (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 \alpha)$$
$$\gamma_1, \gamma_2 \in SL_2(\mathbb{Z})$$

$\Rightarrow$ groupoid $\mathcal{G}_2$ of commensurability relation (when not up to scale)

Convolution algebra:

$$(f_1 * f_2)(g, \rho, \alpha) = \sum_{h \in \Gamma \backslash GL_2^+(\mathbb{Q}) \,:\, h\rho \in M_2(\hat{\mathbb{Z}})} f_1(gh^{-1}, h\rho, h\alpha) f_2(h, \rho, \alpha)$$

where $(gh^{-1}, h\rho, h\alpha) \circ (h, \rho, \alpha) = (g, \rho, \alpha)$

When taking quotient by scaling action

$\mathcal{G}_2 / \mathbb{C}^*$ unlike case of $\mathcal{G}_1 / \mathbb{R}_+^*$

groupoid composition may no longer be well defined because there are lattices: 
with non-trivial symmetries $\rightsquigarrow$ $\mathbb{C}^*$-action not free

$((\Lambda, \phi), (\Lambda', \phi')) \circ ((\Lambda', \phi'), (\Lambda'', \phi''))$
$(\lambda(\Lambda, \phi), \lambda(\Lambda', \phi')) \circ ((\Lambda', \phi'), (\Lambda'', \phi''))$
} should be defined and equal in quotient

but not for case $\Lambda'' = \Lambda = \mathbb{Z} + 2i\mathbb{Z}$; $\Lambda' = \mathbb{Z} + i\mathbb{Z}$, $\lambda = i$

Can still define the algebra of
$\mathcal{G}_2/\mathbb{C}^*$ even if not a groupoid

Take $C^*(\mathcal{G}_2)$ and only functions that
are homogeneous of degree zero in
$\lambda \in \mathbb{C}^*$ $\quad f(\lambda(\Lambda,\phi),\lambda(\Lambda',\phi')) = f((\Lambda,\phi),(\Lambda',\phi'))$
With same convolution product

A more general approach to this problem
of convolution algebras for quotients
of groupoids by non-free actions using
topological stacks (Ha-Paugam)

$$(f_1 * f_2)(g,\rho,z) = \sum_{h\in\Gamma\backslash GL_2^+(\mathbb{Q}):\, h\rho\in M_2(\hat{\mathbb{Z}})} f_1(gh^{-1}, h\rho, h(z))\, f_2(h,\rho,z)$$

$$f^*(g,\rho,z) = \overline{f(g^{-1}, g\rho, g(z))}$$

The functions f deg zero in $\lambda\in\mathbb{C}^*$ and also
satisfy

$$f(\gamma g,\rho,z) = f(g,\rho,z) \quad \text{and}$$
$$f(g\gamma,\rho,z) = f(g,\gamma\rho,\gamma(z)) \quad \text{for } \gamma\in SL_2(\mathbb{Z})$$

Since $SL_2(\mathbb{Z})\times SL_2(\mathbb{Z})$ quotient of groupoid $\tilde{\mathcal{G}}_2$.

Representations

$$G_\rho = \{ g \in GL_2^+(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}}) \}$$

$$\mathcal{H}_\rho = \ell^2(\Gamma \backslash G_\rho)$$

unlike 1-dim case not all the same (there were all $\ell^2(\mathbb{N})$)

$$(\pi_\rho(f)(\xi))(g) = \sum_{h \in \Gamma \backslash G_\rho} f(gh^{-1}, h\rho, h(z))\, \xi(h)$$

Norm

$$\| f \| := \sup_{\rho \in M_2(\hat{\mathbb{Z}})} \| \pi_\rho(f) \|$$

Notice: $\mathcal{A}_2$ algebra non-unital

(because start from functions f_i with compact support i.e. comp. supp in $z \in \mathbb{H}$, finite supp in $g \in \Gamma \backslash GL_2^+(\mathbb{Q})$)

"unit" would be

$$f(g, \rho, z) = \begin{cases} 1 & g \in \Gamma \\ 0 & g \notin \Gamma \end{cases}$$

but not compact support in $\mathbb{H}$

Notice: $\Gamma \backslash G_\rho \longrightarrow$ commensurability class of $\mathbb{Q}$-lattice (ρ, z)

surjects to

not always isomorphism

(e.g. fails at complex multiplication points)

Time evolution

$$\sigma_t(f)(g,\rho,z) = \det(g)^{it} f(g,\rho,z)$$

$$\det(g) = \frac{\mathrm{Covol}(\Lambda')}{\mathrm{Covol}(\Lambda)}$$

same kind of time evolution as in the 1-dim case

Symmetries:

$$GL_2(\mathbb{A}_{\mathbb{Q},f}) = \prod_p{}' GL_2(\mathbb{Q}_p)$$

fact: $GL_2(\mathbb{A}_{\mathbb{Q},f}) = GL_2^+(\mathbb{Q})\, GL_2(\hat{\mathbb{Z}})$

(not unique decomposition
$GL_2^+(\mathbb{Q}) \cap GL_2(\hat{\mathbb{Z}}) = SL_2(\mathbb{Z})$)

$GL_2(\hat{\mathbb{Z}})$ acts by automorphisms

$$\gamma : (\Lambda, \phi) \longmapsto (\Lambda, \phi\circ\gamma) \qquad \gamma \in GL_2(\hat{\mathbb{Z}})$$

(like the $\hat{\mathbb{Z}}^*$ in 1-dim case)

$$(\vartheta_\gamma f)(g,\rho,z) = f(g,\rho\gamma,z) \quad \text{on functions}$$

$$GL_2(\mathbb{A}_{\mathbb{Q},f}) \xrightarrow{\det} GL_1(\mathbb{A}_{\mathbb{Q},f}) = \mathbb{Q}_+^* \times GL_1(\hat{\mathbb{Z}})$$

$\det_{\mathbb{Q}} = \mathbb{Q}_+^*$-component of det

$$\vartheta_m(f)(g,\rho,z) = \begin{cases} f(g, \rho m \det_{\mathbb{Q}}(m)^{-1}, z) & \text{if } \rho m \det_{\mathbb{Q}}(m)^{-1} \in M_2(\hat{\mathbb{Z}}) \text{ and } g\rho m \det_{\mathbb{Q}}(m)^{-1} \in M_2(\hat{\mathbb{Z}}) \\ 0 & \text{otherwise} \end{cases}$$

Action by endomorphism of semigroup $\Sigma = M_2(\hat{\mathbb{Z}}) \cap GL_2(\mathbb{A}_{\mathbb{Q},f})$

diag. sub-semigroup $\mathbb{N} \subset \Sigma$ inner endomorphisms

$\Rightarrow$ induced action on KMS states by quotient

$\mathbb{Q}^* \backslash GL_2(\mathbb{A}_{\mathbb{Q},f})$ of KMS states

Resulting symmetries Autom + Endom mod inner

In terms of lattices

$$(\Lambda, \phi) \longmapsto (\Lambda, \phi \circ \tilde{m}^{-1})$$

$$\tilde{m} = m^{-1} \det_{\mathbb{Q}}(m)$$

Well Known number theory result

$$\mathbb{Q}^* \backslash GL_2(\mathbb{A}_{\mathbb{Q},f}) \cong \mathrm{Aut}(F)$$

Galois group of the Modular field