Counterterms and the β function
(Gross - 't Hooft relations)

Lemma: Given $\gamma_\mu(z)$ with $\gamma_\mu(z) = \gamma_-^{-1}(z)\,\gamma_{+,\mu}(z)$

then

$$\boxed{\gamma_-(z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n} \qquad d_n \in \mathcal{H}^\vee}$$

where

$$\boxed{Y(d_{n+1}) = d_n\, \beta \quad \forall n \geq 1 \qquad \text{with } \beta = \frac{d}{dt} F_t \Big|_{t=0}}$$

and $\boxed{Y(d_1) = \frac{d}{dt} F_t \Big|_{t=0} = \beta}$

Recursive formula for the coefficients d_n

Pf: $x \in \mathcal{H}$

$$\left\langle \frac{d}{dt} F_t \Big|_{t=0}, x \right\rangle = \lim_{z \to 0} z \langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes Y)(\Delta(x)) \rangle$$

This follows from

$$\langle \gamma_-(z)\, \theta_{tz}(\gamma_-(z)^{-1}), x \rangle = \langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes \theta_{tz})(\Delta(x)) \rangle$$

and
$$\frac{d}{dt} F_t \Big|_{t=0} = \frac{d}{dt}\left(\lim_{z \to 0} \gamma_-(z)\, \theta_{tz}(\gamma_-(z)^{-1}) \right)\Big|_{t=0}$$

so that
$$\lim_{z \to 0} \langle \gamma_-(z)^{-1} \otimes \gamma_-(z)^{-1}, (S \otimes \theta_{tz})(\Delta(x)) \rangle = \langle F_t, x \rangle$$

Now use the fact that, when evaluated on an element $x \in \mathcal{H}$ $\quad F_t(x) =$ polynomial in t

$\Rightarrow$ polyn in z $\Rightarrow$ uniform conv $\quad \partial_t \langle F_t, x \rangle|_{t=0} = \lim_{z \to 0} \frac{\partial}{\partial t}(*)$

$$z\langle \gamma_-(z)^{-1}\otimes\gamma_-(z)^{-1}, (S\otimes Y)\Delta(X)\rangle \xrightarrow[z\to 0]{} \langle \tfrac{d}{dt} F_t|_{t=0}, X\rangle$$

holom on $z\in\mathbb{C}\setminus\{0\}$
extends at $z=\infty$ $\quad \gamma_-(\infty)=1$ and $Y(\gamma_-(\infty))=0$
but also hol. at $z=0$ $\Rightarrow$ constant

$$\Rightarrow \quad \langle \gamma_-(z)^{-1}\otimes\gamma_-(z)^{-1}, (S\otimes Y)\Delta(x)\rangle = \frac{1}{z}\langle \tfrac{d}{dt}F_t|_{t=0}, X\rangle$$

$$\| $$

$$\langle \gamma_-(z)\, Y(\gamma_-(z)^{-1}), x\rangle$$

$\Rightarrow$

$$Y(\gamma_-(z)^{-1}) = \frac{1}{z}\,\gamma_-(z)^{-1}\,\frac{d}{dt}F_t|_{t=0} \qquad (*)$$

Also know $Y(\gamma_-(z)^{-1}) = \sum_{n=1}^{\infty} \frac{Y(d_n)}{z^n}$

using

$$\gamma_-(z)^{-1} = 1 + \sum_{n=1}^{\infty}\frac{d_n}{z^n}$$

and $Y(1)=0$

and
$$\frac{1}{z}\left(1+\sum_{n=1}^{\infty}\frac{d_n}{z^n}\right)\frac{d}{dt}F_t|_{t=0} = \frac{\beta}{z}+\sum_{n=2}^{\infty}\frac{d_{n-1}}{z^n}\beta$$

then equality $(*)$ gives

$$\begin{cases} Y(d_1) = \beta \\ Y(d_n) = d_{n-1}\beta \end{cases}$$

<u>Physically</u>: the beta function of a renormalizable theory determines the counterterms

$$\boxed{\text{A. Connes, M.M. Renormalization and motivic Galois theory IMRN (2004) N. 76 4073–4092}}$$

This recursive formula

$$\gamma_-(z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n} \qquad \begin{aligned} Y(d_{n+1}) &= d_n \beta \\ Y(d_1) &= \beta \end{aligned}$$

Can be written as a "time ordered exponential" (Dyson)
(or "Chen iterated integral" or "expansional")

Def:

$G(\mathbb{C})$ complex Lie group $\quad \mathfrak{g}(\mathbb{C}) = \mathrm{Lie}\, G(\mathbb{C})$ Lie algebra

$\alpha : [0,1] \to \mathfrak{g}(\mathbb{C})$ smooth function $\alpha(t)$

$$T e^{\int_a^b \alpha(t)\,dt} := 1 + \sum_{n=1}^{\infty} \int_{0 \le s_1 \le \cdots \le s_n \le b} \alpha(s_1) \cdots \alpha(s_n)\, ds_1 \cdots ds_n$$

with product $\alpha(s_i) \cdot \alpha(s_j)$ in $\mathcal{H}^\vee$ = dual Hopf algebra
and $1 \in \mathcal{H}^\vee$ = counit of $\mathcal{H}$

Prop. Properties of $T e^{\int_a^b \alpha(t)\,dt}$:

(1) When paired with any $X \in \mathcal{H}$ sum is finite

(2) $T e^{\int_a^b \alpha(t)\,dt} \in G(\mathbb{C})$

(3) $T e^{\int_a^b \alpha(t)\,dt} = g(b)$ where $g(t) \in G(\mathbb{C})$ is the unique solution to the differential equation

$$dg(t) = g(t)\,\alpha(t)\,dt \qquad \text{with } g(a) = 1$$

"logarithmic derivative $g(t)^{-1} dg(t)$ "

(4) Multiplicative over sum of paths

$$Te^{\int_a^c \alpha(t)dt} = Te^{\int_a^b \alpha(t)dt} \cdot Te^{\int_b^c \alpha(t)dt}$$

(5) $$\left(Te^{\int_a^b \alpha(t)dt}\right)^{-1} = T'e^{-\int_a^b \alpha(t)dt}$$

where $$T'e^{\int_a^b \alpha(t)dt} := 1 + \sum_{n=1}^{\infty} \int_{0 \le s_1 \le \cdots \le s_n \le 1} \underbrace{\alpha(s_n)\cdots\alpha(s_1)}_{\text{reversed order}} ds_1 \ldots ds_n$$

(the dual Hopf algebra $\mathcal{H}^\vee$ is <u>not</u> commutative)

<u>Pf</u>: (1) $\langle \alpha(s_1)\cdots\alpha(s_m), X\rangle = 0$ for ~~deg~~ all $m > \deg(X)$

Solution of diff eq. (3) $\partial_t \langle Te^{\int_a^t \alpha(s)ds}, X\rangle = \langle Te^{\int_a^t \alpha(s)ds}\alpha(t), X\rangle$

since $Te^{\int_a^t \alpha(s)ds} = 1 + \sum_n \int_{0 \le s_1 \le \cdots \le s_n \le t} \alpha(s_1)\cdots\alpha(s_n)\, ds_1 \ldots ds_n$

element in $G(\mathbb{C})$ (2) To check $Te^{\int_a^b \alpha(t)dt} \in G(\mathbb{C})$

check in $\mathrm{Hom}(\mathcal{H}, \mathbb{C})$ i.e.

$$\langle Te^{\int_a^b \alpha(t)dt}, XY\rangle = \langle Te^{\int_a^b \alpha(t)dt}, X\rangle \langle Te^{\int_a^b \alpha(t)dt}, Y\rangle$$

By induction on sum of degrees of X, Y.

Suppose $\deg(X) = m$, $\deg(Y) = r$ ~~...~~: X, Y homogeneous elements in $\mathcal{H}$ (enough to check for those)

Check that $E(t) = \langle Te^{\int_a^t \alpha(s)ds}, XY\rangle - \langle Te^{\int_a^t \alpha(s)ds}, X\rangle\langle Te^{\int_a^t \alpha(s)ds}, Y\rangle$

satisfies $\partial_t E_t = 0 \quad \forall t$ (then since $=0$ for $t = a$ ok)

$$\partial_t \langle Te^{\int_a^t \alpha(s)ds}, XY\rangle = \langle Te^{\int_a^t \alpha(s)ds}\alpha(t), XY\rangle$$

$$= \langle Te^{\int_a^t \alpha(s)ds} \otimes \alpha(t), \underbrace{\Delta(XY)}_{\Delta(X)\Delta(Y)}\rangle$$

$$\Delta(X) = \Sigma\, X_{(1)} \otimes X_{(2)}$$
$$\Delta(Y) = \Sigma\, Y_{(1)} \otimes Y_{(2)}$$

$$= \langle Te^{\int_a^t \alpha(s)ds}, X_{(1)}Y\rangle \langle \alpha(t), X_{(2)}\rangle$$

$$+ \langle Te^{\int_a^t \alpha(s)ds}, XY_{(1)}\rangle \langle \alpha(t), Y_{(2)}\rangle$$

all other terms are zero since

$\alpha(t) \in \mathfrak{g}(\mathbb{C})$ element in the Lie algebra

$\Rightarrow$ $\alpha(t)$: linear functional $H \to \mathbb{C}$ such that

$$\langle \alpha(t), XY\rangle = \alpha(t)(X)\,\varepsilon(Y) + \varepsilon(X)\,\alpha(t)(Y)$$

but augmentation $\varepsilon(X) = 0$ for $\deg(X) > 0$

and so only terms that remain are

$X_{(2)}Y_{(2)}$ where either $Y_{(2)} = 1$ or $X_{(2)} = 1$

So get only terms

$$\langle Te^{\int_a^t \alpha(s)ds}, X_{(1)}Y_{(1)}\rangle \langle \alpha(t), X_{(2)}Y_{(2)}\rangle \quad \text{as above}$$

$Y \otimes 1$ or

$X \otimes 1$

Then using induction hypothesis → note $\langle \alpha(t), 1\rangle = 0$ so only terms with $\deg X_{(1)} < \deg X$ appear

$$\langle Te^{\int_a^t \alpha(s)ds}, X_{(1)}Y\rangle = \langle Te^{\int_a^t \alpha(s)ds}, X_{(1)}\rangle \langle Te^{\int_a^t \alpha(s)ds}, Y\rangle$$

and similarly for $XY_{(1)}$

$$\Rightarrow \partial_t \langle Te^{\int_a^t \alpha(s)ds}, XY \rangle = \partial_t\left(\langle Te^{\int_a^t \alpha(s)ds}, X\rangle\right) \cdot \langle Te^{\int_a^t \alpha(s)ds}, Y\rangle$$
$$+ \langle Te^{\int_a^t \alpha(s)ds}, X\rangle \, \partial_t\left(\langle Te^{\int_a^t \alpha(s)ds}, Y\rangle\right)$$

hence get $\partial_t E(t) = 0 \quad \forall t$

More general formulation

instead of $[a,b]$ a path $\gamma : [0,1] \to \Omega \subset \mathbb{C}$

$$\omega = \alpha(s,t)\,ds + \eta(s,t)\,dt \qquad (s,t) \in \Omega \subset \mathbb{C} = \mathbb{R}^2$$

$\mathfrak{g}(\mathbb{C})$-valued connection

flat: $\quad \partial_s \eta - \partial_t \alpha + [\alpha, \eta] = 0 \qquad$ (no curvature)

Prop: $Te^{\int_0^1 \gamma^* \omega}$ only depends on the homotopy ~~type~~ class of the path $\gamma : [0,1] \to \Omega$ with fixed endpoints $\gamma(0) = a \quad \gamma(1) = b \in \Omega$.

~~Def~~ Differential equations and monodromy condition

(K, δ) = differential field $\qquad \delta(f+h) = \delta(f) + \delta(h)$
$\qquad \delta(fh) = \delta(f)h + f\,\delta(h)$

subfield of constants $\mathrm{Ker}(\delta) \subset K$

Examples: $K = \mathbb{C}(\{z\})$ convergent Laurent series $(\mathrm{Ker}\,\delta = \mathbb{C})$
$\delta(f) = \frac{d}{dz} f \quad$ or $K = \mathbb{C}[[z]][z^{-1}] = \mathbb{C}((z))$ formal Laurent series

(K, δ) Differential field

$G(K) = \mathrm{Hom}(\mathcal{H}, K)$ has "logarithmic derivative"

$$D : G(K) \longrightarrow \mathfrak{g}(K) = \mathrm{Lie}\, G(K)$$

$$D(g) := g^{-1}\, \delta \circ g \qquad \delta \circ g\,(X) = \delta\,(g(X))$$

$D(g) \in \mathrm{Lie}\, G(K)$ since $\langle D(g), X \rangle = \langle g^{-1} \otimes \delta \circ g, \Delta X \rangle$

and $\Rightarrow$ $\langle D(g), XY \rangle = \langle D(g), X \rangle\, \varepsilon(Y) + \varepsilon(X) \langle D(g), Y \rangle$

Consider then case of $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ $\quad \mathcal{H}_0 = \mathbb{C}$

with $G = \varprojlim G_n$

Given $\omega \in \mathfrak{g}(K) = \mathrm{Lie}\, G(K)$ $\quad K = \mathbb{C}(\{z\})$

- Monodromy representation

  $$M_n(\omega) : \mathbb{Z} \longrightarrow G_n(\mathbb{C})$$

- Condition $M(\omega) = 1$ well defined indep. of $n \geq 0$

Fixed n : $G_n(\mathbb{C})$ fin. dim. Lie group

ω_n = restriction of ω to $\mathcal{H}_n$

$\Rightarrow$ depends only on fin. many elements of K

$\Rightarrow$ finite radius of convergence $\Delta_n^* \subset \mathbb{C}^*$

$$M_n(\omega)(\gamma) := T e^{\int_0^1 \gamma^* \omega_n} \in G_n(\mathbb{C})$$

$\gamma : [0,1] \to \Delta_n^*$

Defines representation

$$M_n(\omega) : \pi_1(\Delta_n^*, z_n) \to G_n(\mathbb{C})$$

The condition $M(\omega)=1$
requires passing to limit $n\to\infty$
(then no common positive radius Δ^*
$\Rightarrow$ no common base point $z\in\Delta^*$)

need to move base pt $z_n \in \Delta_n^*$ for different n
$\Rightarrow$ Conjugation on $\pi_1(\Delta_*^*)$

does not affect $M_n(\omega)=1$ for all n

$\Rightarrow$ condition $M(\omega)=1$ (trivial monodromy)
well defined for $\omega\in \mathfrak{g}(K)$

Differential equation $D(g)=\omega$

Necessary condition for the existence of solutions
$M(\omega)=1$

<u>Example</u>: $G=\mathbb{G}_a$ additive group $D(f)=\delta(f)=f'$ $\omega\in K$

$\operatorname{Res}_{z=0} \omega$ obstruction to existence of solutions to $f'=\omega$

Prop: if $M(\omega)=1$ and G affine group scheme of $H = \bigoplus_{n \geq 0} H_n$, $H_0 = \mathbb{C}$

$\Rightarrow$ $\exists$ solution $g \in G(K)$ of equation

$$D(g) = \omega$$

(So for this type of G also sufficient condition)

Pf: First work with a fixed G_n

define $g_n(z) = T e^{\int_{z_n}^{z} \omega_n}$ ← i.e. along a path $\gamma: [0,1] \to \Delta_n^*$, $\gamma(0) = z_n$, $\gamma(1) = z$

Does not depend on path (because $M(\omega)=1$)

$\Rightarrow$ have solutions $D(g_n) = \omega_n$

need to show can make a consistent choice for all n

g, h two different solutions then gh^{-1} satisfies $D(gh^{-1}) = 0$

i.e. $gh^{-1}(x) \in \mathrm{Ker}(\delta) = \mathbb{C}$

$\Rightarrow$ $gh^{-1} = a \in \mathrm{Hom}(H_n, \mathbb{C}) = G_n(\mathbb{C})$

projections $G_{n+1}(\mathbb{C}) \longrightarrow G_n(\mathbb{C})$

from inclusions $H_n \subset H_{n+1}$

$\Rightarrow$ inductively modify solution $g_{n+1} \in G_{n+1}(K)$ by an element $a_{n+1} \in G_{n+1}(\mathbb{C})$ so that it projects onto $g_n \in G_n(K)$

Gross – 't Hooft relations and time ordered exponential

Thm (CM): $\gamma_-(z) = T e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\,dt}$

from $\gamma_-(z)^{-1} = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n}$ with $Y(d_{n+1}) = d_n \beta$, $Y(d_1) = \beta$

first show

$$d_1 = \int_0^\infty \theta_{-s}(\beta)\,ds \qquad \circledast$$

Since have for any $X \in \mathcal{H}$ with $\varepsilon(X) = 0$

$$Y\left(\int_0^\infty \theta_{-s}(X)\,ds\right) = X$$

if X of deg n
$\theta_s(X) = e^{sn} X$
$\int_0^\infty e^{-sn} ds = \frac{1}{n}$
$Y(\frac{1}{n} X) = X$

then $\forall\, \alpha, \alpha' \in \mathcal{H}'$ with $\alpha' = Y(\alpha)$ and $\langle \alpha, 1\rangle = \langle \alpha', 1\rangle = 0$ get

$$\alpha = \int_0^\infty \theta_{-s}(\alpha')\,ds \qquad (**)$$

$\Rightarrow$ apply to $Y(d_1) = \beta$ to get $\circledast$

Then proceed inductively to get

$$d_n = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta)\, \theta_{-s_2}(\beta) \cdots \theta_{-s_n}(\beta)\, ds_1 \cdots ds_n$$

Using $(**)$ and the $d_n\beta = Y(d_{n+1})$

$$\gamma_-(z)^{-1} = T' e^{\frac{1}{z}\int_0^\infty \theta_{-s}(\beta)ds}$$
$$\Updownarrow$$
$$\Rightarrow \gamma_-(z) = T e^{-\frac{1}{z}\int_0^\infty \theta_{-s}(\beta)ds}$$

$$1 + \sum_{n=1}^{\infty} \frac{1}{z^n} \int_{s_1 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta) \cdots \theta_{-s_n}(\beta)\, ds_1 \cdots ds_n$$

Flat equisingular connections and the Riemann-Hilbert correspondence

$$\gamma_-(z) \rightsquigarrow T e^{-\frac{1}{z}\int_0^\infty \theta_s(\beta)\,ds} \rightsquigarrow$$ unique solution of a diff. equation

Thm (CM): $\gamma_\mu(z)$ loop $\Delta^* \longrightarrow G(\mathbb{C})$ $G = CK$ group scheme

with properties on scaling of μ $\gamma_\mu \in L(G(\mathbb{C}),\mu)$

Then

$$\gamma_\mu(z) = T e^{-\frac{1}{z}\int_\infty^{-z\log\mu} \theta_{-t}(\beta)\,dt} \; \theta_{z\log\mu}(\gamma_{reg}(z))$$

for a unique element $\beta \in Lie\,G(\mathbb{C})$

with Birkhoff factorization

$$\gamma_{\mu+}(z) = T e^{-\frac{1}{z}\int_0^{-z\log\mu} \theta_{-t}(\beta)\,dt} \; \theta_{z\log\mu}(\gamma_{reg}(z))$$

$$\gamma_-(z) = T e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\,dt}$$

Proof: Set $\alpha_\mu(z) := \theta_{z\log\mu}(\gamma_-(z)^{-1})$

Satisfies $\alpha_{e^s\mu}(z) = \theta_{sz}(\alpha_\mu(z))$ scaling property

then take

$\alpha_\mu(z)^{-1}\gamma_\mu(z)$ also satisfies scaling property and regular at $z=0$

true for $\mu = 1$ and by scaling

$$\Rightarrow \quad \alpha_\mu(z)^{-1}\,\gamma_\mu(z) = \theta_{z\log\mu}\big(\gamma_{reg}(z)\big)$$

regular at $z=0$

Moreover since $\gamma_-(z) = T e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\,dt}$

and $\alpha_\mu(z)^{-1} = \theta_{z\log\mu}(\gamma_-(z)) = T e^{-\frac{1}{z}\int_{-z\log\mu}^{\infty} \theta_{-t}(\beta)\,dt}$

$$= T e^{-\frac{1}{z}\int_{-z\log\mu}^{0} \theta_{-t}(\beta)\,dt}\; T e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\,dt}$$

$$\Rightarrow \quad \gamma_\mu(z) = T e^{-\frac{1}{z}\int_0^{-z\log\mu} \theta_{-t}(\beta)\,dt}\;\; \theta_{z\log\mu}\big(\gamma_{reg}(z)\big)$$

$$= \alpha_\mu(z)\,\theta_{z\log\mu}\big(\gamma_{reg}(z)\big)$$

$\Rightarrow$ also expression for Birkhoff factorization

So up to stuff that is convergent at zero

$\gamma_\mu(z)$ determined by an element in $Lie(G) \ni \beta$

(again a rephrasing of Gross-'t Hooft)
counterterms depend only on
the β function

How to make the "regular part"
disappear into an equivalence relation

$\gamma_\mu(z) \sim \gamma'_\mu(z)$ iff have same negative piece of Birkhoff factorization

$$\gamma_-(z) = \gamma'_-(z)$$

In fact this is a gauge equivalence on $\underbrace{\text{connections}}_{\text{Lie}(G)\text{-valued}}$

$$D : G(K) \to \Omega^1(\mathfrak{g})$$

$$D(f) = f^{-1} df$$

$$d : K \to \Omega^1$$

$$d(f) = \delta(f)\, dz = \frac{df}{dz}\, dz \qquad K = \mathbb{C}(\{z\})$$

$$D(fh) = Dh + h^{-1} Df\, h$$

For $\omega \in \Omega^1(\mathfrak{g})$ differential equation $D(f) = \omega$ in $\Omega^1(\mathfrak{g})$

existence of solution under trivial monodromy condition $M(\omega) = 1$

$\omega \in \Omega^1(\mathfrak{g})$ connection on the (trivial) principal G-bundle $\Delta^* \times G$

Def: Gauge equivalence of connections $\omega \sim \omega'$ iff

$$\omega' = Dh + h^{-1}\omega h$$

($f, g \in G(K)$ same singularities if $fg^{-1} \in G(\mathcal{O})$)

for some $h \in G(\mathcal{O})$ $\qquad$ $\mathcal{O} = \mathbb{C}\{z\}$ convergent power series

i.e. solutions have same type of singularities at $z=0$

Prop: $\omega \sim \omega'$ iff solutions f^ω of $D(f) = \omega$ and $f^{\omega'}$ of $D(f) = \omega'$ have the same negative piece of Birkhoff factorization

$$G(K) \ni f^\omega = \underbrace{(f_-^\omega)^{-1}}_{\in G(\tilde{\mathcal{Q}})} \underbrace{f_+^\omega}_{\in G(\mathcal{O})} \qquad \underbrace{f^{\omega'}}_{\in G(K)} = \underbrace{(f_-^{\omega'})^{-1}}_{\in G(\tilde{\mathcal{Q}})} \underbrace{f_+^{\omega'}}_{\in G(\mathcal{O})}$$

$$f_-^\omega \equiv f_-^{\omega'} \qquad \in G(\tilde{\mathcal{Q}})$$

Proof: first check that

$$\omega \sim D((f_-^\omega)^{-1}) \qquad \text{in fact:}$$

$$\omega = D(f^\omega) = D((f_-^\omega)^{-1} f_+^\omega) = D(f_+^\omega) + {f_+^\omega}^{-1} D((f_-^\omega)^{-1}) f_+^\omega$$

$$\Rightarrow \text{if } f_-^\omega = f_-^{\omega'} \Rightarrow D(f_-^\omega) = D(f_-^{\omega'}) \Rightarrow \omega \sim \omega'$$

Converse: $\omega' = Dh + h^{-1}\omega h$ and f^ω sol'n $D(f) = \omega$ $\Rightarrow$ $f^{\omega'} = f^\omega h$ is solution $D(f) = \omega'$, $h \in G(\mathcal{O})$ so same $f_-^\omega = f_-^{\omega'}$.