Standard model, gravity, neutrino mixing, NCG

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joint with A.Chamseddine and A.Connes

Gravity coupled to matter

Flat space $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

Curved space, gravitational potential $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu \, dx^\nu$$

Action principle

$$S_{EH}(g_{\mu\nu}) = \frac{1}{G} \int_M R \sqrt{g} \; d^4 x$$

Gravity minimally coupled to matter :

$$S = S_{EH} + S_{SM}$$

Classical $\rightarrow$ Quantum

$$e^{i \frac{S}{\hbar}}$$

$$
\begin{gathered}
\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu\partial_\nu W^-_\mu - W^-_\nu\partial_\nu W^+_\mu)) - \\
igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu\partial_\nu W^-_\mu - W^-_\mu\partial_\nu W^+_\mu) + A_\mu(W^+_\nu\partial_\nu W^-_\mu - \\
W^-_\nu\partial_\nu W^+_\mu)) - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - \\
Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H\partial_\mu H - 2M^2\alpha_h H^2 - \partial_\mu\phi^+\partial_\mu\phi^- - \frac{1}{2}\partial_\mu\phi^0\partial_\mu\phi^0 - \\
\beta_h\left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2+\phi^0\phi^0+2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h - \\
g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\frac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) - \\
gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \\
\frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \\
\frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + \\
M\left(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+\right) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - \\
W^-_\mu\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\frac{1}{4}g^2W^+_\mu W^-_\mu\left(H^2+(\phi^0)^2+2\phi^+\phi^-\right) - \frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2+(\phi^0)^2+2(2s_w^2-1)^2\phi^+\phi^-\right) - \\
\frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \frac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2-1)Z^0_\mu A_\mu\phi^+\phi^- - \\
g^2s_w^2A_\mu A_\mu\phi^+\phi^- + \frac{1}{2}ig_s\,\lambda^a_{ij}(\bar q^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar e^\lambda(\gamma\partial+m^\lambda_e)e^\lambda - \bar\nu^\lambda(\gamma\partial+m^\lambda_\nu)\nu^\lambda - \bar u^\lambda_j(\gamma\partial+ \\
m^\lambda_u)u^\lambda_j - \bar d^\lambda_j(\gamma\partial+m^\lambda_d)d^\lambda_j + igs_wA_\mu\left(-(\bar e^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar u^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar d^\lambda_j\gamma^\mu d^\lambda_j)\right) + \\
\frac{ig}{4c_w}Z^0_\mu\{(\bar\nu^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar e^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda) + (\bar d^\lambda_j\gamma^\mu(\frac{4}{3}s_w^2-1-\gamma^5)d^\lambda_j) + \\
(\bar u^\lambda_j\gamma^\mu(1-\frac{8}{3}s_w^2+\gamma^5)u^\lambda_j)\} + \frac{ig}{2\sqrt{2}}W^+_\mu\left((\bar\nu^\lambda\gamma^\mu(1+\gamma^5)U^{lep}{}_{\lambda\kappa}e^\kappa) + (\bar u^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right) + \\
\frac{ig}{2\sqrt{2}}W^-_\mu\left((\bar e^\kappa U^{lep\dagger}{}_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar d^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \\
\frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_e(\bar\nu^\lambda U^{lep}{}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar\nu^\lambda U^{lep}{}_{\lambda\kappa}(1+\gamma^5)e^\kappa)\right) + \\
\frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_e(\bar e^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m^\kappa_\nu(\bar e^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1-\gamma^5)\nu^\kappa)\right) - \frac{g}{2}\frac{m^\lambda_\nu}{M}H(\bar\nu^\lambda\nu^\lambda) - \\
\frac{g}{2}\frac{m^\lambda_e}{M}H(\bar e^\lambda e^\lambda) + \frac{ig}{2}\frac{m^\lambda_\nu}{M}\phi^0(\bar\nu^\lambda\gamma^5\nu^\lambda) - \frac{ig}{2}\frac{m^\lambda_e}{M}\phi^0(\bar e^\lambda\gamma^5e^\lambda) - \frac{1}{4}\,\bar\nu_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat\nu_\kappa - \\
\frac{1}{4}\,\bar\nu_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat\nu_\kappa + \frac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar u^\lambda_jC_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar u^\lambda_jC_{\lambda\kappa}(1+\gamma^5)d^\kappa_j)\right) + \\
\frac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar d^\lambda_jC^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar d^\lambda_jC^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j)\right) - \frac{g}{2}\frac{m^\lambda_u}{M}H(\bar u^\lambda_ju^\lambda_j) - \\
\frac{g}{2}\frac{m^\lambda_d}{M}H(\bar d^\lambda_jd^\lambda_j) + \frac{ig}{2}\frac{m^\lambda_u}{M}\phi^0(\bar u^\lambda_j\gamma^5u^\lambda_j) - \frac{ig}{2}\frac{m^\lambda_d}{M}\phi^0(\bar d^\lambda_j\gamma^5d^\lambda_j) + \bar G^a\partial^2G^a + g_sf^{abc}\partial_\mu\bar G^aG^bg^c_\mu + \\
\bar X^+(\partial^2-M^2)X^+ + \bar X^-(\partial^2-M^2)X^- + \bar X^0(\partial^2-\frac{M^2}{c_w^2})X^0 + \bar Y\partial^2Y + igc_wW^+_\mu(\partial_\mu\bar X^0X^- - \\
\partial_\mu\bar X^+X^0) + igs_wW^+_\mu(\partial_\mu\bar YX^- - \partial_\mu\bar X^+Y) + igc_wW^-_\mu(\partial_\mu\bar X^-X^0 - \\
\partial_\mu\bar X^0X^+) + igs_wW^-_\mu(\partial_\mu\bar X^-Y - \partial_\mu\bar YX^+) + igc_wZ^0_\mu(\partial_\mu\bar X^+X^+ - \\
\partial_\mu\bar X^-X^-) + igs_wA_\mu(\partial_\mu\bar X^+X^+ - \\
\partial_\mu\bar X^-X^-) - \frac{1}{2}gM\left(\bar X^+X^+H + \bar X^-X^-H + \frac{1}{c_w^2}\bar X^0X^0H\right) + \frac{1-2c_w^2}{2c_w}igM\left(\bar X^+X^0\phi^+ - \bar X^-X^0\phi^-\right) + \\
\frac{1}{2c_w}igM\left(\bar X^0X^-\phi^+ - \bar X^0X^+\phi^-\right) + igMs_w\left(\bar X^0X^-\phi^+ - \bar X^0X^+\phi^-\right) + \\
\frac{1}{2}igM\left(\bar X^+X^+\phi^0 - \bar X^-X^-\phi^0\right)\,.
\end{gathered}
$$

Main purpose :

Simple mathematical input $\Rightarrow$ Machine (NCG)
$\Rightarrow$ SM+Grav Lagrangian
+ physical predictions

Hint of why NCG :

Symmetries of $S = S_{EH} + S_{SM}$
$G = U(1) \times SU(2) \times SU(3)$

$$\mathcal{G} = \text{Map}(M, G) \rtimes \text{Diff}(M)$$

Is it $\mathcal{G} = \text{Diff}(X)$? Not for a manifold, yes for an NC space

e.g. $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$ $\quad G = PSU(n)$

$$1 \to \text{Inn}(\mathcal{A}) \to \text{Aut}(\mathcal{A}) \to \text{Out}(\mathcal{A}) \to 1$$

$$1 \to \text{Map}(M, G) \to \mathcal{G} \to \text{Diff}(M) \to 1.$$

Spectral triples $(\mathcal{A}, \mathcal{H}, D)$

- involutive algebra $\mathcal{A}$
- representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator D on $\mathcal{H}$
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$

even if $\mathbb{Z}/2$- grading γ on $\mathcal{H}$

$$[\gamma, a] = 0, \ \forall a \in \mathcal{A}, \qquad D\gamma = -\gamma D$$

Example $(C^\infty(M), L^2(M, S), \partial\!\!\!/_M)$

chirality operator γ_5 in 4-dim

Real structure of KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$

antilinear isometry $J : \mathcal{H} \to \mathcal{H}$

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon''\gamma J$$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Commutation $[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$

where $b^0 = Jb^*J^{-1} \qquad \forall b \in \mathcal{A}$

Order one condition

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}$$

Minimal input : left-right symm algebra

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

involution $(\lambda, q_L, q_R, m) \mapsto (\bar{\lambda}, \bar{q}_L, \bar{q}_R, m^*)$

why ? think of as truncation of $\mathbb{C} \oplus M_2(\mathbb{C}) \oplus \cdots M_n(\mathbb{C}) \cdots$

subalgebra $\mathbb{C} \oplus M_3(\mathbb{C})$ integer spin $\mathbb{C}$-alg
subalgebra $\mathbb{H}_L \oplus \mathbb{H}_R$ half-integer spin $\mathbb{R}$-alg

Slogan : algebras better than Lie algebras, more constraints on reps

Adjoint action

$\mathcal{M}$ bimodule over $\mathcal{A}$, $u \in \mathcal{U}(\mathcal{A})$ unitary

$$\mathrm{Ad}(u)\xi = u\xi u^* \qquad \forall \xi \in \mathcal{M}$$

Odd bimodule

$\mathcal{M}$ bimodule for $\mathcal{A}_{LR}$ *odd* iff

$s = (1, -1, -1, 1)$ acts by $Ad(s) = -1$

$\Leftrightarrow$ Rep of $\mathcal{B} = (\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^{op})_p$ as $\mathbb{C}$-alg

$p = \frac{1}{2}(1 - s \otimes s^0)$, with $\mathcal{A}^0 = \mathcal{A}^{op}$

$$\mathcal{B} = \oplus^{4-times} M_2(\mathbb{C}) \oplus M_6(\mathbb{C})$$

Contragredient bimodule of $\mathcal{M}$

$$\mathcal{M}^0 = \{\bar{\xi}\,;\, \xi \in \mathcal{M}\}, \qquad a\,\bar{\xi}\,b = \overline{b^* \xi\, a^*}$$

The bimodule $\mathcal{M}_F$

$\mathcal{M}_F =$ sum all inequiv irred odd $\mathcal{A}_{LR}$-bimodules

- $\dim_{\mathbb{C}} \mathcal{M}_F = 32$
- $\mathcal{M}_F = \mathcal{E} \oplus \mathcal{E}^0$

$$\mathcal{E} = \mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0$$

- $\mathcal{M}_F \cong \mathcal{M}_F^0$ by J_F antilinear

$$J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi}) \,, \quad \forall \xi \,,\ \eta \in \mathcal{E}$$

$$J^2 = 1 \,, \quad \xi\, b = J b^* J\, \xi \,, \quad \forall \xi \in \mathcal{M}_F \,,\ b \in \mathcal{A}_{LR}$$

Sum irreps of $\mathcal{B}$

$$\mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0$$

$$\oplus \mathbf{1} \otimes \mathbf{2}_L^0 \oplus \mathbf{1} \otimes \mathbf{2}_R^0 \oplus \mathbf{3} \otimes \mathbf{2}_L^0 \oplus \mathbf{3} \otimes \mathbf{2}_R^0$$

Grading γ_F

$$\gamma_F = c - J_F\, c\, J_F \,, \quad c = (0, 1, -1, 0) \in \mathcal{A}_{LR}$$

$$J_F^2 = 1 \,, \quad J_F\, \gamma_F = -\, \gamma_F\, J_F$$

$\Rightarrow KO$-dim 6 mod 8

Subalgebra and order one condition

$N = 3$ (number of generations : input)

$$\mathcal{H}_F = \mathcal{M}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F$$

Left action of $\mathcal{A}_{LR}$ sum of reps $\pi|_{\mathcal{H}_f} \oplus \pi'|_{\mathcal{H}_{\bar{f}}}$

$\mathcal{H}_f = \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{E}$ and $\mathcal{H}_{\bar{f}} = \mathcal{E}^0 \oplus \mathcal{E}^0 \oplus \mathcal{E}^0$

no equiv subreps (disjoint)

If D mixes $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}} \Rightarrow$ no order one condition

Problem pair $\mathcal{A} \subset \mathcal{A}_{LR}$ and D off diag

maximal $\mathcal{A}$ where order one condition holds

$$\mathcal{A}_F = \{(\lambda, q_L, \lambda, m) \mid \lambda \in \mathbb{C}\,,\ q_L \in \mathbb{H}\,,\ m \in M_3(\mathbb{C})\}$$

$$\sim \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

unique up to Aut($\mathcal{A}_{LR}$)

Operator $T\ :\ \mathcal{H}_f \to \mathcal{H}_{\bar{f}}$

$$\mathcal{A}(T) = \{b \in \mathcal{A}_{LR} \mid \pi'(b)T = T\pi(b)\,,$$
$$\pi'(b^*)T = T\pi(b^*)\}$$

involutive unital subalgebra of $\mathcal{A}_{LR}$

$\mathcal{A} \subset \mathcal{A}_{LR}$ involutive unital subalgebra of $\mathcal{A}_{LR}$

• restriction of π and π' to $\mathcal{A}$ disjoint $\Rightarrow$ no off diag D for $\mathcal{A}$

• $\exists$ off diag D for $\mathcal{A} \Rightarrow$ pair e, e' min proj in commutants of $\pi(\mathcal{A}_{LR})$ and $\pi'(\mathcal{A}_{LR})$ and operator T

$$e'Te = T \neq 0 \quad \text{and} \quad \mathcal{A} \subset \mathcal{A}(T)$$

Symmetries

Unitary $\mathsf{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^*u = 1\}$
Special unitary

$$\mathsf{SU}(\mathcal{A}_F) = \{u \in \mathsf{U}(\mathcal{A}_F) \,|\, \det(u) = 1\}$$

(det of action of u on $\mathcal{H}_F$)

Up to a finite abelian group

$$\mathsf{SU}(\mathcal{A}_F) \sim \mathsf{U}(1) \times \mathsf{SU}(2) \times \mathsf{SU}(3)$$

Adjoint action of U(1) (in powers of $\lambda \in \mathsf{U}(1)$)

	$\uparrow \otimes \mathbf{1}^0$	$\downarrow \otimes \mathbf{1}^0$	$\uparrow \otimes \mathbf{3}^0$	$\downarrow \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

hypercharges

Classifying Dirac operators for $(\mathcal{A}_F, \mathcal{H}_F, \gamma_F, J_F)$

D_F self adjoint on $\mathcal{H}_F$, comm w J_F, anticomm w γ_F and $[[D,a],b^0] = 0$, $\forall a, b \in \mathcal{A}_F$

Commuting with subalgebra

$$\mathbb{C}_F \subset \mathcal{A}_F \,, \quad \mathbb{C}_F = \{(\lambda, \lambda, 0)\,, \lambda \in \mathbb{C}\}$$

(massless photon)

$$D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$$

$$S = S_1 \oplus (S_3 \otimes 1_3)$$

$$S_1 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix}$$

$$S_3 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 3)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 3)} \\ Y_{(\uparrow 3)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 3)} & 0 & 0 \end{pmatrix}$$

where $Y_{(\downarrow 1)}$, $Y_{(\uparrow 1)}$, $Y_{(\downarrow 3)}$, $Y_{(\uparrow 3)} \in \mathrm{GL}_3(\mathbb{C})$ and Y_R symm

$$T : E_R = \uparrow_R \otimes \mathbf{1}^0 \to J_F \, E_R$$

Moduli space $\mathcal{C}_3 \times \mathcal{C}_1$

$\mathcal{C}_3 =$ pairs $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo

$$Y'_{(\downarrow 3)} = W_1\, Y_{(\downarrow 3)}\, W_3^*\,,\; Y'_{(\uparrow 3)} = W_2\, Y_{(\uparrow 3)}\, W_3^*$$

W_j unitary matrices

$$\mathcal{C}_3 = (K \times K)\backslash(G \times G)/K$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$

$\dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4$ (eigenval, coset 3 angles 1 phase) CKM

$$V = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e_\delta & c_1 c_2 s_3 + s_2 c_3 e_\delta \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e_\delta & c_1 s_2 s_3 - c_2 c_3 e_\delta \end{pmatrix}$$

$\mathcal{C}_1 =$ triplets $(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R)$ with Y_R symmetric modulo

$$Y'_{(\downarrow 1)} = V_1\, Y_{(\downarrow 1)} V_3^*\,,\; Y'_{(\uparrow 1)} = V_2\, Y_{(\uparrow 1)}\, V_3^*\,,$$

$$Y'_R = V_2\, Y_R\, \bar{V}_2^*$$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection forgets Y_R fiber symm matrices mod $Y_R \mapsto \lambda^2 Y_R$

$\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$ (dim fiber 12-1=11)

Dimension KO-dim 6 mod 8
Antisymmetric pairing on K_0

$$\langle e, f \rangle = \mathrm{Tr}(\gamma\, e\, J f J^{-1})$$

$K_0(\mathcal{A}_F)$ free abelian generated by
$e = (1,0,0)$, $e_L = (0,1,0)$ and $f_3 = (0,0,f)$
Pairing decomposes as

$$\begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & -2 \\ -2 & 2 & 0 \end{pmatrix}$$

can obtain a non-degen $K_0 \oplus K_0 \to \mathbb{C} \oplus \mathbb{C}$

Product geometry

$(\mathcal{A}_i, \mathcal{H}_i, D_i, \gamma_i, J_i)$ of KO-dim 4 and 6

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \qquad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2$$

$$\gamma = \gamma_1 \otimes \gamma_2 \qquad J = J_1 \otimes J_2$$

4-dim manifold M and finite geometry F

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F = C^\infty(M, \mathcal{A}_F)$$

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F)$$

$$D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$$

Inner fluctuations of the metric

Right $\mathcal{A}$-module structure on $\mathcal{H}$

$$\xi\, b = b^0\, \xi\,, \quad \forall\, \xi \in \mathcal{H}\,, \quad b \in \mathcal{A}$$

Unitary group, adjoint representation

$$\xi \in \mathcal{H} \to \mathrm{Ad}(u)\, \xi = u\, \xi\, u^*\,, \quad \forall\, \xi \in \mathcal{H}$$

Inner fluctuations

$$D \to D_A = D + A + \varepsilon'\, J\, A\, J^{-1}$$

A self-adjoint operator

$$A = \sum a_j [D, b_j]\,, \quad a_j, b_j \in \mathcal{A}$$

Properties of inner fluctuations $(\mathcal{A}, \mathcal{H}, D, J)$

• Gauge potential $A \in \Omega^1_D$, $A = A^*$
Unitary $u \in \mathcal{A}$, then

$$\mathrm{Ad}(u)(D + A + \varepsilon' J\, A\, J^{-1})\mathrm{Ad}(u^*) =$$

$$D + \gamma_u(A) + \varepsilon' J\, \gamma_u(A)\, J^{-1}$$

where $\gamma_u(A) = u\,[D, u^*] + u\, A\, u^*$

• $D' = D + A$ (with $A \in \Omega^1_D$, $A = A^*$) then

$$D' + B = D + A', \quad A' = A + B \in \Omega^1_D$$

$\forall B \in \Omega^1_{D'}$ $B = B^*$

• $D' = D + A + \varepsilon' J\, A\, J^{-1}$ then

$$D' + B + \varepsilon' J\, B\, J^{-1} = D + A' + \varepsilon' J\, A'\, J^{-1} \quad A' = A + B \in \Omega^1_D$$

$\forall B \in \Omega^1_{D'}$ $B = B^*$

Spectral action (Chamseddine, Connes)

Observables in gravity $\int_M F(K)\sqrt{g}\, d^4x$

(scalar invariant function, non-localized)

Spectral data $\Rightarrow$ action functional

$$\mathrm{Trace}\,(f(D/\Lambda))$$

D Dirac, Λ mass scale, $f > 0$ even function

Simple dimension spectrum $\Rightarrow$ expansion

$$\mathrm{Trace}\,(f(D/\Lambda)) \sim \sum_k f_k \Lambda^k ⨍ |D|^{-k} + f(0)\,\zeta_D(0) + o(1),$$

with $f_k = \int_0^\infty f(v)\, v^{k-1}\, dv$

Relation between asymptotic expansion

$$\text{Trace}\,(e^{-t\Delta}) \sim \sum a_\alpha\, t^\alpha \qquad (t \to 0)$$

and the ζ function

$$\zeta_D(s) = \text{Trace}\,(\Delta^{-s/2}) \tag{1}$$

– Non-zero term a_α with $\alpha < 0 \Rightarrow$ *pole* of ζ_D at -2α with

$$\text{Res}_{s=-2\alpha}\, \zeta_D(s) = \frac{2\, a_\alpha}{\Gamma(-\alpha)}$$

– No $\log t$ terms $\Rightarrow$ regularity at 0 for ζ_D with

$$\zeta_D(0) = a_0$$

Get first statement from

$$|D|^{-s} = \Delta^{-s/2} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{-t\Delta}\, t^{s/2-1}\, dt$$

with $\int_0^1 t^{\alpha+s/2-1}\, dt = (\alpha + s/2)^{-1}$. Second from

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)} \sim \frac{s}{2}, \quad s \to 0$$

contrib to $\zeta_D(0)$ from pole part at $s = 0$ of

$$\int_0^\infty \text{Tr}(e^{-t\Delta})\, t^{s/2-1}\, dt$$

given by $a_0 \int_0^1 t^{s/2-1}\, dt = a_0 \frac{2}{s}$

Product geometry : $M \times F$ prod metric

M Riemannian spin 4-manifold

F finite noncommutative geometry KO-dim $= 6$

1. Unimod subgr of $\mathcal{U}(\mathcal{A})$ adjoint rep Ad(u) on $\mathcal{H}$ is gauge group of SM
2. Unimodular inner fluctuations $\Rightarrow$ gauge bosons of SM
3. Full standard model (neutrino mixing and seesaw) min coupled to gravity from

   $$S = \mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle, \quad \tilde{\xi} \in \mathcal{H}^+_{cl},$$

   $D_A =$ Dirac w/ unimodular inner fluctuations

   $J =$ real structure

   $\mathcal{H}^+_{cl} =$ classical spinors, Grassmann variables

The bosonic part of the spectral action is

$$
\begin{aligned}
S &= \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, c + \frac{f_0}{4}\, d) \int \sqrt{g}\, d^4x (2) \\
&+ \frac{96\, f_2\, \Lambda^2 - f_0\, c}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{10\, \pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x \\
&+ \frac{(-2\, a\, f_2\, \Lambda^2 + e\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{2\, \pi^2} \int a\, |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
&- \frac{f_0}{12\, \pi^2} \int a\, R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{2\, \pi^2} \int (g_3^2\, G_{\mu\nu}^i\, G^{\mu\nu i} + g_2^2\, F_{\mu\nu}^\alpha\, F^{\mu\nu\alpha} \\
&+ \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x \\
&+ \frac{f_0}{2\, \pi^2} \int b\, |\varphi|^4\, \sqrt{g}\, d^4x
\end{aligned}
$$

where

$$R^* R^* = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R_{\mu\nu}^{\alpha\beta} R_{\rho\sigma}^{\gamma\delta}$$

is the Euler characteristic.

• Rescaling of the Higgs field φ (kinetic terms normalized)

• Normalize Higgs kinetic energy : rescale φ to

$$\mathbf{H} = \frac{\sqrt{a\, f_0}}{\pi}\, \varphi$$

kinetic term becomes

$$\int \frac{1}{2} |D_\mu \mathbf{H}|^2 \sqrt{g}\, d^4x$$

Normalization of the kinetic terms ⇒ relation coupling constants g_1, g_2, g_3 and coeff f_0

$$\frac{g_3^2\, f_0}{2\pi^2} = \frac{1}{4}, \qquad g_3^2 = g_2^2 = \frac{5}{3}\, g_1^2 \,.$$

Then bosonic action :

$$
\begin{aligned}
S = & \int \left(\frac{1}{2\kappa_0^2} R + \alpha_0 \, C_{\mu\nu\rho\sigma} \, C^{\mu\nu\rho\sigma} + \quad \gamma_0 + \tau_0 \, R^* R^* \right. \\
+ & \; \frac{1}{4} G^i_{\mu\nu} \, G^{\mu\nu i} + \frac{1}{4} F^\alpha_{\mu\nu} \, F^{\mu\nu\alpha} + \; \frac{1}{4} B_{\mu\nu} \, B^{\mu\nu} \\
+ & \left. \; \frac{1}{2} |D_\mu \, \mathbf{H}|^2 - \mu_0^2 |\mathbf{H}|^2 - \xi_0 \, R \, |\mathbf{H}|^2 + \lambda_0 |\mathbf{H}|^4 \right) \sqrt{g} \, d^4 x,
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{1}{\kappa_0^2} = & \quad \frac{96 \, f_2 \, \Lambda^2 - f_0 \, c}{12 \, \pi^2} \\
\mu_0^2 = & \quad 2 \, \frac{f_2 \, \Lambda^2}{f_0} - \frac{e}{a} \\
\alpha_0 = & \; - \frac{3 \, f_0}{10 \, \pi^2} \\
\tau_0 = & \; \frac{11 \, f_0}{60 \, \pi^2} \\
\gamma_0 = & \quad \frac{1}{\pi^2} (48 \, f_4 \, \Lambda^4 - f_2 \, \Lambda^2 \, c + \frac{f_0}{4} \, d) \\
\lambda_0 = & \; \frac{\pi^2}{2 \, f_0} \frac{b}{a^2} \\
\xi_0 = & \; \frac{1}{12}
\end{aligned}
$$

Predictions

– Unification of couplings

$$\frac{g_3^2\, f_0}{2\pi^2} = \frac{1}{4}, \qquad g_3^2 = g_2^2 = \frac{5}{3}\, g_1^2\,.$$

– See-saw mechanism (neutrino masses with large $M_R \sim \Lambda$).

– Mass relation at unification

$$\sum_\sigma (m_\nu^\sigma)^2 + (m_e^\sigma)^2 + 3\,(m_u^\sigma)^2 + 3\,(m_d^\sigma)^2 = 8\, M^2$$

$Y_2(S) = 4g^2$

$$Y_2 = \sum_\sigma (y_\nu^\sigma)^2 + (y_e^\sigma)^2 + 3\,(y_u^\sigma)^2 + 3\,(y_d^\sigma)^2$$

– $\Rightarrow$ value of top quark mass agrees w/ experim (1.04 times if neglect other Yukawa couplings)

– Higgs scattering parameter

$$\tilde{\lambda}(\Lambda) = g_3^2\, \frac{b}{a^2}.$$

$\Rightarrow$ Higgs mass ~ 170 GeV

numerical solution RG eqs w/ boundary value $\lambda_0 = 0.356$ at $\Lambda = 10^{17}$ GeV $\Rightarrow \lambda(M_Z) \sim 0.241$

– Newton constant ($f_2 \sim 5 f_0$)

– No proton decay (unlike GUT)

The mass relation

from the equality of the Yukawa coupling terms $\mathcal{L}_{Hf}$

After Wick rotation to Euclidean + chiral transf $U = e^{i\frac{\pi}{4}\gamma_5} \otimes 1$ same iff

$$\begin{aligned}
(k_{(\uparrow 3)})_{\sigma\kappa} &= \frac{g}{2M}\, m_u^\sigma\, \delta_\sigma^\kappa \\
(k_{(\downarrow 3)})_{\sigma\kappa} &= \frac{g}{2M}\, m_d^\mu\, C_{\sigma\mu}\delta_\mu^\rho C_{\rho\kappa}^\dagger \\
(k_{(\uparrow 1)})_{\sigma\kappa} &= \frac{g}{2M}\, m_\nu^\sigma\, \delta_\sigma^\kappa \\
(k_{(\downarrow 1)})_{\sigma\kappa} &= \frac{g}{2M}\, m_e^\mu\, {U^{lep}}_{\sigma\mu}\delta_\mu^\rho {U^{lep}}^\dagger_{\rho\kappa}
\end{aligned}$$

δ_i^j = Kronecker delta

Constraint

$\mathrm{Tr}(k^*_{(\uparrow 1)}k_{(\uparrow 1)} + k^*_{(\downarrow 1)}k_{(\downarrow 1)} + 3(k^*_{(\uparrow 3)}k_{(\uparrow 3)} + k^*_{(\downarrow 3)}k_{(\downarrow 3)})) = 2\,g^2$

$\Rightarrow$ mass matrices satisfy

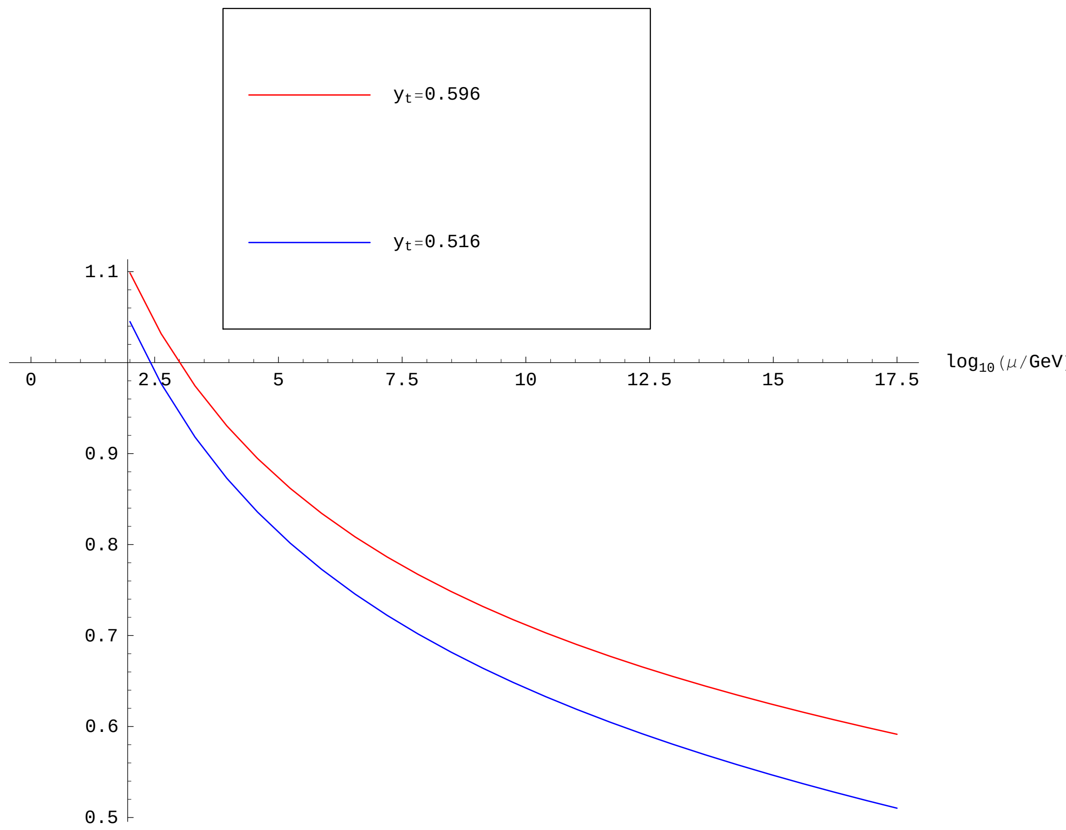
$$\sum_\sigma (m_\nu^\sigma)^2 + (m_e^\sigma)^2 + 3\,(m_u^\sigma)^2 + 3\,(m_d^\sigma)^2 = 8\,M^2$$

Running of top Yukawa coupling

$$\frac{v}{\sqrt{2}}(y^{\sigma}_{\cdot}) = (m^{\sigma}_{\cdot}),$$

$$\frac{dy_t}{dt} = \frac{1}{16\pi^2}\left[\frac{9}{2}y_t^3 - \left(a\,g_1^2 + b\,g_2^2 + c\,g_3^2\right)y_t\right],$$

$$(a,b,c) = (\frac{17}{12}, \frac{9}{4}, 8)$$



<u>See-saw mechanism</u> $D = D(Y)$ Dirac

$$\begin{pmatrix} 0 & M_\nu^* & M_R^* & 0 \\ M_\nu & 0 & 0 & 0 \\ M_R & 0 & 0 & \bar{M}_\nu^* \\ 0 & 0 & \bar{M}_\nu & 0 \end{pmatrix}$$

on subspace $(\nu_R, \nu_L, \bar{\nu}_R, \bar{\nu}_L)$

Largest eigenvalue of $M_R \sim \Lambda$ unif scale

Take $M_R = x\, k_R$ in flat space, Higgs vacuum v small (w/resp to unif scale)

$$\partial_u \mathrm{Tr}(f(D_A/\Lambda)) = 0 \qquad u = x^2$$

$$x^2 = \frac{2\, f_2\, \Lambda^2\, \mathrm{Tr}(k_R^* k_R)}{f_0\, \mathrm{Tr}((k_R^* k_R)^2)}$$

Dirac mass $M_\nu \sim$ Fermi energy v

$$\frac{1}{2}(\pm m_R \pm \sqrt{m_R^2 + 4\, v^2})$$

two eigenvalues $\sim \pm m_R$ and two $\sim \pm \frac{v^2}{m_R}$

Compare with estimates

$(m_R)_1 \geq 10^7 GeV\,, \quad (m_R)_2 \geq 10^{12} GeV\,, \quad (m_R)_3 \geq 10^{16} GeV$

Running of coupling constants

At one loop :

$$\beta_{g_i} = (4\pi)^{-2}\, b_i\, g_i^3, \qquad \text{with} \quad b_i = (\frac{41}{6}, -\frac{19}{6}, -7),$$

$$\begin{aligned}
\alpha_1^{-1}(\Lambda) &= \alpha_1^{-1}(M_Z) - \frac{41}{12\pi} \log \frac{\Lambda}{M_Z} \\
\alpha_2^{-1}(\Lambda) &= \alpha_2^{-1}(M_Z) + \frac{19}{12\pi} \log \frac{\Lambda}{M_Z} \\
\alpha_3^{-1}(\Lambda) &= \alpha_3^{-1}(M_Z) + \frac{42}{12\pi} \log \frac{\Lambda}{M_Z}
\end{aligned}$$

M_Z mass of Z^0 boson

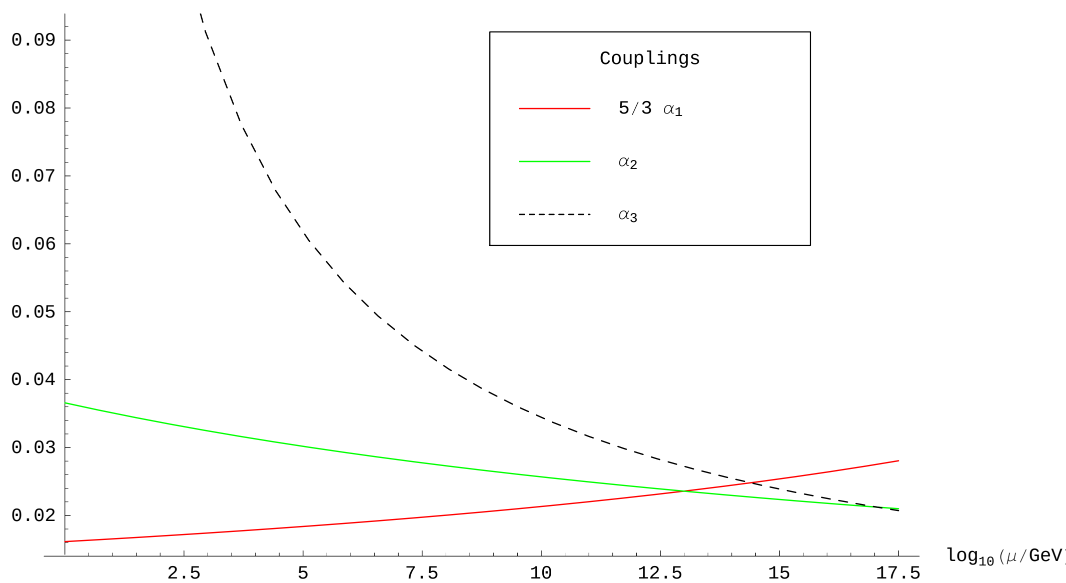
Higgs scattering parameter

$$\frac{f_0}{2\,\pi^2} \int b\,|\varphi|^4 \sqrt{g}\, d^4x = \frac{\pi^2}{2\,f_0} \frac{b}{a^2} \int |\mathbf{H}|^4 \sqrt{g}\, d^4x$$

$\Rightarrow$ *Another relation* at unification

$$\tilde{\lambda}(\Lambda) = g_3^2 \frac{b}{a^2}$$

($\tilde{\lambda}$ is the $|\mathbf{H}|^4$ coupling)



Couplings
5/3 α_1
α_2
α_3
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
2.5
5
7.5
10
12.5
15
17.5
$\log_{10}(\mu/\text{GeV})$

Running of Higgs scattering parameter

$$\frac{d\lambda}{dt} = \lambda\gamma + \frac{1}{8\pi^2}(12\lambda^2 + B)$$

$$\gamma = \frac{1}{16\pi^2}(12y_t^2 - 9g_2^2 - 3g_1^2) \quad B = \frac{3}{16}(3g_2^4 + 2g_1^2 g_2^2 + g_1^4) - 3y_t^4$$

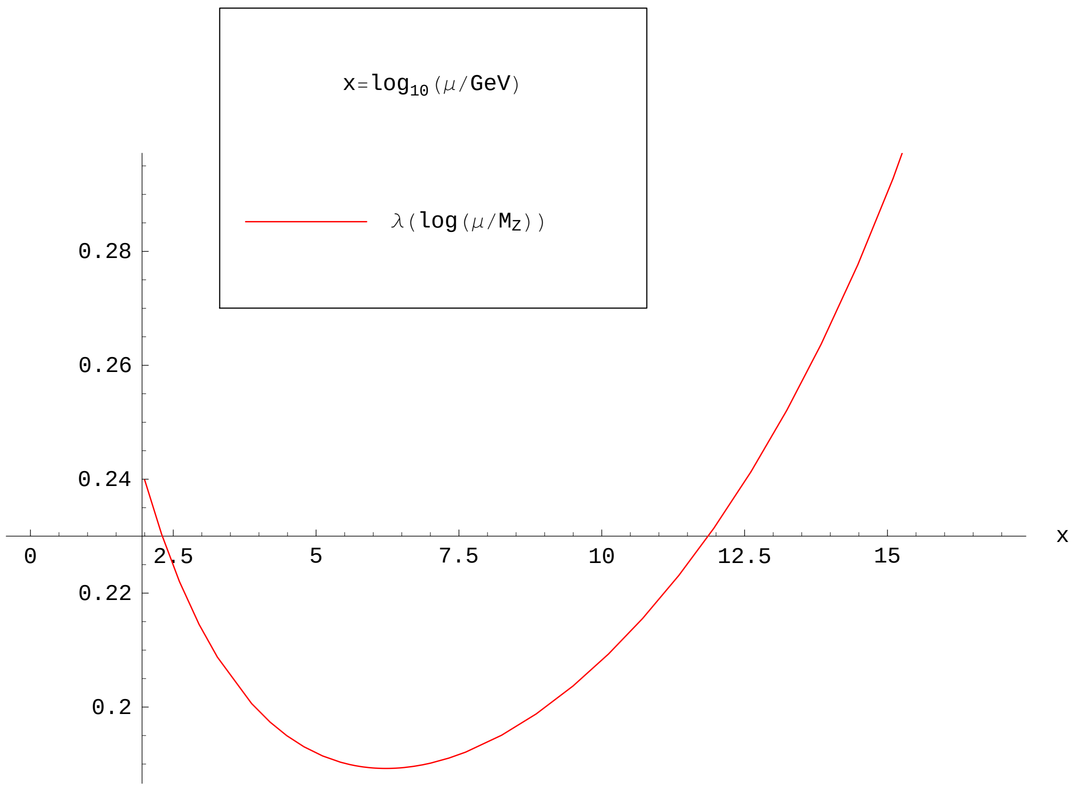
Higgs mass estimate

$$m_H^2 = 8\lambda \frac{M^2}{g^2}, \quad m_H = \sqrt{2\lambda}\,\frac{2M}{g}$$

numerical solution (neglecting other Yukawa couplings)

$\lambda(M_Z) \sim 0.241$ and Higgs mass ~ 170 GeV

(w/correction from see-saw ~ 168 GeV)


x=log$_{10}$(μ/GeV)
λ(log(μ/M$_Z$))
0.28
0.26
0.24
0.22
0.2
0
2.5
5
7.5
10
12.5
15
x