Hopf algebras and affine group schemes

R = commutative ring with unit (e.g. $\mathbb{Z}$, $\mathbb{Z}[x]$, $\mathbb{R}$, $\mathbb{C}$, $\mathcal{O}_K$, ...)

$GL_n(R)$ = invertible $n\times n$-matrices w/ entries in R
$SL_n(R)$ = " w/ det $=1$
$\mathbb{G}_m(R) = GL_1(R) = R^*$ = invertible elements in R
$\mathbb{G}_a(R) = R$ w/ additive structure
etc. "procedure to construct a group that works for any R"

Also if $\varphi: R \to S$ ring homomorphism
then $\Rightarrow$ group homomorphism

$GL_n(R) \to GL_n(S)$ etc.
$SL_n(R) \to SL_n(S)$ ...

e.g. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(R) \mapsto \begin{pmatrix} \varphi(a) & \varphi(b) \\ \varphi(c) & \varphi(d) \end{pmatrix} \in SL_2(S)$

$ad - bc = 1$
$\varphi(a)\varphi(d) - \varphi(b)\varphi(c) = \varphi(ad-bc) = \varphi(1) = 1$

- To any unital ring R (commutative) $\Rightarrow$ group
- To any morphism of rings $\Rightarrow$ morphism of groups

Functor $G : \mathcal{R} \to \mathcal{G}$
category of unital comm. rings to category of groups

- $R \in Obj(\mathcal{R}) \mapsto G(R) \in Obj(\mathcal{G})$
- $\varphi \in Mor(\mathcal{R}) \mapsto G(\varphi) \in Mor(\mathcal{G})$

Affine group schemes are functors of this sort that are defined by solutions of polynomial equations

(or more generally limits of such)
projective limits of alg. groups

K = field (char 0) $\quad \mathbb{Q}, \mathbb{C}, \mathbb{R}, \mathbb{Q}_p \ldots$

$\mathcal{A}$ = category of commutative unital k-algebras

Given a k-algebra $A \in Obj(\mathcal{A})$: can view it as defined by equations

$\{x_\alpha\}$ generators of A $\qquad \{f_i\}$ = Polynomials in

Polynomial ring $k[X_\alpha]$ s.t. $X_\alpha \longmapsto x_\alpha$

$\{f_i\}$ generate Kernel of this map)

i.e. elements of A are solutions of equations $f_i = 0$ $\qquad$ "general solutions x_α of $f_i = 0$"

Given a set of equations $f_i = 0$ can look for solutions in a given algebra $R \in Obj(\mathcal{A})$

Set $F(R)$ = solutions to $\{f_i = 0\}$ in R

- A way to obtain solutions in R ; algebra homomorphism $\varphi: A \to R$ maps x_α to solutions in R, $\varphi(x_\alpha)$

$\Rightarrow \mathrm{Hom}_{\mathcal{A}}(A, R) \longrightarrow F(R)$ injective map of sets

- x_α generators of A are "most general possible solutions" for equations $f_i = 0$ $\Rightarrow$ map surjective $\mathrm{Hom}_{\mathcal{A}}(A,R) = F(R)$

F = functor : $\mathcal{A} \to$ Sets

Representable : $\exists\, A \in Obj(\mathcal{A})$ s.t. $F(R) = Hom_{\mathcal{A}}(A, R)$

<u>Def</u>: An affine group scheme is a representable functor $G : \mathcal{A} \to \underset{\mathcal{G}}{\text{Groups}}$

- Natural transformations of functors:

$G_1 : \mathcal{A} \to \mathcal{G} \qquad G_2 : \mathcal{A} \to \mathcal{G}$

η = natural transformation $\eta : G_1 \to G_2$

$\forall\, \underset{B}{A} \in Obj(\mathcal{A})$

$\forall\, \varphi \in Mor(\mathcal{A})$

$\varphi : A \to B$

$$\begin{tikzcd} G_1(A) \arrow[r, "\eta_A"] \arrow[d, "G_1(\varphi)"'] & G_2(A) \arrow[d, "G_2(\varphi)"] \\ G_1(B) \arrow[r, "\eta_B"'] & G_2(B) \end{tikzcd}$$

All these diagrams commute

<u>Example</u> : det : $GL_n \to G_m$

Yoneda's Lemma:

$E, F : \mathcal{A} \to$ Sets

$E(R) = Hom_{\mathcal{A}}(A, R)$

$F(R) = Hom_{\mathcal{A}}(B, R)$

then natural transformations $\eta : E \to F$

$\iff$ algebra homomorphisms

$\varphi : B \to A$ (direction reversal!)

Affine group scheme:

$$G : \mathcal{A} \to \text{Sets}$$

$$G(R) = \mathrm{Hom}_{\mathcal{A}}(A, R)$$

s.t. these sets are actually groups!

What does this condition mean in terms of A?

- A is a unital algebra with additional structure

COMMUTATIVE HOPF ALGEBRA

Group: G set with operations

- $\cdot : G \times G \to G$ multiplication
- associative
- 1_G = unit
- inverses

A is a unital commutative algebra with

- comultiplication $\Delta : A \to A \otimes A$
- counit (augmentation) $\varepsilon : A \to k$
- antipode $S : A \to A$

satisfying properties:

Coassociativity:

- $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta \quad : \cancel{A} A \to A \otimes_k A \otimes_k A$

i.e.
$$\begin{tikzcd} A\otimes A \arrow[r, "id\otimes\Delta"] & A\otimes A\otimes A \\ A \arrow[u, "\Delta"] \arrow[r, "\Delta"'] & A\otimes A \arrow[u, "\Delta\otimes id"'] \end{tikzcd}$$
commutes

- $(id\otimes\varepsilon)\Delta = id = (\varepsilon\otimes id)\Delta \;:\; A \to A$

$$\begin{tikzcd} A \arrow[r, "\simeq"] & k\otimes A \\ A \arrow[u, equal, "id"] \arrow[r, "\Delta"'] & A\otimes A \arrow[u, "\varepsilon\otimes id"'] \end{tikzcd}$$

$m : A\otimes A \to A$ algebra multiple
1 = unit

- $m(id\otimes S)\Delta = m(S\otimes id)\Delta = 1\cdot\varepsilon$

$$\begin{tikzcd} A\otimes A \arrow[r, "m(S\otimes id)"] & A \\ A \arrow[u, "\Delta"] \arrow[r, "\varepsilon"'] & k \arrow[u, "1"'] \end{tikzcd}$$

<u>Notation</u>: will use A for algebras
and H for Hopf algebras

$G(R)$ group $\qquad G(R) = \mathrm{Hom}_{\mathcal{A}}(H, R)$

$\phi : H \to R \qquad \phi(xy) = \phi(x)\phi(y) \qquad \phi(1) = 1$

group structure:

$$\phi_1 * \phi_2 \,(x) = \langle \underset{\substack{\cap \\ \mathrm{Hom}(H\otimes H,\, R)}}{\phi_1 \otimes \phi_2}, \Delta(x) \rangle = \phi_1 \otimes \phi_2 \,(\Delta(x))$$

$$= \sum \phi_1(x_{(1)})\, \phi_2(x_{(2)})$$

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} \in H \otimes H$$

- group multiplication is associative because Δ is coassociative

inverse:

$\phi \in \mathrm{Hom}(H, R) \qquad \phi^{-1}(x) = \phi(S(x))$

$$\phi * \phi^{-1} = \phi^{-1} * \phi = \varepsilon$$

unit of $G(R)$ from counit of H

$\varepsilon : H \to K \qquad \varepsilon(x) = \sum S(x_{(1)}) \cdot x_{(2)}$

$\qquad \Delta(x) = \sum x_{(1)} \otimes x_{(2)}$

$H \to k \to R$

Affine group schemes are representable functors $\quad G : \mathcal{A} \to \mathcal{G}$

where $\quad G(A) = \mathrm{Hom}_{\mathcal{A}}(H, A)$

with H a Hopf algebra (commutative)

Examples :

- The additive group $\mathbb{G}_a$

$$H = k[t] \quad \text{with} \quad \Delta(t) = t\otimes 1 + 1\otimes t$$
$$S(t) = -t \qquad \varepsilon(t) = 0$$

- The multiplicative group $\mathbb{G}_m$

$$H = k[t, t^{-1}] \quad \text{with} \quad \Delta(t) = t\otimes t$$
$$\varepsilon(t) = 1$$
$$S(t) = t^{-1}$$

- The group of roots of unity μ_n

Kernel of homomorphism $\mathbb{G}_m \to \mathbb{G}_m$ given by rising to n-th power

$$H = k[t]/(t^n - 1) \quad \left(= k[t, t^{-1}]/(t^n - 1) \right)$$

$$\Delta(t) = t\otimes t \qquad \varepsilon(t) = 1 \qquad S(t) = t^{-1}$$

characters of G

Note (1) any element $x \in H$ in a Hopf algebra such that $\Delta(x) = x\otimes x$ $(\varepsilon(x) = 1 \quad S(x) = x^{-1})$ is called "group-like"

a group-like element $\Rightarrow$ homom. $G \to \mathbb{G}_m$

(2) elements $x \in H$ with $\Delta(x) = x\otimes 1 + 1\otimes x$ $(\varepsilon(x) = 0 \quad S(x) = -x)$ are called "primitive"

Primitive element $\Rightarrow$ homom. $G \longrightarrow \mathbb{G}_a$

More examples:

- GL_n $\qquad H = k[x_{i,j}, t]_{i,j=1,\dots,n} \Big/ (\det(x_{i,j})t-1)$

$$\Delta(x_{i,j}) = \sum_k x_{i,k} \otimes x_{k,j}$$

- Linear algebraic group$/_k$ $\quad G \subset GL_n$ Zariski closed

$H =$ fin. ~~gen~~. algebra $/_k$

- $\mathcal{H}$ any commutative $\left(\begin{array}{l}\text{pos. graded } \mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n \\ \text{connected } \mathcal{H}_0 = k\end{array}\right)$

$\exists\ \mathcal{H}_i \subset \mathcal{H}$ fin gen alg$/k$

$\mathcal{H} = \bigcup_i \mathcal{H}_i \qquad \Delta(\mathcal{H}_i) \subset \mathcal{H}_i \otimes \mathcal{H}_i$

$\qquad\qquad\qquad S(\mathcal{H}_i) \subset \mathcal{H}_i$

$\Rightarrow \quad G = \varprojlim_i G_i \qquad G_i \subset GL_{n_i}$ linear alg. groups

The Lie algebra of an affine group scheme

Covariant functor $\mathfrak{g} : \mathcal{A} \longrightarrow \mathcal{L}ie$

$\mathcal{L}ie$ = category of Lie algebras$/k$

k-vector spaces w/ bracket $[\,,\,]$

Jacobi identity

$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0$$

$\mathfrak{g}(A)$ = Lie algebra of linear maps

$$L : \mathcal{H} \longrightarrow A$$

s.t.

$$L(XY) = L(X)\varepsilon(Y) + \varepsilon(X)L(Y)$$

$$[L_1, L_2](x) = (L_1 \otimes L_2 - L_2 \otimes L_1)(\Delta(x))$$

$$= \langle L_1 \otimes L_2 - L_2 \otimes L_1 , \Delta(x) \rangle$$

When is $\mathfrak{g}$ enough to reconstruct G ?

- Usually not: e.g. $\mathbb{G}_m$ and $\mathbb{G}_a$ same $\mathfrak{g}$

- Milnor-Moore theorem

when $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ $\quad \mathcal{H}_0 = k$ $\quad$ (k of char 0)

$\mathcal{H}_n$ = fin dim vector spaces$/k$

$\mathcal{H}^\vee$ = dual Hopf algebra $\quad$ $\mathcal{L}$ = Lie algebra of primitive elts of $\mathcal{H}^\vee$

$$\mathcal{H} \cong U(\mathcal{L})^\vee$$

Comments: [Primitively generated Hopf alg. (char 0) is univ. enveloping alg of Lie alg of prim. el'ts

1) Primitive elements form a Lie algebra

$\Delta(x) = x \otimes 1 + 1 \otimes x$ $\qquad\qquad$ $P(\mathcal{H})$

$[x_1, x_2] = x_1 x_2 - x_2 x_1$ (multiplication in $\mathcal{H}$)

2) Univ. enveloping algebra of a Lie algebra

tensor algebra $T(\mathcal{L})$ (seeing $\mathcal{L}$ as vector space)

quotient $U(\mathcal{L}) = T(\mathcal{L}) /$ relations $a \otimes b - b \otimes a = [a,b]$

associative algebra ; Hopf w/ $\Delta(x) = x \otimes 1 + 1 \otimes x$, $S(x) = -x$, $\varepsilon(x) = 0$

(Note : A assoc. alg. $\rightsquigarrow \mathcal{L}_A$ Lie $[a,b] = ab - ba$)

$P(U(\mathcal{L})) = \mathcal{L}$

3) Dual Hopf algebra $\mathcal{H}^\vee$

linear dual of $\mathcal{H}$

$L : \mathcal{H} \to K$ linear functions

multipl. on $\mathcal{H} \Longleftrightarrow$ comultipl on $\mathcal{H}^\vee$

comultipl on $\mathcal{H} \Longleftrightarrow$ multipl on $\mathcal{H}^\vee$

unit $\Longleftrightarrow$ counit

antipode $\Longleftrightarrow$ antipode

Hopf algebra of Feynman graphs
(Connes-Kreimer Hopf algebra)

T = renormalizable quantum field theory

$\mathcal{H} = \mathcal{H}(T)$ free commutative algebra (over $\mathbb{C}$ or $\mathbb{Q}$)

generated by elements of the form (Γ, w)

Γ = a 1PI Feynman graph of the theory T

w = a monomial in $\mathcal{L}$ = Lagrangian (not only interaction monomials) of deg = "valence" = # external edges of Γ

Coproduct:

$$\boxed{\Delta(\Gamma) = \sum_{\gamma \in \bar{\mathcal{V}}(\Gamma)} \gamma \otimes \Gamma/\gamma} = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

where $\Gamma/\gamma = (\Gamma/\gamma, w)$ same as for Γ since external legs unchanged

and γ = prod of components $(\tilde{\gamma}_i, w_i)$ $\quad w_i = \chi(\gamma_i) \in T$

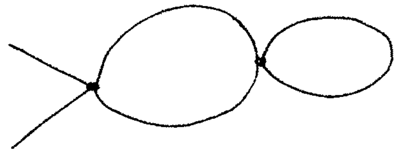
T = collection of all monomials in Lagr.

(γ, χ) $\quad \gamma$ = subgraph (possibly multi-connected) made of a set of <u>internal</u> edges of Γ

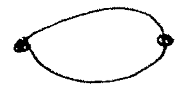
$\chi : \left\{ \begin{array}{l} \text{connected} \\ \text{components} \\ \text{of } \gamma \end{array} \right\} \longrightarrow T$ = monomials in Lagrangian

$\tilde{\gamma} = \left\{ \begin{array}{l} \gamma \text{ as internal lines and vertices} \\ \text{external lines } \partial_i^{-1}(v) \cap \gamma^c \end{array} \right.$

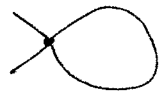
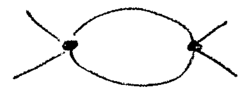
Example:

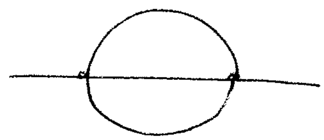


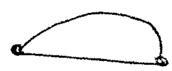
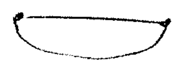
graph for ϕ^4

subgraphs  and 

γ γ'

then $\tilde{\gamma}$  and 

Similarly 

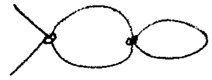
Subgraphs  and 

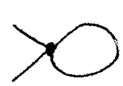
give $\tilde{\gamma}$ 

The function χ has the property that

$\deg(\chi(\gamma))$ = valence vertex in Γ/γ obtained when contracting γ

= # external edges of $\tilde{\gamma}$

e.g. in  contracting 

leaves  vertex of valence 4

In case of vertices of valence 2, as in  when contracting , χ distinguishes between $\frac{1}{2}(\partial\phi)^2$ and $\frac{m^2}{2}\phi^2$ cases

Note: the coproduct

$$\Delta(\Gamma) = \overbrace{\Gamma\otimes 1 + 1\otimes\Gamma + \sum_\gamma}\ \gamma \otimes \Gamma/\gamma$$

is linear in second variable and polynomial in first

$\gamma = \gamma_1 \cdots \gamma_n$ if γ_i connected components

w/ γ_i shorthand for $(\tilde{\gamma}_i, \chi(\gamma_i))$

and Γ/γ for $(\Gamma/\gamma, w)$

Theorem $\mathcal{H}(T)$ is a Hopf algebra

- $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta$ coassociativity

enough to check on 1PI graphs Γ (generators)

Set $\bar{\mathcal{V}}(\Gamma) = \mathcal{V}(\Gamma) \cup \{\gamma = \emptyset\} \cup \{\gamma = \Gamma^{(1)}_{int}\}$

Define: $\gamma \leq \Gamma$ iff $\gamma \in \bar{\mathcal{V}}(\Gamma)$ $\qquad \gamma < \Gamma$ if $\gamma \in \bar{\mathcal{V}}(\Gamma)$ $\gamma \neq \Gamma^{(1)}_{int}$

$$\Delta(\Gamma) = \sum_{\gamma \leq \Gamma} \tilde{\gamma} \otimes \Gamma/\gamma \qquad \text{(includes } \Gamma\otimes 1 + 1\otimes\Gamma \text{ part)}$$

$$(\Delta \otimes id)\Delta(\Gamma) = \sum_{\gamma \leq \Gamma} \Delta\tilde{\gamma} \otimes \Gamma/\gamma$$

$$\Delta\tilde{\gamma} = \prod_i \Delta\tilde{\gamma}_i \qquad = \sum_{\gamma' \leq \tilde{\gamma}} \tilde{\gamma}' \otimes \tilde{\gamma}/\gamma'$$

componentwise

Note: $\gamma' \in \bar{\mathcal{V}}(\Gamma) \Leftrightarrow \gamma'_j \in \bar{\mathcal{V}}(\tilde{\gamma}_j)$

$$(\Delta \otimes id)\Delta\Gamma = \sum_{\gamma' \leq \gamma \leq \Gamma} \tilde{\gamma}' \otimes \tilde{\gamma}/\gamma' \otimes \Gamma/\gamma$$

$\gamma \in \bar{\mathcal{V}}(\Gamma)$ $\quad \gamma' \in \bar{\mathcal{V}}(\Gamma)$ $\quad \gamma' \subset \gamma$

Rewriting $\Delta\Gamma = \sum_{\gamma'} \tilde{\gamma}' \otimes \Gamma/\gamma'$

$$(id \otimes \Delta)\Delta\Gamma = \sum_{\gamma' \leq \Gamma} \tilde{\gamma}' \otimes \Delta(\Gamma/\gamma')$$

then want to show: $\Delta(\Gamma/\gamma') = \sum_{\gamma \leq \Gamma,\ \gamma' \subset \gamma} \tilde{\gamma}/\gamma' \otimes \Gamma/\gamma$

internal lines of Γ/γ' = complement of γ' in $\Gamma^{(1)}_{int}$

$\gamma \supset \gamma'$: subset of $(\Gamma/\gamma)^{(1)}_{int}$

$\gamma \mapsto \gamma'' = \gamma \setminus \gamma'$ $\qquad \Delta(\Gamma/\gamma') = \sum \tilde{\gamma}'' \otimes (\Gamma/\gamma')/\gamma''$

$$\begin{cases} (\Gamma/\gamma')/\gamma'' = \Gamma/\gamma \\ \tilde{\gamma}/\gamma' = \tilde{\gamma}'' \quad \text{if } \gamma'' = \gamma \setminus \gamma' \end{cases}$$

when adding ext. legs

Then need to check maps X to J also match

For counit and antipode

Notice that $\mathcal{H}(\mathcal{T}) = \bigoplus_{n \geq 0} \mathcal{H}_n$

where grading is by "loop number"

(or better # internal edges) $\deg(\Gamma) = b_1(\Gamma)$

$$\begin{cases} \deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i) & \text{on } \text{~~components~~ products} \\ \deg(1) = 0 \end{cases}$$

$$\deg(\Gamma) = \deg(\gamma) + \deg(\Gamma/\gamma)$$

Also $\mathcal{H}_0 = k$ connected

Then antipode defined inductively

$$\Delta(x) = \sum x_{(1)} \otimes x_{(2)} = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$$

$$\deg(x') < \deg(x) \quad \deg(x'') < \deg(x)$$

$$\Rightarrow \quad S(X) = -X - \sum S(x')x''$$

$(\ \varepsilon(X) = 0 \quad \deg X \geq 0\ )$

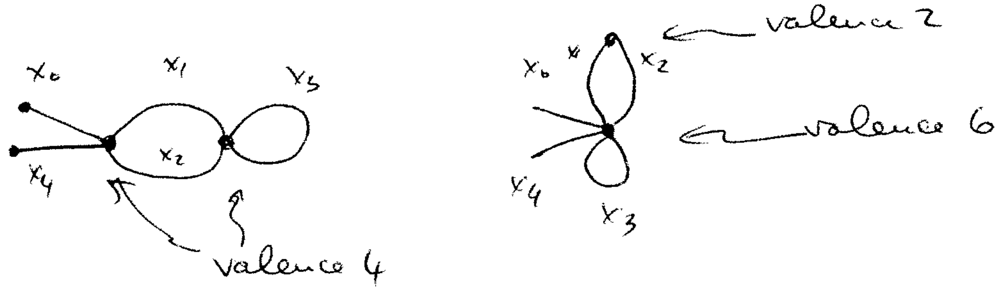
Lie algebra (insertion Lie algebra)

$$[(\Gamma, w), (\Gamma', w')] = \sum_{v\,:\, i(v)=w'} (\Gamma \circ_v \Gamma', w) - \sum_{v'\,:\, i(v')=w} (\Gamma' \circ_{v'} \Gamma, w')$$

inserting in all possible ways a graph into another one

<u>Problem</u>: "motivic" lifts of the Connes-Kreimer Hopf algebra (or of the insertion Lie algebra)

Different graphs can give the same graph hypersurface



$$\Psi_\Gamma(x_0, x_1, x_2, x_3, x_4) = (x_1 + x_2)\, x_3$$

hypersurface in $\mathbb{P}^4$