Input also the μ-dependence geometrically

A principal $\mathbb{C}^* = G_m(\mathbb{C})$ - bundle B

$$G_m \to B \to \Delta$$ over the infinitesimal disk Δ

Note: trivial bundle, but do not fix a choice of a trivialization $\sigma : \Delta \to B$

Then take $P = B \times G$ trivial (and trivialized) principal G-bundle over B

Physical meaning of these data:

G = group dual to CK Hopf algebra of Feynman graphs

$z \in \Delta$ = complexified dimension of Dim Reg

$\mathbb{C}^*$ = fibre over $z \in \Delta$ is $\mu^{z\hbar}$

Choice of a unit of mass μ = choice of a section $\sigma : \Delta \to B$ of fibration

value $\sigma(0) = \hbar$

G_m - action on B by $b \mapsto u(b)$ in a trivialization

$$b = (z, \sigma(z)) \quad u(b) = (z, u(\sigma(z)))$$

on $P = B \times G$

$$u(b,g) = (u(b), u^Y(g))$$

Equisingular connections:

Def: A flat connection on P^0 is equisingular iff

- G_m-invariant: $\omega(z, u(v)) = u^Y(\omega(z,v))$ $\forall u \in G_m$
- Different sections $\sigma_1, \sigma_2 : \Delta \to B$ with same $\sigma_1(0) = \sigma_2(0)$ have $\sigma_1^*(\gamma) \sim \sigma_2^*(\gamma)$ for γ solution of $D(\gamma) = \omega$

Notation: $V = \pi^{-1}(0) \subset B$ fiber over $z = 0$
$B^0 = B \setminus V$ complement
$P^0 = B^0 \times G$

The equivalence relation $\sigma_1^*(\gamma) \sim \sigma_2^*(\gamma)$ is the one of having the same negative piece of the Birkhoff factorization as before

i.e. $\sigma_1^*(\gamma)$ and $\sigma_2^*(\gamma)$ have the same type of singularity at $z = 0$ (equisingular)

The first condition

$$\omega(z, u(v)) = u^Y(\omega(z,v))$$

is a rephrasing of the scaling property of loops

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z))$$

while the second condition

$$\sigma_1^*(\gamma) \sim \sigma_2^*(\gamma)$$

is a geometric reformulation of the condition

$$\frac{\partial}{\partial \mu} \gamma_-(z) = 0$$

Equivalence relation on equisingular connections

$\omega \sim \omega'$ on P° iff

$$\omega' = Dh + h^{-1}\omega h$$

where h is a $\mathbb{G}_m$-invariant map regular in B

$h : B \longrightarrow G$

Thm (CM): There is a bijective correspondence between equivalence classes of flat equisingular G-connections on P° and elements $\beta \in \mathfrak{g}(\mathbb{C}) = \mathrm{Lie}\, G(\mathbb{C})$ with

(1) $\omega \sim D\gamma$

$$(*) \quad \gamma(z,v) = \mathrm{T}\, e^{-\frac{1}{2}\int_0^v u^Y(\beta)\,\frac{du}{u}}$$

fixing a trivialization $\sigma : \Delta \to B$

integration along straight path $u = tv \quad t \in [0,1]$

(2) $(*)$ Independent of choice of trivialization $\sigma : \Delta \to B$ with given $\sigma(0)$.

Pf: First show vanishing of two monodromy representations $M_{\Delta^*}(\omega)$ and $M_{\mathbb{C}^*}(\omega)$ around the two generators of $\pi_1(B^\circ)$

(This ensures existence of solutions to $D(\gamma) = \omega$)

Δ^* infinitesimal disk $\Delta^* = \bigcap_n \Delta_n^*$ corresp. to G_n

- Monodromy $M_{\{z_0\}\times\mathbb{C}^*}(\omega)$ is trivial
  because ω extends to $\{z_0\}\times\mathbb{C}$ (only over $\{z=0\}$ singular)

- Monodromy $M_{\Delta^*\times\{u\}}(\omega)$ also trivial
  in fact can pick $u=0$ but there
  restriction of ω to $\Delta^*\times\{0\}$ is identically zero
  (also indep of where choose u $\Rightarrow$ trivial for all u)

$\Rightarrow$ equation $D\gamma = \omega$ has solutions

Notice:

$$\omega(z,u) = u^Y(a(z))\,dz + u^Y(b(z))\,\frac{du}{u}$$

$a, b \in \mathfrak{g}(K)$ flat: $\frac{db}{dz} - Y(a) + [a,b] = 0$

positivity of grading $\Rightarrow$ ω extends to $\Delta^*\times\mathbb{C}$

Given $(z_0, 0) \in \Delta_n^* \times \{0\}$ construct solution using
path from $(z_0,0)$ to $(z,0)$ and then
path (z, tv) $t\in[0,1]$ $\gamma = \gamma_1 \cup \gamma_2$

$$\gamma^*\omega = \underset{0}{\gamma_1^*\omega} + \underset{u^Y(b(z))}{\gamma_2^*\omega}$$

$$\gamma(z,v) = T\, e^{\int_0^v u^Y(b(z))\,\frac{du}{u}}$$

satisfies $\gamma(z, wu) = w^Y(\gamma(z,u))$ $w\in\mathbb{C}^*$

$\gamma(z,u) = u^Y\gamma(z)$ $\gamma(z) = \gamma(z,v)\big|_{v=1}$

Since $D\gamma = \omega$ $\Rightarrow$ $\gamma(z)^{-1}d\gamma(z) = a(z)\,dz$, $\gamma(z)^{-1}Y\gamma(z) = b(z)$

Section $\sigma_s(z) = (z, e^{sz}) \qquad z \in \Delta$ (5)

$$\sigma_s^*(\gamma)(z) =: \gamma_s(z) = \theta_{sz}\, \gamma(z)$$

and $\frac{\partial}{\partial s} \gamma_{s-}(z) = 0$

i.e. get loop $\gamma_\mu(z)$ $\mu = e^s$ in the right class of "physical" loops

then know from previous results $\exists\, \beta \in Lie(G(\mathbb{C}))$ s.t.

$$\gamma(z,1) = T e^{-\frac{1}{z}\int_\infty^0 \theta_{-t}(\beta)\,dt} \cdot \gamma_{reg}(z)$$

so that

$$\gamma(z, v) = v^Y\left(T e^{-\frac{1}{z}\int_\infty^0 \theta_{-t}(\beta)\,dt}\right) v^Y(\gamma_{reg}(z))$$

$$v^Y\left(T e^{-\frac{1}{z}\int_\infty^0 \theta_{-t}(\beta)\,dt}\right) = T e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}$$

$$\Rightarrow \gamma(z,v) = \left(T e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}\right) v^Y(\gamma_{reg}(z))$$

use this as gauge equivalence h

$$\Rightarrow \omega \sim D\left(T e^{-\frac{1}{z}\int_0^v u^Y(\beta)\,\frac{du}{u}}\right)$$

- If $\omega' \sim \omega$ $\quad \omega' = Dh + h^{-1}\omega h \qquad \gamma_2(z,u) = \gamma_1(z,u)\, h(z,u)$

$\Rightarrow \gamma_i$ same neg. part of Birkhoff $\Rightarrow \beta_2 = \beta_1$ (from equality of residues at zero)

- If have $\gamma(z,v) = T e^{-\frac{1}{z}\int_0^v u^Y(\beta)\frac{du}{u}}$

  then $\omega = D\gamma$ is equisingular (enough check on one solutions: others differ by $g \in G(\mathbb{C})$)

Given a section

$$\sigma : \Delta \to B \qquad v(z)\sigma(z) \text{ varying section}$$

$$\gamma_v(z) = T e^{-\frac{1}{z}\int_0^{v(z)} u^Y(\beta)\frac{du}{u}}$$

$$= \underbrace{\left(T e^{-\frac{1}{z}\int_0^1 u^Y(\beta)\frac{du}{u}}\right)}_{\substack{\text{polyn } z^{-1} \\ \text{when paired} \\ \text{w/ } x \in \mathcal{H}}} \underbrace{\left(T e^{-\frac{1}{z}\int_1^{v(z)} u^Y(\beta)\frac{du}{u}}\right)}_{\text{regular at } z=0}$$

This shows indep. of choice of trivialization of (*)

The universal Hopf algebra and group $\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$

principal bundles

$P = B \times G$

vector bundles

$E = B \times V$

$$\rho : G \to GL(V)$$

Can accommodate simultaneously different G's (for different theories)

Def: (E, W) filtered vector bundle E on B
increasing filtration

$$W^{-n-1}(E) \subset W^{-n}(E)$$

and trivialization of associated graded

$$Gr_n^W(E) = W^{-n}(E) \big/ W^{-n-1}(E)$$

A W-connection ∇ on E:

connection on $E^\circ = E|_{B^\circ}$ (i.e. singular on $\pi^{-1}(0)$)

Such that

1) ∇ compatible w/ filtration: restricts to a connection on each $W^{-n}(E^\circ)$

2) ∇ induces the trivial connection on $Gr_n^W(E)$

Equivalence: $\nabla_1 \sim \nabla_2$ on E° (W-equivalence)

iff $\exists$ $T \in Aut(E)$ s.t. T preserves the filtration, induces identity on $Gr_n^W(E)$, and

$$T \circ \nabla_1 = \nabla_2 \circ T$$

intertwines the connections

Equisingular : ∇

V = fin. dim. $\mathbb{Z}$-graded vector space

$$V = \bigoplus_{n \in \mathbb{Z}} V_n$$

$E = B \times V$ $\quad\quad$ (E, W) with

$$W^{-n}(E) = \bigoplus_{m \geq n} V_m$$

$\mathbb{G}_m$-action induced by grading

Def: ∇ a W-connection on $E = B \times V$ equisingular iff $\mathbb{G}_m$-invariant and

$\nabla \eta = 0$ solutions (fundam. system of solutions = basis of flat sections)

$$\sigma_1^*(\eta) \sim \sigma_2^*(\eta) \quad \text{for } \sigma_1(0) = \sigma_2(0)$$

$$\sigma_i : \Delta \longrightarrow B$$

A category of flat equisingular vector bundles (E, ∇)

$\mathcal{E}$ Objects of $\mathcal{E}$: pairs $\Theta = [V, \nabla]$

W-equivalence class of ∇

Morphisms $\mathrm{Hom}_{\mathcal{E}}(\Theta, \Theta')$

$T : V \to V'$ compatible w/ grading and s.t.

on the bundle $(E' \oplus E)^*$ the W-connection

$$\nabla_1 := \begin{pmatrix} \nabla' & 0 \\ 0 & \nabla \end{pmatrix} \text{ and}$$

$$\nabla_2 = \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix}^{-1} \nabla_1 \begin{pmatrix} 1 & T \\ 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{pmatrix}$$

are W-equivalent: $\nabla_2 \sim \nabla_1$

To avoid a problem with morphisms of filtered vector spaces (not abelian category)

- If G dual to $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$
  ω flat equisingular on $P = B \times G$

$\Rightarrow$ given $\xi : G \to GL(V)$ repres.

(ω, ξ) determine an element $\Theta = [E, \nabla] \in \mathrm{Obj}(\mathcal{E})$

and $\omega \sim \omega' \Rightarrow$ same Θ

Notice : an affine group scheme G is completely determined by its category of finite dimensional linear representations $\mathcal{R}_G$

So replacing $\cancel{P} B \times G$ by all the possible
$E = B \times V$ with $\xi : G \to GL(V)$
maintains same information

$\Rightarrow$ Way to recover each G for each particular theory inside a common $\mathcal{E}$ category

Theorem (CM) : $\boxed{\mathcal{E} \cong \mathcal{R}_{\mathbb{U}^*}}$ finite dim linear representations

$\mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$ where

$\mathcal{L}_{\mathbb{U}} = \mathcal{F}(1,2,3,\ldots)_\bullet$ = free graded Lie algebra with one generator e_{-n} in each degree $n \geq 1$

$\mathcal{H}_{\mathbb{U}} = U(\mathcal{F}(1,2,3,\ldots)_\bullet)^\vee$ Hopf algebra

Renormalization group

$$e = \sum_{n=1}^{\infty} e_{-n} \quad \text{(pronilpotent Lie algebra)}$$

$rg : \mathbb{G}_a \to \mathbb{U}$ additive subgroup generated by e

Beta function $\beta = \sum_{n=1}^{\infty} \beta_n \in \mathrm{Lie}\, G$

components of $\deg = n$ of beta function

$\Rightarrow \quad e_{-n} \longmapsto \beta_n$

morphism from

$\mathrm{Lie}\, \mathbb{U} \longrightarrow \mathrm{Lie}\, G$

integrates to morphism $\mathbb{U} \longrightarrow G$

then $\mathbb{U}$ maps to all G's of different physical theories through their β-function

"Universal counterterms"