The modular field

$\mathbb{H}$ = upper half plane $= \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

$\Gamma = SL_2(\mathbb{Z}) \qquad \Gamma(N) = \{ \gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv 1 \mod N \}$

$$(f|_k \alpha)(z) = \det(\alpha)^{k/2} f\left(\frac{az+b}{cz+d}\right)(cz+d)^{-k}$$

$\uparrow \qquad \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{R})$

"slash operation"

quotient map

$p : \mathbb{H} \to {}_{\Gamma}\backslash\mathbb{H} = X_\Gamma$

$X_\Gamma = {}_{\Gamma}\backslash\mathbb{H}$ modular curve (level $N=1$) Γ acting by fract. lin. transf. on $\mathbb{H}$

$f|\gamma = f \qquad f$ funct. on X_Γ is a modular function if

$f \circ p : \mathbb{H} \to \mathbb{C}$ holomorphic and

$|f \circ p(z)| \le C_1 e^{C_2 \mathrm{Im}(z)}$ for some $C_1, C_2 > 0$ and for $\mathrm{Im}(z) \to \infty$

q-expansion $\qquad q^n \tilde{f}(q)$ bounded near $q = 0$

with $q = e^{2\pi i z} \qquad \tilde{f}(q) = f(p(z))$

$$\Rightarrow \tilde{f}(q) = a_{-n} q^{-n} + \dots + a_0 + a_1 q + \dots + a_n q^n + \dots$$
$$= \sum_{k \ge -n} a_k q^k$$

Level N: $\quad F_N(\mathbb{C}) \qquad f|\gamma = f \quad \forall \gamma \in \Gamma(N)$

f holom. and $|(f|\gamma)(z)| \le C_1 e^{C_2 \mathrm{Im}(z)}$

$F_N \subset F_N(\mathbb{C})$ gen. over $\mathbb{Q}$ by j & f_u $\quad u \in (\tfrac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \quad u \neq 0$

Equivalently F_N gen. over $\mathbb{Q}$ by $f \in F_N(\mathbb{C})$
with $\mathbb{Q}$-expansion in powers of $q^{1/N}$
has coefficients in $\mathbb{Q}(\zeta_N)$ $\quad \zeta_N = e^{2\pi i/N}$

Modular field $\quad F = \bigcup_N F_N$

Result (Shimura) $\quad \mathrm{Aut}(F) = \mathbb{Q}^* \backslash GL_2(\mathbb{A}_{\mathbb{Q},f})$

Arithmetic "subalgebra" of the GL_2-system
in fact will be a subalg. of unbounded multipliers

$\tilde{\mathcal{U}}_2 \subset GL_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ $\qquad \mathcal{U}_2 = (\Gamma \times \Gamma) \backslash \tilde{\mathcal{U}}_2$

$(g, \rho, z) : g\rho \in M_2(\hat{\mathbb{Z}})$

$f \in C(\mathcal{U}_2)$ finite support in g
if supported on a finite subset of $\Gamma \backslash GL_2^+(\mathbb{Q}) \ni g$

$\Downarrow$

Defines an unbounded multiplier on the C^*-algebra $\mathcal{A}_2$
(acting by same convolution product that defines algebra product in $\mathcal{A}_2$)

$f \in C(\mathcal{U}_2)$ has level N iff
$f(g, \rho, z)$ only depends on $(g, p_N(\rho), z)$

$p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ projections of ρ.

Suppose f level N and finite support in $g \in_\Gamma \backslash GL_2^+(\mathbb{Q})$
then completely specified by finitely many
continuous functions $f_{g,m} \in C(\mathbb{H})$ $\quad m \in M_2(\mathbb{Z}/N\mathbb{Z})$

with $\quad f(g,\rho,z) = f_{g,\,p_N(\rho)}(z)$

invariance:

$$f(\gamma_1 g \gamma_2, \rho, z) = f(g, \gamma_2 \rho, \gamma_2(z)) \qquad \gamma_1, \gamma_2 \in \Gamma$$

$$\Rightarrow \qquad f_{g,m}|\gamma = f_{g,m} \qquad \forall\, \gamma \in \Gamma(N) \cap g^{-1}\Gamma g$$

$\mathcal{A}_{2,\mathbb{Q}}$ <u>Arithmetic algebra</u>: $\quad f \in C(\mathcal{U}_2)$ s.t.

- f has finite support in $g \in_\Gamma \backslash GL_2^+(\mathbb{Q})$
- f has finite level in $\rho \in M_2(\hat{\mathbb{Z}})$
- The functions $f_{g,m}(z) \in F$ (are holomorphic and w/ growth condition)

Plus a condition that
rules out constant functions
equal to root of 1 : Want to obtain a good "intertwining condition" for KMS$_\infty$-states

↑

Want that if one root of 1
appears in q-expansion of f then all
its Galois conjugates under $Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$
also appear (cyclotomic condition)

Convolution product of $\mathcal{A}_2$ extends to
product on $\mathcal{A}_{2,\mathbb{Q}}$ $\Rightarrow$ algebra

KMS states extend from $\mathcal{A}_2$ to $\mathcal{A}_{2,\mathbb{Q}}$

Embeddings of the modular field in $\mathbb{C}$

Can realize F as a subfield of $\mathbb{C}$
by evaluation

$f \longmapsto f(\tau)$ at a point $\tau \in \mathbb{H}$ that is <u>generic</u>

then $\{f(\tau)\}$ generate an embedded copy of $F \hookrightarrow \mathbb{C}$

when the j-function has value $j(\tau)$ that is transcendental $j(\tau) \notin \overline{\mathbb{Q}}$

$$\wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\gamma \in \Lambda \setminus \{0\}} \left(\frac{1}{(z+\gamma)^2} - \frac{1}{\gamma^2} \right)$$

$$\wp'^2 = 4\wp^3 - g_2 \wp - g_3$$

$$\Delta = g_2^3 - 27 g_3^2 \qquad \mathbb{C}/\Lambda \ni z \mapsto (\wp(z), \wp'(z)) \in E_\Lambda$$

$$y^2 = 4x^3 - g_2 x - g_3$$

$$j = 1728 \, \frac{g_2^3}{\Delta}$$

$$j = \frac{1}{q} + 744 + 196884\, q + 21493760\, q^2 + \cdots$$

j-function

<u>Notice</u>: a case of non-generic points
$\tau \in \mathbb{C}$ complex multiplication elliptic curve E_τ
$\Rightarrow f \mapsto f(\tau)$ generate $\mathbb{K}^{ab}$
where $\mathbb{K} = \mathbb{Q}(\tau)$ is the imaginary quadratic field of the CM point

KMS states of the GL_2-system

Partition function:

<u>Thm</u>: • Each invertible $(\Lambda, \phi) = L$ (i.e. $\rho \in GL_2(\hat{\mathbb{Z}})$) determines a positive energy representation π_L of $\mathcal{A}_2$

• Partition function $Z(\beta) = \zeta(\beta)\zeta(\beta-1)$

• Extremal KMS states at low temperature $\underline{\beta > 2}$ (Gibbs form)

$$\varphi_{\beta,L}(f) = Z(\beta)^{-1} \sum_{m \in \Gamma \backslash M_2^+(\mathbb{Z})} f(1, m\rho, m(z)) \det(m)^{-\beta}$$

• Extremal KMS-states for $\beta > 2$

$$\mathcal{E}_\beta \cong {}_{GL_2^+(\mathbb{Q})}\backslash GL_2(\mathbb{A}_{\mathbb{Q},f}) \times \mathbb{H}$$

$$= {}_{GL_2(\mathbb{Q})}\backslash GL_2(\mathbb{A}_{\mathbb{Q}}) / \mathbb{C}^*$$ Shimura variety of GL_2

In fact repres

π_L on $\mathcal{H}_L = \ell^2(\Gamma \backslash G_L)$ $\quad (H_L \xi)(g) = \log \det(g)\, \xi(g)$

if L = invertible then $G_L = M_2^+(\mathbb{Z})$

and $Spec(H_L) \subset [0, \infty)$

eigenspaces:

$\{m \in M_2^+(\mathbb{Z}) : \det m = n\} \bmod \Gamma$

matrices of form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $a, d \geq 1$ $0 \leq b < d$ with $\det = n$

$= \sigma_1(n) = \sum_{d \mid n} d$ $\quad Z(\beta) = Tr(e^{-\beta H}) = \sum_n \frac{\sigma_1(n)}{n^\beta} = \zeta(\beta)\zeta(\beta-1)$

At zero temperature:

$$\mathcal{E}_\infty \cong GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}_\mathbb{Q}) / \mathbb{C}^*$$

Action of symmetries on KMS_∞-states

$$\varphi_{\infty,L}(f) = f(1,\rho,\tau) \qquad L = (\Lambda,\phi) = (\rho,\tau)$$
$$\rho \in GL_2(\hat{\mathbb{Z}})$$

$$\varphi_{\infty,L}(\theta_\alpha(f)) = Gal_\tau(\rho\alpha\rho^{-1})\, \varphi_{\infty,L}(f)$$

$\alpha \in GL_2(\mathbb{A}_{\mathbb{Q},f})$

Action of Gal group of F_τ = image of modular field under specialization at τ

$\mathbb{Q}^*$ acts by inner

$\Rightarrow$ $\mathbb{Q}^* \backslash GL_2(\mathbb{A}_{\mathbb{Q},f})$ action

intertwines with action of Galois group

Higher temperatures $\qquad \beta = 2$ first phase transition

$1 < \beta < 2$ unique KMS state (Laca, Larsen, Neshveyev)

then $\beta = 1$ second phase transition

$\beta < 1$ NO KMS states

A general procedure to produce Bost–Connes type quantum statistical mechanical systems

Algebraic version:

A_α = finite dimensional commutative algebras over a field $\mathbb{K}$ (e.g. a number field)

$A = \varinjlim_\alpha A_\alpha$ $\quad X_\alpha = \mathrm{Spec}(A_\alpha)$ a zero-dimensional alg. variety over $\mathbb{K}$

$X = \varprojlim_\alpha X_\alpha$ a pro-variety

S = an abelian semigroup of endomorphisms of A
s.t. $\rho \in S$ isom $\rho: A \xrightarrow{\simeq} eA$
range of projection $e = \rho(1)$ $\quad e^2 = e$

$\mathcal{A}_{\mathbb{K}} = A \rtimes S$ $\qquad$ Presentation

Presentation of A + additional generators

$\rho \in S$: $\quad U_\rho$ and U_ρ^* with relations

$$U_\rho^* U_\rho = 1 \qquad U_\rho U_\rho^* = \rho(1)$$

$$U_{\rho_1 \rho_2} = U_{\rho_1} U_{\rho_2} \qquad U^*_{\rho_2 \rho_1} = U^*_{\rho_1} U^*_{\rho_2}$$

$$U_\rho a = \rho(a) U_\rho \qquad a U_\rho^* = U_\rho^* \rho(a)$$

As a vector space $\mathcal{A}_{\mathbb{K}} = A \rtimes S$ span of monomials $U^*_{\rho_1} a U_{\rho_2}$

Analytic version: $\mathcal{X} = X(\bar{\mathbb{K}}) = \varprojlim_\alpha X_\alpha(\bar{\mathbb{K}})$ topol. space (Cantor set)

C^*-algebra $\quad C(\mathcal{X}) \rtimes S$

$\mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{K}} \otimes_{\mathbb{K} \subset \mathbb{C}} \mathbb{C}$ $\qquad \mathcal{A}_{\mathbb{K}}$ = subalgebra of arithmetic elements

These noncommutative spaces (endomotives) form a category with morphisms given by correspondences (bimodules)

Galois action : $X(\bar{\mathbb{K}}) = \mathrm{Hom}(A, \bar{\mathbb{K}})$ as $\mathbb{K}$-algebras

$\gamma \in \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ $\quad \chi \in X(\bar{\mathbb{K}})$ action:

$A \xrightarrow{\chi} \bar{\mathbb{K}} \xrightarrow{\gamma} \bar{\mathbb{K}}$ commutes w/ $\rho \in S$ since

$A \xrightarrow{\rho} A \xrightarrow{\chi} \bar{\mathbb{K}} \xrightarrow{\gamma} \bar{\mathbb{K}}$ $\Rightarrow$ action on $\mathcal{A}_{\bar{\mathbb{K}}} = A \rtimes S$

Note: Counting measures on finite sets $X_\alpha(\bar{\mathbb{K}}) \Rightarrow$ state φ on $C(\mathcal{X}) \rtimes S$

Many examples can be constructed from self-maps of algebraic varieties

(Y, y_0) variety over $\mathbb{K}$ $\quad s \in S \quad s: Y \to Y$, $s(y_0) = y_0$

such that finite to one : i.e. well defined $\deg(s)$ and unramified at y_0

$X_s = \{ y \in Y : s(y) = y_0 \}$

$\xi_{s,s'} : X_{sr = s'} \longrightarrow X_s \qquad X_{sr} \ni y \mapsto r(y) \in X_s$

projective system $\quad X = \varprojlim_{s \in S} X_s$

$\mathcal{X} = X(\bar{\mathbb{K}}) \quad \rightsquigarrow \quad C(\mathcal{X}) \rtimes S$

e.g. Bost-Connes obtained in this way from

$Y = \mathbb{G}_m$ $\quad S: \; s_n : u \mapsto u^n$ $\quad$ endomorphisms of the multiplicative grp.

Dual system: If have $(\mathcal{A}, \sigma)$ $\quad \sigma$ = time evolution

Dual system $\quad \hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$ $\quad (\hat{\mathcal{A}}, \theta)$ $\leftarrow$ scaling action

$$X = \int_{\mathbb{R}} x(t)\, U_t\, dt \qquad X * Y = \int_{\mathbb{R}} (x*y)(t)\, U_t\, dt$$

$$(x*y)(t) = \int_{\mathbb{R}} x(t)\, \sigma_t(y(s-t))\, dt$$

$$\lambda \in \mathbb{R}_+^* \qquad \theta_\lambda(X) = \int \lambda^{it}\, x(t)\, U_t\, dt$$

Given an endomotive $\quad \underset{\text{alg. over } \mathbb{K}}{\mathcal{A}_{\mathbb{K}} \rtimes S} \qquad \underset{C^*\text{-alg.}}{C(X(\bar{\mathbb{K}})) \rtimes S}$

To get time evolution: Tomita-Takesaki theory (NC spaces are always dynamical)

$\mathcal{A}$ C^*-alg. $\quad \mathcal{A}'' = \mathcal{M}$ von Neumann alg. i.e. weak closure in GNS repres. given by state φ $\quad \langle a, b \rangle = \varphi(a^* b)$

$S : \mathcal{M}\xi \to \mathcal{M}\xi \subset \mathcal{H}_\varphi$ dense $\qquad \xi = 1 + I \quad I = \ker\varphi = \{a : \varphi(a^*a) = 0\}$

$a\xi \mapsto S(a\xi) = a^*\xi$ $\qquad$ anti linear operator

has polar decomposition $\quad S = J \Delta^{1/2}$ $\quad \Delta$ linear positive self-adj

$J = J^* = J^{-1}$ antilinear

Tomita $\begin{cases} J \mathcal{M} J = \mathcal{M}' \\ \Delta^{it} \mathcal{M} \Delta^{-it} = \mathcal{M} \end{cases}$ $\qquad \sigma_t(a) = \Delta^{-it} a \Delta^{it}$ time evol.

(Connes: unique up to inner)

Dual system & KMS states w/ this time evolution
(in Bost-Connes case get the one obtained before)

Ω_β = extremal low temperature KMS states

$$\varphi(a) = \frac{\mathrm{Tr}(\pi_\varepsilon(a)\, e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

Can change
$H \mapsto H + \log\lambda$
$\lambda \in \mathbb{R}_+^*$

$\mathbb{R}_+^* \to \tilde{\Omega}_\beta \to \Omega_\beta$ bundle (ε, H) $\varepsilon \in \Omega_\beta$

Elements $(\varepsilon, H) \in \tilde{\Omega}_\beta$ determine representations of the dual system $\hat{\mathcal{A}} = \mathcal{A} \rtimes_\sigma \mathbb{R}$

$$\pi_{(\varepsilon, H)}(X) = \int_{\mathbb{R}} \pi_\varepsilon(x(t))\, e^{itH}\, dt$$

$$X = \int_{\mathbb{R}} x(t)\, U_t\, dt$$

Scaling action $\pi_{(\varepsilon,H)}(\theta_\lambda(X)) = \pi_{(\varepsilon, H+\log\lambda)}(X)$

Under suitable conditions $\pi_{(\varepsilon,H)}(X)$ are Trace class

map $\hat{\mathcal{A}} \xrightarrow{\delta} C(\tilde{\Omega}_\beta)$
$X \mapsto \mathrm{Tr}(\pi_{(\varepsilon,H)}(X))$

"inclusion of classical pts in the NC space"

Not an algebra homomorphism

<u>But</u> $\exists$ Category of NC spaces (<u>cyclic modules</u>)
- abelian category
- Tr is a morphism (along w/ alg. homom, bimodules, ...)

Since abelian category: $\exists$ $\mathrm{Coker}(\delta)$ (complement of classical pts)

$HC_*(\mathrm{CoKer}(\delta))$ has $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$ action and $\mathbb{R}_+^*$ action

Bost-Connes case: $\hat{\mathbb{Z}}^* \times \mathbb{R}_+^* = C_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$ action

χ character of $\hat{\mathbb{Z}}^* \Rightarrow$ projector p_χ on $(\mathrm{Coker}\,\delta)$

$\mathcal{H}_\chi = HC_*(p_\chi \mathrm{Coker}(\delta))$ $\qquad e_\chi = \int_{\hat{\mathbb{Z}}^*} \chi(g) Z_g \, dg$ correspond

$Z_g = \{(x, g^{-1}x)\}$

$p_\chi = \int_{\hat{\mathbb{Z}}} g\, \chi(g)\, dg$

Thm: Infinitesimal generator D_χ of $\mathbb{R}_+^*$-action on $\mathcal{H}_\chi$ has spectrum the zeros of the L-function w/ Grössencharakter L_χ

($\chi = 1$ Riemann zeta function)

There is a trace formula for the action of $C_{\mathbb{Q}}$ on $HC_*(\mathrm{Coker}(\delta))$.

That gives the Weil explicit formula (relation between zeros of ζ and counting of primes)

$C_{\mathbb{Q}} \cong \Omega_\beta \qquad \hat{\mathcal{A}} = C^*(\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}^*)$

C^*- alg. Isom. Dual of BC system

$$\hat{\mathcal{A}}_1 \xrightarrow{\iota'} C^*(\mathcal{G}_1)$$

$\mathcal{G}_1$ = grpoid of 1-dim $\mathbb{Q}$-lattices up to commens (not up to scaling)

$$X = \int x(t)\, U_t\, dt$$

$x \in \mathcal{A}_1$

$$\iota(X)(k, \rho, \lambda) = \int x(t)(k, \rho)\ \lambda^{it}\, dt$$