- $$Z(J) = \mathcal{N} \int \exp\left(\frac{i}{\hbar}\left(S(\phi) + \langle J, \phi\rangle\right)\right) \mathcal{D}[\phi]$$

$\langle J, \phi \rangle = \int J(x)\phi(x)\,dx$

$$\mathcal{N} = \left(\int \exp\left(\frac{i}{\hbar} S(\phi)\right) \mathcal{D}[\phi]\right)^{-1}$$

Generating function for the Green functions

$$Z(J) = \sum_{N=0}^{\infty} \frac{i^N}{N!\,\hbar^N} \int J(x_1)\cdots J(x_N)\, G_N(x_1,\dots,x_N)\, dx_1\cdots dx_N$$

(usually choose units where $\hbar=1$)

$$G_N(x_1\dots x_N) = \mathcal{N}\int \phi(x_1)\cdots\phi(x_N)\, e^{\frac{iS(\phi)}{\hbar}}\, \mathcal{D}[\phi]$$

expansion in Feynman graphs:

$$G_N(x_1,\dots,x_N) = \sum_{\Gamma} \int \frac{V(\Gamma)(p_1,\dots,p_N)}{\#\mathrm{Aut}(\Gamma)}\, e^{i(x_1p_1+\dots+x_Np_N)} \prod_j \frac{dp_j}{(2\pi)^D}$$

All Feynman graphs of the theory (also non-connected)

- Passing to connected graphs:

$$iW(J) = \log Z(J) = \sum_{N=0}^{\infty} \frac{i^N}{N!\,\hbar^N} \int J(x_1)\cdots J(x_N)\, G_{N,c}(x_1\dots x_N)\, dx_i$$

$$G_{N,c}(x_1,\dots,x_N) = \sum_{\Gamma\ \text{connected}} \frac{V(\Gamma)(p_1,\dots,p_N)}{\#\mathrm{Aut}(\Gamma)}\, e^{i(x_1p_1+\dots+x_Np_N)} \prod_j \frac{dp_j}{(2\pi)^D}$$

- Eliminate "vacuum bubbles" = graphs w/ no external legs by dividing by $\mathcal{N}^{-1} = \int \exp\left(\frac{i\,S(\phi)}{\hbar}\right) \mathcal{D}[\phi]$

(no $J \Rightarrow$ no external legs)

Reason why from $W = \sum_{\Gamma \text{ connected}} \frac{V(\Gamma)}{\#\mathrm{Aut}(\Gamma)}$

get $e^W = \sum_\Gamma \frac{V(\Gamma)}{\#\mathrm{Aut}(\Gamma)}$

is because $V(\Gamma_1 \sqcup \Gamma_2) = V(\Gamma_1)\, V(\Gamma_2)$

and $\#\mathrm{Aut}(\Gamma) = \prod_j (n_j)! \prod_j \#\mathrm{Aut}(\Gamma_j)^{n_j}$

for $\Gamma = \bigsqcup_j \Gamma_j$ connected components (w/ n_j = multiplicity of component j)

Further simplification of combinatorics of graphs: (1PI)

Γ is 1PI (one-particle-irreducible)

$\begin{cases} \bullet\ \Gamma \text{ not a tree} \\ \bullet\ \Gamma \text{ cannot be disconnected by cutting a single edge} \end{cases}$

<u>Effective action:</u>

$$S_{eff}(\phi) = \langle \phi, J \rangle - W(J)$$

and evaluate at J = solution of variational equation

$$\frac{\delta W}{\delta J} = \phi \qquad \left(\underline{\text{i.e.}} \text{ so that } \frac{\delta}{\delta J}\left(\langle \phi, J\rangle - W(J)\right) = 0\right)$$

"Legendre transform of W"

write as: $S_{eff}(\phi) = \left(\langle \phi, J\rangle - W(J)\right)\Big|_{J = J(\phi)}$

If Knew an explicit form for $S_{eff}(\phi)$ could do all QFT at semiclassical level since

Claim:

$$\frac{\int \mathcal{O}(\phi)\, e^{-S(\phi)}\, \mathcal{D}[\phi]}{\int e^{-S(\phi)}\, \mathcal{D}[\phi]} = \mathcal{O}(\phi_{cl})\, e^{-S_{eff}(\phi_{cl})}$$

classical solution obtained from stationary phase approximation from $e^{-\frac{S_{eff}(\phi)}{T}}$ as $T \longrightarrow 0$

i.e. ϕ_{cl} solution of Euler-Lagrange equations for S_{eff}

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \varphi)} - \frac{\delta \mathcal{L}}{\delta \varphi} = 0 \quad \text{for } S(\phi) = \int \mathcal{L}(\phi)\, dx$$

However do NOT know explicitly $S_{eff}(\phi)$!

Only as an asymptotic series in graphs

$$S_{eff}(\phi) = S(\phi) - \sum_{\substack{\Gamma \text{ 1PI} \\ \text{graphs}}} \frac{U(\Gamma)(\phi)}{\#\mathrm{Aut}(\Gamma)}$$

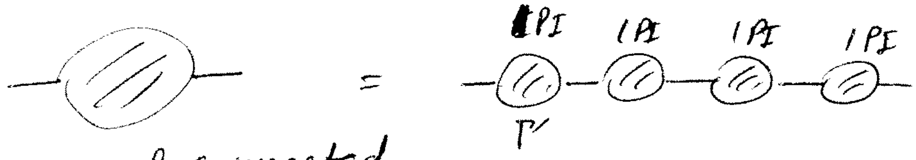
$$U(\Gamma)(\phi) = \frac{1}{N!} \int_{\sum_{j=1}^{N} p_j = 0} \hat{\phi}(p_1) \cdots \hat{\phi}(p_N)\, U(\Gamma(p_1, \dots, p_N)) \prod_j \frac{dp_j}{(2\pi)^D}$$

$U(\Gamma(p_1, \dots, p_N))$ determined by Feynman rules as before with

$$V(\Gamma(p_1, \dots, p_N)) = \varepsilon(\Gamma)\, U(\Gamma(p_1, \dots, p_N))$$

(factor involving external edges propagators

Reason why only 1PI: (example of ϕ^3 theory)

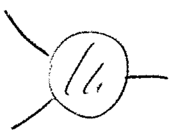
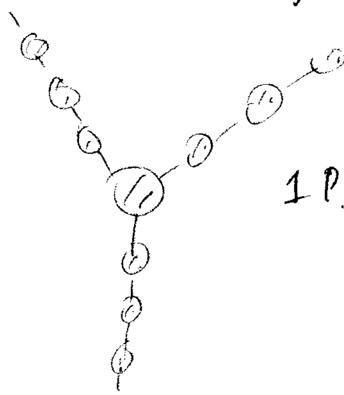
Γ 

general connected graph 2 ext. legs

$$\Rightarrow \quad V(\Gamma(p,-p)) = (p^2+m^2)^{-(q+1)}\, U(\Gamma'(p,-p))^q$$

if same Γ'

reduce to case of 1PI $\Longleftarrow$ similar formula for different

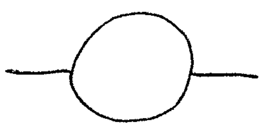
Similarly:  = 

3 ext legs

1 PI graphs

Example: where the problem of divergences first appear

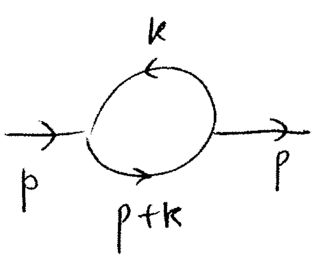
Take ϕ^3 $D=6$ (sufficiently simple & generic though not physically significant)



This is a Feynman graph of the theory

with momenta

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2} + \frac{g}{\text{[illegible]}}\phi^3$$



gives by Feynman rules

$$g^2(2\pi)^{-D}\int \frac{1}{k^2+m^2}\,\frac{1}{(p+k)^2+m^2}\, d^Dk$$

In dimension $D=4$ or $D=6$ divergent

Schwinger parameters

$$\int \frac{1}{k^2+m^2}\,\frac{1}{(p+k)^2+m^2}\,d^Dk = \int\int_{s>0,\,t>0} e^{-s(k^2+m^2)}\, e^{-t((p+k)^2+m^2)}\,ds\,dt\,d^Dk$$

$\Vert$ exchange order

$$\int_{s>0,\,t>0}\int e^{-s(k^2+m^2)-t((p+k)^2+m^2)}\,d^Dk\,ds\,dt$$

then complete square : write first

$$-Q(k) = -\lambda\left((k+xp)^2 + ((x-x^2)p^2+m^2\right)$$

for $s=(1-x)\lambda \quad t = x\lambda$

then in variable $q = k+xp$ get Gaussian

$$\int e^{-\lambda q^2}\,d^Dq = \pi^{D/2}\,\lambda^{-D/2}$$

Thus have (after the two changes of variables above)

$$\int_0^1\int_0^\infty e^{-(\lambda(x-x^2)p^2+\lambda m^2)}\int e^{-\lambda q^2}\,d^Dq\;\lambda\,d\lambda\,dx$$

$$= \pi^{D/2}\int_0^1\int_0^\infty e^{-\lambda((x-x^2)p^2+m^2)}\,\lambda^{-D/2}\,\lambda\,d\lambda\,dx$$

$$\lambda\,((x-x^2)p^2+m^2) = u$$

$$= \pi^{D/2}\int_0^\infty e^{-u}\,u^{-\frac{D}{2}+1}\,du \int_0^1 ((x-x^2)p^2+m^2)^{D/2-2}\,dx$$

$$= \pi^{D/2}\,\Gamma(2-D/2)\,F(D)$$

$$\Gamma(s+1) = \int_0^\infty dt \; t^s e^{-s}$$

$\Gamma(s+1) = s\,\Gamma(s) \quad \Rightarrow \quad \Gamma(n+1) = n!$ poles at neg. integers and zero

Schwinger parameters $\rightsquigarrow$ divergence in terms of poles of Γ-function

$\Gamma\left(2 - \frac{D}{2}\right)$ for $D \in 4 + 2\mathbb{N}$ has a pole

(Note: at odd integers divergence apparent: $\Gamma(2-\frac{D}{2})$ finite value)

Dimensional regularization

$D_{\text{integer}} \mapsto D_{\text{integer}} + z \qquad z \in \mathbb{C}^*$

(or $z \in \Delta^*$ small disk around $z=0$)

$$\int \frac{1}{k^2+m^2} \, \frac{1}{(p+k)^2+m^2} \, dk^{D+z} = \pi^{\frac{D+z}{2}} \, \Gamma\left(2 - \frac{D+z}{2}\right) \cdot F(D+z)$$

polynomial in P (local)

meromorphic function on $\Delta^* \ni z$

with a pole at $z=0$

$$f(z) = \sum_{k=-N}^{\infty} a_k z^k$$ Laurent series (if convergent on some Δ^* small disk)

Polar part $(Tf)(z) = \sum_{k=-N}^{-1} a_k z^k$

$(f - Tf)(z) = \sum_{k=0}^{\infty} a_k z^k$ power series: finite value at $z=0$

Dimensional Regularization (general procedure)

- Define integral of Gaussian in dimension $z \in \mathbb{C}$ by

$$\int e^{-\lambda x^2} d^z x := \pi^{z/2} \lambda^{-z/2}$$

- Feynman integral

$$I_P(k) = \frac{P(k,p)}{\prod_j (F_j(k,p)^2 + m_j^2)}$$

$P =$ polynomial

$F_j(k,p) = \sum_i a_{j,i}(k_j) + \sum_l b_{j,l}(p_j)$

linear forms w/ integer coefficients

product of quadratic forms:

$q_1 \cdots q_n$

Schwinger parameters:

$$\int P(k,p)\left(\int_{(\mathbb{R}^+)^n} e^{-\sum t_j q_j} \underbrace{dt_1 \cdots dt_n}_{dt}\right) d^z k$$

reverse order: (formal)

$$\int \left(\int P(k,p)\, e^{-\sum t_j q_j(k,p)}\, d^z k\right) dt$$

Complete square in $q_j(k,p)$

$$\underset{\displaystyle \sum t_j q_j(q,0)}{\overset{\|}{A(t)(q)}} + \sum t_j q_j(k(t),p)$$

$q = k - k(t)$

$k(t) = \min \sum t_j q_j(k)$
(rational funct. of t)
$A(t)^{-1} \ell(t,p)$

$$\int e^{-A(t)(q)} d^z q = \pi^{\frac{Lz}{2}} \det(A(t))^{-z/2}$$

$\Rightarrow$ Get Feynman integral in form

$$U(\Gamma(p_1,\dots,p_N)) = \int I_\Gamma(k_1,\dots,k_L,p_1,\dots,p_N)\, d^{D-z}k_1 \dots d^{D-z}k_L :=$$

$$\int e^{-\sum t_j\, q_j(k(t),p)} \underbrace{f(p,t)}_{\text{polynomial in } t \,\&\, p} \det(A_\Gamma(t))^{-\left(\frac{D-z}{2}\right)-n}\, dt$$

General question: Does this expression define a meromorphic function in the complex plane?

$\boxed{\text{Theorem: The Taylor coefficients at } p=0 \text{ of } U(\Gamma(p_1,\dots,p_N)) \text{ admit meromorphic continuation to all of } \mathbb{C}}$

Minimal subtraction: $L(z)$ Laurent series

$TL(z)$ polar part : $(L-TL)(z)$ regular at $z=0$

Apply to Laurent series obtained from Taylor coeff at $p=0$ of $U(\Gamma(p_1,\dots,p_N))$

Energy scale dependence of $U(\Gamma(p_1,\dots,p_N))(z)$

$$= \int I_\Gamma(p_1,\dots,p_N,k_1,\dots,k_L)\, d^{D-z}k_1 \cdots d^{D-z}k_L \;\mu^{zL}$$

μ = physical units of a mass (energy)

$U_\mu^z(\Gamma(p_1,\dots,p_N))$ functions of

$z \in \Delta^*$ and $\mu^z \in \mathbb{C}^*$

$: \mu^z$

$\Delta^* \times \mathbb{C}^*$ or rather fibration

$0 \;\; z \;\; \Delta^*$

$$\mathbb{C}^* \longrightarrow B^* \longrightarrow \Delta^* \qquad \mathbb{C}^* \longrightarrow B \longrightarrow \Delta$$

$$B^* = B|_{\Delta^*}$$

Schwinger parameters, Feynman trick and algebraic geometry of Feynman graphs

Feynman trick

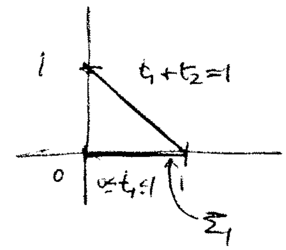
$$\frac{1}{ab} = \int_0^1 \frac{dt}{(ta+(1-t)b)^2}$$

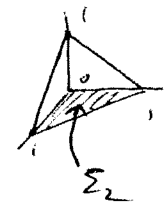
simple change of variables:

$$\int_0^1 \frac{dt}{(t(a-b)+b)^2} = \frac{1}{a-b}\int_b^a \frac{du}{u^2} = \frac{-1}{a-b}\,\frac{1}{u}\Big|_b^a = \frac{-1}{a-b}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab}$$

$$u = (a-b)t + b \qquad dt = \frac{1}{a-b}\,du$$

Generalization





$$\frac{1}{a_1\cdots a_n} = \int_{\Sigma_n} \frac{(n-1)!\quad dt_1\cdots dt_{n-1}}{(t_1a_1+\cdots+t_na_n)^n}$$

$$\Sigma_n = \left\{ (t_1,\ldots,t_{n-1}) : \sum_{i=1}^{n-1} t_i \le 1 \right\}$$

$$t_n = 1 - \sum_{i=1}^{n-1} t_i$$

$$= (n-1)! \int_0^\infty\!\!\int_0^\infty \frac{dt_1\cdots dt_n \quad \delta(1-\sum_{i=1}^n t_i)}{(t_1a_1+\cdots+t_na_n)^n}$$

Proof: Use Stokes theorem (induction on n)

$$\partial\Sigma_n = \bigcup_{i=1}^{n-1} \Sigma_{n-1}(i) \cup D_{n-1}$$

$$\Sigma_{n-1}(i) = \{ (t_1,\ldots,t_{n-1}) : t_i = 0 \} \qquad D_{n-1} = \{ (t_1,\ldots,t_{n-1}) : \sum_{i=1}^{n-1} t_i = 1 \}$$

$$\int_\Sigma \nabla\cdot F = \int_{\partial\Sigma} F\cdot\eta \qquad F_n = \sum_i (-1)^i \frac{dt_i}{(\sum_i t_i a_i)^{n-1}}$$

Or better use Schwinger parameters:

$$\frac{1}{a_1\cdots a_n} = \int_0^\infty\cdots\int_0^\infty e^{-(s_1a_1+\cdots+s_na_n)}\, ds_1\cdots ds_n = \overset{(n-1)!}{\overset{\shortparallel}{\Gamma(n)}} \cdot \int \frac{\delta(1-\sum_{i=1}^n t_i)\, dt_1\cdots dt_n}{(t_1a_1+\cdots+t_na_n)^n}$$

substitutions $\quad s_i = \eta t_i \quad \eta = \sum_{i=1}^n s_i \quad \sum t_i = 1$

More general formula

$$\frac{1}{a_1^{k_1}\cdots a_n^{k_n}} = \frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)}\int e^{-(s_1a_1+\cdots+s_na_n)} s_1^{k_1-1}\cdots s_n^{k_n-1}\,ds_1\cdots ds_n$$

from $\frac{1}{a^k} = \frac{1}{\Gamma(n)}\int_0^\infty e^{-sa} s^{k-1}\,ds$

Schwinger parameters and Feynman trick:

$$\frac{1}{a_1^{k_1}a_2^{k_2}} = \frac{1}{\Gamma(k_1)\Gamma(k_2)}\int e^{-(s_1a_1+s_2a_2)} s_1^{k_1-1} s_2^{k_2-1}\,ds_1\,ds_2$$

$0\le t\le 1$

set $\eta = s_1+s_2$ and $s_1 = t\eta$ $s_2 = (1-t)\eta$

$$\frac{1}{a_1^{k_1}a_2^{k_2}} = \frac{\Gamma(k_1+k_2)}{\Gamma(k_1)\Gamma(k_2)}\int_0^1 \frac{t^{k_1-1}(1-t)^{k_2-1}}{(ta_1+(1-t)a_2)^{k_1+k_2}}\,dt$$

Feynman trick special case

$$\frac{1}{a_1^{k_1}\cdots a_n^{k_n}} = \frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)}\int_0^\infty\!\!\cdots\!\int_0^\infty e^{-(s_1a_1+\cdots+s_na_n)} s_1^{k_1-1}\cdots s_n^{k_n-1}\,ds_1\cdots ds_n$$

$$= \frac{\Gamma(k_1+\cdots+k_n)}{\Gamma(k_1)\cdots\Gamma(k_n)}\int_{\Sigma_n} \frac{t_1^{k_1-1}\cdots t_n^{k_n-1}\,\delta(1-\sum_{i=1}^n t_i)}{(t_1a_1+\cdots+t_na_n)^{k_1+\cdots+k_n}}\,dt_1\cdots dt_n$$

Graph polynomial: loop variables $d^D k_1 \cdots d^D k_L$ (12)

L = loop # $= b_1(\Gamma)$

$$\int \frac{d^D k_1 \cdots d^D k_\ell}{q_1 \cdots q_n}$$

q_i = quadratic form $\quad q_i = p_i^2 + m_i^2$

momentum variable associated to each edge

Momentum conservation at each vertex $\delta(\sum p_i)$ from Feynman rules

$\Rightarrow$ change of variables

loop variables

$$p_i = u_i + \sum_r \eta_{ir} k_r \qquad \eta_{ir} = \begin{cases} +1 & \text{edge } i \text{ in loop } r \text{ same orientation} \\ -1 & \text{opposite or.} \\ 0 & \text{edge } i \notin \text{loop } r \end{cases}$$

with condition

$$\sum_i t_i u_i \eta_{ir} = 0 \quad \text{for all } r$$

Then $q_i = u_i^2 + m_i^2 + \sum_{r,r'} \eta_{ir} \eta_{ir'} k_r k_{r'}$

Then after Feynman trick

$$\int_{\Sigma_n} dt_1 \cdots dt_n \, \delta(1-\Sigma t_i) \int \frac{d^D k_1 \cdots d^D k_L}{(t_1 q_1 + \cdots + t_n q_n)^n}$$

$M_\Gamma(t)_{rr'} = \sum_i t_i \eta_{ir} \eta_{ir'}$

$k^T M_\Gamma(t) k = \sum_{r,r'} M_\Gamma(t) k_r k_{r'}$

$$\| $$

$$\int \frac{d^D k_1 \cdots d^D k_L}{\left(\sum_i t_i(u_i^2+m_i^2) + k^T M_\Gamma(t) k\right)^n} = \frac{1}{\left(\sum_i t_i(u_i^2+m_i^2)\right)^n} \int \frac{d^D k_1 \cdots d^D k_L}{(1 + k^T X_\Gamma(t) k)}$$

$$X_\Gamma(t) = \frac{M_\Gamma(t)}{\sum t_i(u_i^2+m_i^2)}$$

$$u_i = k_i \cdot \left(\sum t_i (u_i^2 + m_i^2) \right)^{-1/2}$$

$$d^D k_i = \left(\sum t_i (u_i^2 + m_i^2) \right)^{D/2} du_i \qquad \text{so get :}$$

$$\frac{1}{\left(\sum_i t_i (u_i^2 + m_i^2) \right)^{n-\frac{DL}{2}}} \int \frac{d^D u_1 \cdots d^D u_L}{(1 + \underbrace{u^T M_\Gamma(t) u}_{\sum_r \lambda_r u_r^2})^n}$$

$$v_r = \lambda_r^{1/2} u_r$$

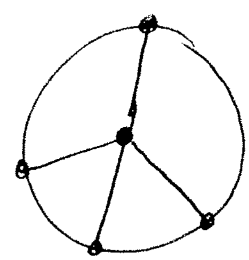
$$= \frac{\int \frac{d^D v_1 \cdots d^D v_L}{(1+\sum_r v_r^2)^n}}{\left(\sum_i t_i (u_i^2+m_i^2) \right)^{n-\frac{DL}{2}}} \frac{1}{\left(\det M_\Gamma(t) \right)^{D/2}}$$

When Γ graph satisfying $n = \frac{DL}{2}$ log divergent

(e.g. $D=4$ $n = 2L$ #edges $= 2 \cdot$ #loops)

then Feynman integral

$$C \cdot \int_{\Sigma_n} \frac{dt_1 \cdots dt_n \; \delta(1-\sum_i t_i)}{\det(M_\Gamma(t))^{D/2}}$$



examples: wheel with n spokes

$$\det M_\Gamma(t) = \Psi_\Gamma(t) = \sum_{T \text{ spanning trees}} \prod_{\text{edges not in } T} t_e$$

$\psi_\Gamma(t)$ Kirchhoff polynomial of the graph

$$\int_\Sigma \frac{\Omega}{\psi_\Gamma(t)^{D/2}}$$

Divergences from where

$X_\Gamma = \{ t : \psi_\Gamma(t) = 0 \}$ intersects Σ

Graph hypersurface

Algebraic geometry of alg. variety X_Γ gives information on Feynman integral

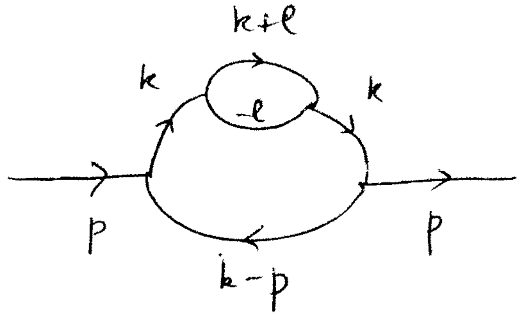
chain of integration in

$\Sigma \subset \mathbb{P}^n \setminus X_\Gamma$ $\quad n = \#$ edges

if $\Sigma \cap X_\Gamma = \emptyset$ otherwise rel. to $\Sigma \cap X_\Gamma$

$$H^{n-1}(\mathbb{P}^n \setminus X_\Gamma)$$

A more complicated example of divergence
(Subdivergences)



for ϕ^3 theory $m=0$ (massless)
$D=6$

(Since massless also infrared divergence: only concentrate on ultraviolet problem)

$$U(\Gamma(p,-p)) = (2\pi)^{-2D} \int \frac{1}{k^4}\,\frac{1}{(k-p)^2}\,\frac{1}{(k+l)^2}\,\frac{1}{l^2}\, d^D l\, d^D k$$

Integral in l as before  case (subdivergence)

Get
$$\int \frac{1}{(k+l)^2 l^2}\, d^D l = k^{D-4}\, \pi^{D/2}\, \Gamma\left(2-\frac{D}{2}\right) \int_0^1 (x-x^2)^{\frac{D}{2}-2} dx$$

Now use
$$\int_0^1 (x-x^2)^{\frac{D}{2}-2} dx = \frac{\Gamma\left(\frac{D}{2}-1\right)^2}{\Gamma(D-2)}$$

See infrared divergence due to $m=0$ from poles at $\frac{D}{2}-1 \in -\mathbb{N}$

Left w/ integral $\displaystyle\int (k^2)^{\frac{D}{2}-4} \frac{1}{(k-p)^2}\, d^D k = I$

Use: $$x^{D/2-4} = \Gamma\left(4-\frac{D}{2}\right)^{-1} \int_0^\infty e^{-tx}\, t^{3-\frac{D}{2}}\, dt$$

This gives

$$I = \Gamma(4-\tfrac{D}{2})^{-1} \int e^{-t_1 k^2 - t_2 (k-p)^2} t_1^{3-D/2} \, dt_1 \, dt_2 \, d^D k$$

Set $t_1 = \lambda s \quad t_2 = \lambda(1-s)$ so that

$$t_1 k^2 + t_2 (\lambda - p)^2 = \lambda q^2 + \lambda (s - s^2) p^2 \qquad q = k - (1-s) p$$

$$I = \Gamma(4-\tfrac{D}{2})^{-1} \pi^{D/2} \int e^{-\lambda (s-s^2) p^2} \lambda^{3-D} s^{3-\frac{D}{2}} \lambda \, d\lambda \, ds$$

integration in λ as before gives another Γ-function

$$\Gamma(5-D) \quad ((s-s^2)p^2)^{D-5}$$

then integration in s gives as in $(x-x^2)^{\frac{D}{2}-2}$ before another product of Γ's

Get:

$$U(\Gamma(p,-p)) = (4\pi)^{-D} \frac{\Gamma(2-\frac{D}{2}) \Gamma(\frac{D}{2}-1)^3 \Gamma(5-D) \Gamma(D-4)}{\Gamma(D-2) \Gamma(4-\frac{D}{2}) \Gamma(\frac{3D}{2}-5)} (p^2)^{D-5}$$

Note two facts: • Double pole at $D=6$

from $\Gamma(2-\frac{D}{2})$ and $\Gamma(5-D)$

$D = 6 - z$ expand in z:

• Coefficient from $\left(\frac{p^2}{\mu^2}\right)^{-z} = \sum \frac{(-z)^n}{n!} \log^n\left(\frac{p^2}{\mu^2}\right)$

$-g^4 (4\pi)^{-6} \frac{1}{18} p^2 (\log(\frac{p^2}{\mu^2}) + \text{const.})$ NOT POLYNOMIAL IN P

BPHZ procedure:

preparation:

in example above just one subdivergence (massless case)

γ:

inside Γ:

w/ quotient Γ/γ:

$$\bar{R}^z_\mu(\Gamma) := U^z_\mu(\Gamma) + C(\gamma)\, U^z_\mu(\Gamma/\gamma)$$

$$C(\gamma) = -T\, U^z(\Gamma)$$

T = polar part of Laurent series

Note: two possible valence two vertices

from $\frac{1}{2}(\partial\phi)^2$ and $\frac{m^2}{2}\phi^2$ terms when $m \neq 0$

In $\displaystyle\int \frac{1}{(k+\ell)^2}\frac{1}{\ell^2}\, d^D\ell \sim -\frac{1}{3}\pi^3 k^2 \frac{1}{z}$ has local coeff.

$$\underbrace{\int \frac{1}{(k+\ell)^2}\frac{1}{\ell^2}\, d^D\ell + \frac{1}{3}\pi^3 k^2 \frac{1}{z}}_{\text{Replace this instead of}}$$

now convergent $\sim k^2 \log k^2$

$$\int \frac{1}{k^4}\frac{1}{(k-p)^2}\frac{1}{(k+\ell)^2}\frac{1}{\ell^2}\, d^D\ell\, d^Dk$$

Get $U(\Gamma) + C(\gamma)\, U(\Gamma/\gamma)$

Now get order one pole and local coeff. of residue

$\boxed{\text{General BPHZ}}$

preparation

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\substack{\gamma \subsetneq \Gamma \\ \text{subdiv.}}} C(\gamma)\, U(\Gamma/\gamma) \Rightarrow \bar{R}^z_\mu(\Gamma(p_1, \ldots, p_N))$$

where Counterterms

$$C(\Gamma) = -T\left(U(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, U(\Gamma/\gamma) \right)$$

Recursive formula

Renormalized value

$$R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma)$$

$$= U(\Gamma) + C(\Gamma) + \sum_{\gamma \subsetneq \Gamma} C(\gamma)\, U(\Gamma/\gamma)$$

- Order one poles
- local coefficients (Polyn in p)

---

massive case: $$C(\gamma)(p) = m^2\, C(\gamma_{(0)}) + p^2\, C(\gamma_{(1)})$$

$\frac{1}{2}(\partial\phi)^2$ $\quad$ $\frac{1}{2}\, m\, \phi^2$