Solvmanifolds and noncommutative tori with real multiplication

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Background:

Hilbert modular surfaces

$$X = (\mathbb{H} \times \mathbb{H})/\Gamma$$

$\Gamma = \mathsf{SL}_2(O_{\mathbb{K}})$, real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{d})$

w/ singularities, link 3-dim solvmanifolds

1970s: Hirzebruch conjecture: signature defects

$$\sigma(X) = \int_X L \ - \eta(0)$$

$$\eta(0) = L(\Lambda', V, 0)$$

Shimizu L-function

$$L(\Lambda, V, s) = \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} \mathsf{sign}(N(\mu)) \, |N(\mu)|^{-s}$$

1983: proved Atiyah–Donnelly–Singer using APS

$$\eta(s) = \sum_{\lambda \neq 0} \mathsf{sign}(\lambda)|\lambda|^{-s}$$

(also higher rank totally real fields)

Atiyah's question:

Eta function $\eta(s)$ splits as $L(\Lambda, V, s)$ plus stuff

Does the arithmetic part come from a *non-commutative geometry*?

3-dimensional solvmanifold "model up to homotopy"

Answer: Yes!

Notation:

$\mathbb{K} = \mathbb{Q}(\sqrt{d})$ real quadratic field

$\alpha_i : \mathbb{K} \hookrightarrow \mathbb{R}$, $i = 1, 2$ real embeddings

$L \subset \mathbb{K}$ lattice $\Rightarrow$ lattice $\Lambda \subset \mathbb{R}^2$

$$L \ni \ell \mapsto (\alpha_1(\ell), \alpha_2(\ell)) \subset \mathbb{R}^2$$

$\Lambda = (\alpha_1, \alpha_2)(L)$

$V = \epsilon^{\mathbb{Z}}$ totally positive units preserving L

$$V = \{u \in O_{\mathbb{K}}^* \,|\, uL \subset L, \ \alpha_i(u) \in \mathbb{R}_+^*\}$$

$(L = O_{\mathbb{K}} \Rightarrow \epsilon =$ fundamental unit$)$

Acts of V on Λ

$\lambda = (\alpha_1(\ell), \alpha_2(\ell)) \mapsto (\epsilon \alpha_1(\ell), \epsilon' \alpha_2(\ell)) = (\epsilon \alpha_1(\ell), \epsilon^{-1} \alpha_2(\ell))$

Solvmanifold

$$S(\Lambda, V) = \Lambda \rtimes_\epsilon V$$

action of V on Λ by

$$A_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon' \end{pmatrix} \in \mathsf{SL}_2(\mathbb{R})$$

$$S(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}^2 \rtimes \mathbb{R}$$

action of $\mathbb{R}$ on $\mathbb{R}^2$ by

$$\Theta_t(x, y) = (e^t x, e^{-t} y), \quad \Theta_t \in \mathsf{SL}_2(\mathbb{R})$$

$$X_\epsilon = S(\Lambda, V) \backslash S(\mathbb{R}^2, \mathbb{R})$$

$\pi_1(X_\epsilon) = S(\Lambda, V)$

Topology

$$H_1(X_\epsilon, \mathbb{Z}) = \Lambda/(1 - A_\epsilon)\Lambda \oplus \mathbb{Z}$$

abelianization of $\pi_1(X_\epsilon) = S(\Lambda, V)$

Surjective homomorphism

$$\pi(\lambda, n) = (\lambda \mod (1 - A_\epsilon)\Lambda, n)$$

Kernel commutators

Poincaré duality

$$H^0(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \quad H^1(X_\epsilon, \mathbb{Z}) = \mathbb{Z}$$
$$H^3(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \quad H^2(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \oplus \text{Coker}(1 - A_\epsilon)$$

$\text{Ker}(1 - A_\epsilon) = 0$ and $\text{Coker}(1 - A_\epsilon)$ torsion

Action: (equivalent formulation)

Basis $\{1, \theta\}$ of $\alpha_1(L) \subset \mathbb{R}$

$$\varphi_\epsilon = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathsf{SL}_2(\mathbb{Z})$$

with $\epsilon = a + b\theta$, $\epsilon\theta = c + d\theta$

Fixed points $1/\theta$ and $1/\theta'$

Identifications

$$S(\Lambda, V) = \mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}$$

$$(\lambda, \epsilon^k) \mapsto (\lambda_1 = n + m\theta, k)$$

$$(\lambda, \epsilon^k) \mapsto (\lambda_2 = n + m\theta', -k)$$

Solvmanifold and universal family

$\mathbb{H}$ = hyperbolic plane, $\mathcal{U}$ = universal family

$$\mathcal{U} \xrightarrow{\pi} \mathbb{H}, \quad \pi^{-1}(\tau) = E_\tau$$

elliptic curves, up to equivalence descends to

$$\mathcal{U}/\mathsf{SL}_2(\mathbb{Z}) \to \mathbb{H}/\mathsf{SL}_2(\mathbb{Z})$$

Galois conjugate θ, θ' for lattice Λ

Geodesic $\ell_{\theta,\theta'} \Rightarrow$ closed geodesic in $\mathbb{H}/\mathsf{SL}_2(\mathbb{Z})$

$$\text{length}(\bar{\ell}_{\theta,\theta'}) = \log \epsilon$$

unit generating V acting on Λ

X_ϵ = restriction of $\mathcal{U}/\mathsf{SL}_2(\mathbb{Z})$ to geodesic $\bar{\ell}_{\theta,\theta'}$

Real multiplication NC tori

$$VU = e^{2\pi i \theta} UV$$

C^*-algebra $\mathcal{A}_\theta$

Morita equivalences (Connes, Rieffel)

$$\mathcal{A}_{\theta_1} \simeq \mathcal{A}_{\theta_2} \Leftrightarrow g \in \mathsf{SL}_2(\mathbb{Z})$$

$\theta_1 = g\theta_2$

Quadratic irrationalities: $\mathbb{Q}(\theta)$ real quadratic

$$\exists g \in \mathsf{SL}_2(\mathbb{Z}), \quad g\theta = \theta$$

Self Morita equivalences

Manin: parallel to elliptic curves with complex multiplication

Twisted group algebras

Γ finitely generated discrete group

$$\sigma : \Gamma \times \Gamma \to U(1)$$

multiplier, 2-cocycle

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)$$
$$\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$$

Hilbert space $\ell^2(\Gamma)$ left/right σ-regular representations

$$L_\gamma^\sigma f(\gamma') = f(\gamma^{-1}\gamma')\sigma(\gamma, \gamma^{-1}\gamma')$$

$$R_\gamma^\sigma f(\gamma') = f(\gamma'\gamma)\sigma(\gamma', \gamma)$$

$$L_\gamma^\sigma L_{\gamma'}^\sigma = \sigma(\gamma, \gamma')L_{\gamma\gamma'}^\sigma, \quad R_\gamma^\sigma R_{\gamma'}^\sigma = \sigma(\gamma, \gamma')R_{\gamma\gamma'}^\sigma$$

left σ-regular commutes with right $\bar{\sigma}$-regular

$\mathbb{C}(\Gamma, \sigma)$ = twisted group ring, gen by R_γ^σ
$C_r^*(\Gamma, \sigma)$ = norm closure, (reduced) twisted group C^*-algebra

NC tori as twisted group algebras

$$\mathcal{A}_\theta = C_r^*(\mathbb{Z}^2, \sigma)$$

$$\sigma((n,m),(n',m')) = \exp(-2\pi i(\xi_1 nm' + \xi_2 mn'))$$

$$\theta = \xi_2 - \xi_1$$

$$U = R^\sigma_{(0,1)} \text{ and } V = R^\sigma_{(1,0)}$$

$$Uf(n,m) = e^{-2\pi i\xi_2 n} f(n, m+1)$$

$$Vf(n,m) = e^{-2\pi i\xi_1 m} f(n+1, m)$$

$$UV = e^{2\pi i\theta} VU$$

Some freedom to choose ξ_1, ξ_2

Invariance under $\mathsf{SL}_2(\mathbb{Z})$:

Cocycle $\sigma((n, m), (n', m'))$ satisfies

$$\sigma((n, m), (n', m')) = \sigma((n, m)\varphi, (n', m')\varphi)$$

for all $\varphi \in \mathsf{SL}_2(\mathbb{Z})$ iff

$$\xi_2 = -\xi_1$$

$\Rightarrow$ Set $\xi_2 = \theta/2 = -\xi_1$

$$\sigma_\theta((n, m), (k, r)) = \exp(\pi i \theta (nr - mk))$$

$$= \exp(\pi i \theta (n, m) \wedge (k, r))$$

for

$$(a, b) \wedge (c, d) := \det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

NC tori:

$$
\begin{aligned}
\mathbb{T}_{\Lambda,1} &= C^*(\mathbb{Z}^2, \sigma_\theta) &&= C^*(\Lambda, \sigma_{\theta(\theta'-\theta)^{-1}}) \\
\mathbb{T}_{\Lambda,2} &= C^*(\mathbb{Z}^2, \sigma_{\theta'}) &&= C^*(\Lambda, \sigma_{\theta'(\theta'-\theta)^{-1}})
\end{aligned}
$$

For elliptic curves: natural object E/O_K^*

For NC tori: V acts on $\mathbb{T}_{\Lambda,i}$ (automorphisms)

$$
v_\epsilon^k(R_{(n,m)}^\sigma) = R_{(n,m)\varphi_\epsilon^k}^\sigma
$$

$$
v_\epsilon^k(R_\lambda^\sigma) = R_{A_\epsilon^k(\lambda)}^\sigma
$$

Crossed product $\mathbb{T}_{\Lambda,i} \rtimes_i V$

Twisted group algebra of $S(\Lambda, V)$

Given $\sigma((n,m),(k,r)) = \exp(\pi i \theta(nr - mk))$

$\tilde{\sigma}((n,m,k),(n',m',k')) := \sigma((n,m),(n',m')\varphi_\epsilon^k)$

is a multiplier for $S(\Lambda, V)$

$$C^*(\Lambda, \sigma) \rtimes V = C^*(S(\Lambda, V), \tilde{\sigma})$$

$$R^{\tilde{\sigma}}_{(n,m,k)} \mapsto R^{\sigma}_{(n,m)} v_\epsilon^k$$

$\mathbb{T}_{\Lambda,V,i} = \mathbb{T}_{\Lambda,i} \rtimes_i V$

$$
\begin{aligned}
\mathbb{T}_{\Lambda,V,1} &= C^*(\mathbb{Z}^2, \sigma_\theta) \rtimes_{v_1} \mathbb{Z} &&= C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_\theta) = \\
&\;\; C^*(\Lambda, \sigma_{\theta(\theta'-\theta)^{-1}}) \rtimes_{v_1} V &&= C^*(S(\Lambda, V), \tilde{\sigma}_{\theta(\theta'-\theta)^{-1}}) \\
\mathbb{T}_{\Lambda,V,2} &= C^*(\mathbb{Z}^2, \sigma_{\theta'}) \rtimes_{v_2} \mathbb{Z} &&= C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_{\theta'}) = \\
&\;\; C^*(\Lambda, \sigma_{\theta'(\theta'-\theta)^{-1}}) \rtimes_{v_2} V &&= C^*(S(\Lambda, V), \tilde{\sigma}_{\theta'(\theta'-\theta)^{-1}})
\end{aligned}
$$

Note: $S(\Lambda, V)$ amenable: $C_r^*(S(\Lambda, V), \tilde{\sigma}) = C_{max}^*(S(\Lambda, V), \tilde{\sigma})$

Homotopy quotient (Baum–Connes)

$X_\epsilon =$ homotopy quotient of $C^*(S(\Lambda, V), \tilde\sigma)$

K-theory computation $\mathcal{A} = C^*(\Lambda, \sigma)$
$$K_0(C^*(S(\Lambda, V), \tilde\sigma)) \cong \Lambda$$
$$K_1(C^*(S(\Lambda, V), \tilde\sigma)) \cong \Lambda \oplus \Lambda/(1 - A_\epsilon)\Lambda$$
(Pimsner–Voiculescu six terms exact sequence)

$$
\begin{array}{ccc}
K_0(\mathcal{A}) \xrightarrow{1-\alpha_*} K_0(\mathcal{A}) \longrightarrow K_0(\mathcal{A} \rtimes \mathbb{Z}) \\
\uparrow{\partial} \qquad\qquad\qquad\qquad\qquad \downarrow{\partial} \\
K_1(\mathcal{A} \rtimes \mathbb{Z}) \longleftarrow K_1(\mathcal{A}) \xleftarrow{1-\beta_*} K_1(\mathcal{A})
\end{array}
$$

Twist: no effect on K-theory
$$K_i(C^*(S(\Lambda, V), \tilde\sigma)) \cong K_i(C^*(S(\Lambda, V))$$
use homotopy $\exp(-t\pi i\theta(mn' - nm'))$, $t \in [0, 1]$

Kasparov assembly map isomorphism

$$\mu : K^1(X_\epsilon) \xrightarrow{\cong} K_0(C^*(S(\Lambda, V)))$$

$$\mu : K^0(X_\epsilon) \xrightarrow{\cong} K_1(C^*(S(\Lambda, V)))$$

(mapping torus, Thom isomorphism)
$$K_{i+1}(C(X_\epsilon)) = K_i(C(T^2) \rtimes_{A_\epsilon} \mathbb{Z})$$
$\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}$ satisfies BC (Chabert-Echteroff)

An arithmetic quantum Hall effect

discrete group Γ acting on contractible space $\tilde{X}$ w/ compact quotient $X = \tilde{X}/\Gamma$

base point $x_0 \in \tilde{X}$, crystal charged ions $\Gamma x_0 \Rightarrow$ electron-ion interaction

Hamiltonian $\Delta + \mathcal{V}$ on $L^2(\tilde{X})$ with $T_\gamma \Delta = \Delta T_\gamma$ and $\mathcal{V}$ invariant

Magnetic field: closed 2-form ω w/ $\gamma^* \omega = \omega$

global magnetic potential $\omega = d\chi$, hermitian connection $\nabla = d - i\chi$ with $\nabla^2 = i\omega$

$$\chi - \gamma^* \chi = d\phi_\gamma$$

$$\phi_\gamma(x) = \int_{x_0}^{x} \chi - \gamma^* \chi$$

$$\phi_\gamma(x) - \phi_{\gamma'}(\gamma x) - \phi_{\gamma\gamma'}(x)$$

independent of $x \in \tilde{X}$ and $\phi_\gamma(x_0) = 0$

$$\sigma(\gamma, \gamma') = \exp(-i\phi_\gamma(\gamma' x_0))$$

multiplier $\sigma : \Gamma \times \Gamma \to U(1)$

Magnetic Laplacian:

$$\Delta^\chi = \nabla^*\nabla = (d - i\chi)^*(d - i\chi)$$

magnetic translations

$$T_\gamma^\phi \Delta^\chi = \Delta^\chi T_\gamma^\phi$$

$$T_\gamma^\phi f(x) = \exp(i\phi_\gamma(x))f(\gamma^{-1}x)$$

Independent electron approximation:
V invariant under T_γ^ϕ

Discrete model: Hilbert space $\ell^2(\Gamma)$

$$\mathcal{R} = \sum_{i=1}^{r} R_{\gamma_i} \quad \text{with} \quad R_{\gamma_i}f(\gamma) = f(\gamma\gamma_i)$$

Laplacian $\Rightarrow \Delta_{discr} = r - \mathcal{R}$ random walk operator
($r = \#$ symm set if generators)
Harper operator

$$\mathcal{H}_\sigma = \sum_{i=1}^{r} R_{\gamma_i}^\sigma, \quad R_{\gamma_i}^\sigma \in \mathbb{C}(\Gamma, \sigma)$$

$$\Delta_{discr}^\chi = r - \mathcal{H}_\sigma$$

magnetic Laplacian; potential $V \in \mathbb{C}(\Gamma, \sigma)$

Harper operators

Noncommutative torus:

$$\mathcal{H}_\sigma = U + U^* + V + V^*$$

(interesting spectral theory: Hofstadter butterfly, rational/irrational θ)

For X_ϵ:

$$\mathcal{H}_{\tilde{\sigma}} = U + U^* + V + V^* + W + W^*$$

$U = R^{\tilde{\sigma}}_{(0,1,0)}$, $V = R^{\tilde{\sigma}}_{(1,0,0)}$ and $W = R^{\tilde{\sigma}}_{(0,0,1)}$

Problem: band structure, gaps in the spectrum

range of the trace on K-theory

$\mathcal{U}(\Gamma, \sigma) =$ von Neumann alg: weak closure $\mathbb{C}(\Gamma, \sigma)$

$$P_E = \mathbf{1}_{(-\infty, E]}(\mathcal{H}_{\sigma,V}) \in \mathcal{U}(\Gamma, \sigma)$$

spectral projections of $\mathcal{H}_{\sigma,V}$ satisfy

$$P_E \in C^*_r(\Gamma, \sigma)$$

when E in a gap in the spectrum

$\Rightarrow$ Counting of projections in $C^*_r(\Gamma, \sigma)$

Counting gaps $\Leftrightarrow$ count proj's mod equiv

$$[\mathrm{tr}] : K_0(C_r^*(\Gamma, \sigma))) \to \mathbb{R}$$

where $\tau(a) = \langle a\delta_1, \delta_1 \rangle_{\ell^2(\Gamma)}$ on $C_r^*(\Gamma, \sigma)$ and

$\mathrm{tr} = \tau \otimes \mathrm{Tr} : \{P \in C_r^*(\Gamma, \sigma) \otimes \mathcal{K}) \,|\, P^* = P,\ P^2 = P\} \to \mathbb{R}$

Estimate by: $[\mathrm{tr}](K_0(C_r^*(\Gamma, \sigma))) \cap [0, 1]$

Compute via a (twisted) index theorem:

$$[\mathrm{tr}](K_0(C^*(S(\wedge, V), \tilde{\sigma}))) = \mathbb{Z} + \mathbb{Z}\theta(\theta' - \theta)^{-1}$$

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Spectral flow: $K_1(C(X_\epsilon)) \ni [g]$, $g : X_\epsilon \to \mathsf{GL}_N(\mathbb{C})$

$$\beta(g) = g^{-1}dg \in \Omega^1(X_\epsilon, gl_N(\mathbb{C}))$$

connections $\nabla_u = d + u\beta(g)$ on $X_\epsilon \times \mathbb{C}^N \Rightarrow$ Chern–Simons form (odd Chern character)

$$Ch(g) := cs(d, d + \beta(g)) = \int_0^1 \mathsf{Tr}(\frac{d}{du}(\nabla_u)e^{\nabla_u^2})du$$

$$Ch(g) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \mathsf{Tr}(\beta(g)^{2k+1})$$

Odd Fredholm module pairing w/ $[g] \in K_1$

$$\langle D, [g] \rangle = SF(D, g^{-1}Dg)$$

Dirac operator $\Rightarrow$ APS (mapping torus)

$$SF(\not{\partial}, g^{-1}\not{\partial}g) = -\frac{1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A}(X_\epsilon)Ch(g)$$

Twisted index theorem:

range of trace on $K_0(C^*(S(\Lambda, V), \tilde{\sigma}))$

$$[\text{tr}](\mu_{\tilde{\sigma}}[g]) = \frac{-1}{(2\pi i)^2} \int_{X_\epsilon} \widehat{A} \, e^{\omega_\epsilon} \, Ch(g)$$

$$\mu_{\tilde{\sigma}} : K^1(X_\epsilon) \to K_0(C^*(S(\Lambda, V), \tilde{\sigma}))$$

twisted Kasparov isomorphism $[g] \in K^1(X_\epsilon)$

ω_ϵ closed 2-form on X_ϵ of cocycle $\tilde{\sigma}$:

2 cocycle: $\tilde{\sigma} = \exp(2\pi i \zeta)$

$$\zeta((\lambda, k), (\eta, r)) = \frac{1}{4\pi i} \int_{\mathcal{R}} \omega$$

ω closed 2-form on $T^2 = \mathbb{R}^2/\Lambda$ flux

$$\int_{T^2} \omega = 2\pi i \theta(\theta' - \theta)^{-1}$$

parallelogram $\mathcal{R} = \{0, A_\epsilon^k(\eta), \lambda, \lambda + A_\epsilon^k(\eta)\}$

Sketch of proof:

$P^\pm$ = projs on L^2-kernel of $\mathcal{D}_g \mathcal{D}_g^*$ and $\mathcal{D}_g^* \mathcal{D}_g$

$$\mathcal{D}_g P^+ = 0 \qquad \mathcal{D}_g^* P^- = 0$$

$$\mathcal{D}_g = \frac{\partial}{\partial u} + D_u$$

on $\tilde{X}_\epsilon \times [0,1]$ with $D_u = \tilde{\partial}_u \otimes \nabla$

$$\tilde{\partial}_u = (1-u)\tilde{\partial} + ug^{-1}\tilde{\partial} g, \quad [g] \in K^1(X_\epsilon)$$

$P^\pm$ have smooth kernels $P^\pm(x,y)$

$$e^{-i\phi_\gamma(x)} P^\pm(\gamma x, \gamma y) e^{i\phi_\gamma(y)} = P^\pm(x,y)$$

$\Rightarrow P^\pm(x,x)$ is $\Gamma = S(\Lambda, V)$-invariant

$$\mathrm{tr}(P^\pm) = \int_{X_\epsilon \times S^1} tr P^\pm((x,t),(x,t))\, dx\, dt,$$

von Neumann trace, $tr\, P^\pm(x,x)$ ptwise trace

$$\mathrm{Ind}_{L^2}(\mathcal{D}_g) = \mathrm{tr}(P^+) - \mathrm{tr}(P^-)$$

Set $\bar{P}^{\pm}(x, y) = \int_{S^1} tr P^{\pm}((x,t),(y,t))\, dt$

$\int_{X_\epsilon \times S^1} tr P^{\pm}((x,t),(x,t))\, dx\, dt = \int_{X_\epsilon} tr \bar{P}^{\pm}(x,x) dx$

projections $\bar{P}^{\pm}$ in von Neumann algebra $\mathcal{U}(\Gamma, \tilde{\sigma})$

(after compact perturbation)

$$\mathsf{Ind}_{(\Gamma,\tilde{\sigma})}(\mathcal{D}_g) = [\bar{P}^+] - [\bar{P}^-] \in K_0(C_r^*(\Gamma, \tilde{\sigma}))$$

Twisted Kasparov map

$$\mu_{\tilde{\sigma}}[g] = \mathsf{Ind}_{(\Gamma,\tilde{\sigma})}(\mathcal{D}_g)$$

L^2-index:

$\mathsf{Ind}_{L^2}(\mathcal{D}_g) = \mathsf{tr}(\bar{P}^+) - \mathsf{tr}(\bar{P}^-) = \mathsf{tr}(\mathsf{Ind}_{(\Gamma,\tilde{\sigma})}(\mathcal{D}_g))$

Heat kernel $e^{-t\mathcal{D}^2}$

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_g^* \\ \mathcal{D}_g & 0 \end{pmatrix} \quad \text{with} \quad \mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_g^* \mathcal{D}_g & 0 \\ 0 & \mathcal{D}_g \mathcal{D}_g^* \end{pmatrix}$$

$$\lim_{t \to \infty} \text{tr}_s(e^{-t\mathcal{D}^2}) = \text{tr}(P^+) - \text{tr}(P^-)$$

$$\frac{\partial}{\partial t} \text{tr}_s(e^{-t\mathcal{D}^2}) = -\text{tr}_s(\mathcal{D}^2 e^{-t\mathcal{D}^2}) = \text{tr}_s([\mathcal{D}e^{-t\mathcal{D}^2}, \mathcal{D}]) = 0$$

$$\text{tr}(P^+) - \text{tr}(P^-) = \lim_{t \to \infty} \text{tr}_s(e^{-t\mathcal{D}^2}) = \lim_{t \to 0} \text{tr}_s(e^{-t\mathcal{D}^2})$$

$$= \frac{-1}{(2\pi i)^2} \int_{X_\epsilon \times S^1} \hat{A} \, Ch(\nabla_u) = \frac{-1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A} \, e^{\omega_\epsilon} \, Ch(g),$$

$Ch(\nabla_u) = tr(\beta e^{(d + u\beta)^2}), \quad \beta = g^{-1} dg,$

$\int_{S^1} Ch(\nabla_u) = Ch(g)$

Range computation:

3-manifold X_ϵ

$$\widehat{A}(X_\epsilon) = 1 - \frac{1}{24}p_1(X_\epsilon) + \cdots$$

$$e^{\omega_\epsilon} = 1 + \omega_\epsilon + \frac{1}{2}\omega_\epsilon^2 + \cdots$$

$$Ch(g) = -\frac{1}{6}\mathsf{Tr}(\beta(g)) + \frac{1}{5!}\mathsf{Tr}(\beta^3(g)) + \cdots$$

terms up to dim=3

$$\frac{1}{(2\pi)^2}\int_{X_\epsilon}\left(\frac{-1}{6}\mathsf{Tr}(\beta(g)) \wedge \omega + \frac{1}{5!}\mathsf{Tr}(\beta(g)^3)\right)$$

$$\frac{1}{(2\pi)^2}\int_{X_\epsilon}\frac{1}{5!}\mathsf{Tr}(\beta(g)^3) = \frac{1}{(2\pi)^2}\int_{X_\epsilon}Ch(g)$$

(untwisted odd Chern character)

$$\frac{1}{(2\pi)^2}\frac{-1}{6}\int_{X_\epsilon}\mathsf{Tr}(\beta(g)) \wedge \omega_\epsilon \in \mathbb{Z}R(\omega)$$

$R(\omega) =$ range of linear form

$$T_\omega : [g] \mapsto \frac{1}{(2\pi)^2}\frac{-1}{6}\int_{X_\epsilon}\mathsf{Tr}(\beta(g)) \wedge \omega_\epsilon \in \mathbb{R}$$

$$\oint_C Ch_1(g) = 2\pi i \deg(g|_C) \in 2\pi i \mathbb{Z}$$

with $Ch_1(g) = \frac{-1}{6}\mathsf{Tr}(\beta(g))$ and $C \in H_1(X_\epsilon, \mathbb{Z})$

$$\frac{1}{2\pi i} \oint_{X_\epsilon} Ch_1(g) \wedge PD(C) \in \mathbb{Z}$$

$$PD(C) \in H^2(X_\epsilon, \mathbb{Z}) \hookrightarrow H^2(X_\epsilon, \mathbb{R})$$

$$\omega_\epsilon = 2\pi i \theta(\theta' - \theta)^{-1}\bar{\omega}_\epsilon, \quad \bar{\omega}_\epsilon \in H^2(X_\epsilon, \mathbb{Z})$$

$$\bar{\omega}_\epsilon(v, w) = A_\epsilon^k(\eta) \wedge \lambda,$$

for $v = ((0,0),(\lambda, k))$ and $w = ((0,0),(\eta, r))$

$$\frac{1}{(2\pi)^2} \oint_{X_\epsilon} Ch_1(g) \wedge \omega_\epsilon$$

$$= -\frac{1}{2\pi i}\theta(\theta' - \theta)^{-1} \oint_{PD(\bar{\omega}_\epsilon)} Ch_1(g)$$

$$= \theta(\theta' - \theta)^{-1} \deg(g|_{PD(\bar{\omega}_\epsilon)}) \in \theta(\theta' - \theta)^{-1}\mathbb{Z}$$

Dirac operator on X_ϵ

$\{dt, e^t dx, e^{-t} dy\}$ basis of cotangent bundle
of $S(\mathbb{R}^2, \mathbb{R}, \epsilon) = \mathbb{R}^2 \rtimes_\epsilon \mathbb{R}$

$$\partial\!\!\!/_{X_\epsilon} = c(dt)\frac{\partial}{\partial t} + c(e^t dx)\frac{\partial}{\partial x} + c(e^{-t} dy)\frac{\partial}{\partial y}$$

$$\partial\!\!\!/_{X_\epsilon} = \frac{\partial}{\partial t}\sigma_0 + e^t\frac{\partial}{\partial x}\sigma_1 + e^{-t}\frac{\partial}{\partial y}\sigma_2$$

$$\partial\!\!\!/_{X_\epsilon} = \begin{pmatrix} \frac{\partial}{\partial t} & e^{-t}\frac{\partial}{\partial y} - ie^t\frac{\partial}{\partial x} \\ e^{-t}\frac{\partial}{\partial y} + ie^t\frac{\partial}{\partial x} & -\frac{\partial}{\partial t} \end{pmatrix}$$

σ_i, for $i = 0, 1, 2$ Pauli matrices

Commutative spectral triple

$$(C^\infty(X_\epsilon), L^2(X_\epsilon, S), \partial\!\!\!/_{X_\epsilon})$$

Isospectral deformations (Connes–Landi)

Given $(C^\infty(X), L^2(X,S), \partial\!\!\!/_X)$ with $T^2 \subset \mathrm{Isom}(X)$

$$\pi(f) = \sum_{n,m\in\mathbb{Z}} \pi(f_{n,m})$$

$$\alpha_\tau(\pi(f_{n,m})) = e^{2\pi i(n\tau_1+m\tau_2)}\,\pi(f_{n,m})$$

$\forall \tau = (\tau_1, \tau_2) \in T^2$

$$\alpha_\tau(T) = U(\tau)TU(\tau)^*, \quad \forall T \in \mathcal{B}(\mathcal{H}), \forall \tau \in T^2$$

$U(\tau)\psi(x) = \psi(\tau^{-1}(x))$ on $\mathcal{H} = L^2(X,S)$

$$U(\tau) = \exp(2\pi i\tau L) = \exp(2\pi i(\tau_1 L_1 + \tau_2 L_2))$$

w/ L_1 and L_2 the infinitesimal generators

Algebra in $\mathcal{B}(\mathcal{H})$ gen by

$$\pi_{\xi_1,\xi_2}(f) = \sum_{n,m} \pi(f_{n,m}) e^{-2\pi i(\xi_1 n L_2 + \xi_2 m L_1)}$$

$$\pi_{\xi_1,\xi_2}(f_{n,m})\pi_{\xi_1,\xi_2}(h_{k,r}) = e^{-2\pi i(\xi_1 nr + \xi_2 mk)}\pi(f_{n,m})\pi(h_{k,r})$$

$\Rightarrow$ cocycle

$$\sigma((n,m),(k,r)) = \exp(-2\pi i(\xi_1 nr + \xi_2 mk))$$

Spectral triple $(C^\infty(X)_{\xi_1,\xi_2}, L^2(X,S), \partial\!\!\!/_X)$

Deforming X_ϵ isospectrally

Fibration $T^2 \to X_\epsilon \to S^1$ torus action

$$U(\tau)\psi((x,y),t) = \psi((x + e^t\tau_1, y + e^{-t}\tau_2), t)$$

preserves the metric $dt^2 + e^t dx^2 + e^{-t}dy^2$

$$U(\tau)\partial_{X_\epsilon}U(\tau)^* = \partial_{X_\epsilon}$$

Infinitesimal generators

$$2\pi L_1 = e^t\frac{\partial}{\partial x}, \quad 2\pi L_2 = e^{-t}\frac{\partial}{\partial y}$$

$$U(\tau) = \exp(2\pi i(\tau_1 L_1 + \tau_2 L_2))$$

$$E_\lambda((x,y),t) := e^{2\pi i\langle\ominus_{-t}(x,y),\lambda\rangle}$$

$\ominus_{-t}(x,y) = (e^{-t}x, e^t y)$ and $\langle(a,b),\lambda\rangle = a\lambda_1 + b\lambda_2$

$$\Xi_u(\lambda, L_1, L_2) := \exp\left(i\pi\frac{u}{(\theta' - \theta)}\lambda \wedge (L_1, L_2)\right)$$

Connes-Landi deformation $C^\infty(X_\epsilon)_u$
subalg of $\mathcal{B}(L^2(X_\epsilon, S)$ gen by

$$\pi_u(f) = E_\lambda \Xi_u(\lambda, L_1, L_2)$$

$\xi_2 = u/2 = -\xi_1$

$$\pi(R^\sigma_\lambda) = E_\lambda \, \Xi_u(\lambda, L_1, L_2)$$

$$\pi(R^\sigma_\lambda)\pi(R^\sigma_\eta) = \sigma(\lambda, \eta)\pi(R^\sigma_{\lambda+\eta})$$

$$\sigma(\lambda, \eta) = \exp(2\pi i u_\theta \, \lambda \wedge \eta)$$

for $u = \theta$ and $u = \theta'$ get $\mathbb{T}_{\Lambda, i}$

$$\Rightarrow (C^*(\Lambda, \sigma), L^2(X_\epsilon, S), \not{\partial}_{X_\epsilon})$$

Representation on $L^2(X_\epsilon, S)$ of $C^*(S(\wedge, V), \tilde{\sigma})$

$$(U(k \log \epsilon)\psi)((x, y), t) = \psi(A_\epsilon^k(x, y), t)$$

$$= \psi((x, y), t - k \log \epsilon)$$

$$\pi(R_{(\lambda, k)}^{\tilde{\sigma}}) = E_\lambda \, \Xi_u(\lambda, L_1, L_2) \, U(k \log \epsilon)$$

$$\pi(R_{(\lambda, k)}^{\tilde{\sigma}})\pi(R_{(\eta, r)}^{\tilde{\sigma}}) = \tilde{\sigma}((\lambda, k), (\eta, r))\pi(R_{(\lambda, k)(\eta, r)}^{\tilde{\sigma}})$$

$$(\lambda, k)(\eta, r) = (\lambda + A_\epsilon^k(\eta), k + r) \in S(\wedge, V)$$

$$\tilde{\sigma}((\lambda, k), (\eta, r)) = \exp\left(2\pi i \frac{u_\theta}{2}\lambda \wedge A_\epsilon^k(\eta)\right)$$

Unitary equivalences (after ADS)

- Fourier modes on the fibers

$$\partial\!\!\!/_{X_\epsilon}\psi_\lambda = (\frac{\partial}{\partial t}\sigma_0 + 2\pi i\lambda_1\sigma_1 + 2\pi i\lambda_2)\psi_\lambda$$

$$\pi(R_\eta^\sigma)\psi_\lambda = e^{2\pi i u\,\eta\wedge\lambda}\psi_{\eta+\lambda}$$

$$[\partial\!\!\!/_{X_\epsilon}, \pi(R_\eta^\sigma)] = (\eta_1\sigma_1 + \eta_2\sigma_2)\,R_\eta^\sigma$$

- $\mathcal{U}\,\psi_\lambda = \sigma_\lambda\,\psi_\lambda$: $\sigma_\lambda = $ prod σ_i, $i = 1, 2$, $\lambda_i < 0$

$$\mathcal{U}\widehat{\partial\!\!\!/}_{X_\epsilon}\mathcal{U}^* = \text{sign}(N(\lambda))(\frac{\partial}{\partial t}\sigma_0 + 2\pi i|\lambda_1|\sigma_1 + 2\pi i|\lambda_2|\sigma_2)$$

$$\mathcal{U}\hat{\pi}(R_\eta^\sigma)\mathcal{U}^* : \sigma_\lambda\psi_\lambda \mapsto \sigma_{\lambda+\eta}\psi_{\lambda+\eta}$$

- For $\lambda = A_\epsilon^k(\mu) \neq 0$, $\mu \in \mathcal{F}_V$

$$\tilde{\mathcal{U}}(\sigma_\lambda\psi_\lambda)(t) = \sigma_\lambda\psi_\lambda(t - \log\frac{|\mu_1|}{|N(\mu)|^{1/2}})$$

$$\tilde{\partial} = \tilde{\partial}^{(0)} + \sum_{\mu \in (\Lambda \setminus \{0\})/V} \tilde{\partial}^{(\mu)}$$

$$\tilde{\psi}_\lambda := \tilde{\mathcal{U}}(\sigma_\lambda \psi_\lambda) = \sigma_\lambda \psi_{|N(\lambda)|^{1/2}(\text{sign}(\lambda_1)\epsilon^k, \text{sign}(\lambda_2)\epsilon^{-k})}$$

Unitarily equivalent: $\tilde{\pi}(R^\sigma_\eta)\,\tilde{\psi}_\lambda = \tilde{\psi}_{\lambda+\eta}$

$$\tilde{\partial}^{(\mu)}\,\tilde{\psi}_{A^k_\epsilon(\mu)} = \text{sign}(N(\mu))|N(\mu)|^{1/2}\cdot$$

$$\left(|N(\mu)|^{-1/2}\frac{\partial}{\partial t}\sigma_0 + 2\pi i \epsilon^k \sigma_1 + 2\pi i \epsilon^{-k} \sigma_2\right) \tilde{\psi}_{A^k_\epsilon(\mu)}$$

$$\tilde{\partial}^{(\mu)} = D_\mu B_\mu$$

$$D_\mu\,\tilde{\psi}_{A^k_\epsilon(\mu)} = \text{sign}(N(\mu))|N(\mu)|^{1/2}\,\tilde{\psi}_{A^k_\epsilon(\mu)}$$

$$B_\mu\,\tilde{\psi}_{A^k_\epsilon(\mu)} = \left(|N(\mu)|^{-1/2}\frac{\partial}{\partial t}\sigma_0 + 2\pi i \epsilon^k \sigma_1 + 2\pi i \epsilon^{-k}\sigma_2\right)\tilde{\psi}_{A^k_\epsilon(\mu)}$$

Foliations, transverse measures, DOs

$$\Theta_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Fixed point $(0,0)$, stable manifold axis $(0,y)$, unstable manifold axis $(x,0)$

$$(x,y) = (s_1 + s_2\theta, s_1 + s_2\theta'), \quad (s_1, s_2) \in \mathbb{R}^2/\mathbb{Z}^2$$

Kronecker foliations

$$L_\theta = \{s_1 + s_2\theta\}, \quad L_{\theta'} = \{s_1 + s_2\theta'\}$$

operators $e^t\frac{\partial}{\partial x}$ and $e^{-t}\frac{\partial}{\partial y}$ leaf-wise derivations
factors e^t and e^{-t} scaling of transverse measure

$$\delta_\theta : \psi_{n,m} \mapsto (n + m\theta)\,\psi_{n,m}$$

$$\delta_{\theta'} : \psi_{n,m} \mapsto (n + m\theta')\,\psi_{n,m}$$

$$\slashed{D}_{\theta,\theta'} = \begin{pmatrix} 0 & \delta_{\theta'} - i\delta_\theta \\ \delta_{\theta'} + i\delta_\theta & 0 \end{pmatrix}$$

$D\!\!\!\!/_{\theta,\theta',0}$ on complement of zero modes

$$D\!\!\!\!/_{\theta,\theta',0} = \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} D\!\!\!\!/^{\mu}_{\theta,\theta'}$$

$$D\!\!\!\!/^{\mu}_{\theta,\theta'}\,\psi_{A^k_\epsilon(\mu)} = (\lambda_1\sigma_1 + \lambda_2\sigma_2)\,\psi_{A^k_\epsilon(\mu)}$$

adiabatic limit of

$$\widehat{\partial\!\!\!/}_{X_\epsilon} : \psi_\lambda \mapsto (\frac{\partial}{\partial t}\sigma_0 + 2\pi i\lambda_1\sigma_1 + 2\pi i\lambda_2\sigma_2)\,\psi_\lambda$$

Unitary equivalence

$$\tilde{D\!\!\!\!/}^{\mu}_{\theta,\theta'}\,\tilde\psi_{A^k_\epsilon(\mu)} = \mathsf{sign}(N(\mu))\,|N(\mu)|^{1/2}\,(\epsilon^k\sigma_1 + \epsilon^{-k}\sigma_2)\,\tilde\psi_{A^k_\epsilon(\mu)}$$

$$\tilde{D\!\!\!\!/}^{\mu}_{\theta,\theta'} = D^{\mu}_\theta B_\theta$$

$$D^{\mu}_\theta\,\tilde\psi_{A^k_\epsilon(\mu)} = \mathsf{sign}(N(\mu))\,|N(\mu)|^{1/2}\,\tilde\psi_{A^k_\epsilon(\mu)}$$

$$B_\theta\,\tilde\psi_{A^k_\epsilon(\mu)} = (\epsilon^k\sigma_1 + \epsilon^{-k}\sigma_2)\,\tilde\psi_{A^k_\epsilon(\mu)}$$

Lorentzian geometry

$$N(\lambda) = \lambda_1\lambda_2 = (n+m\theta)(n+m\theta') \Leftrightarrow \Box = p_0^2 - p_1^2$$

wave operator

$$\mathcal{D}_\lambda = \begin{pmatrix} 0 & \mathcal{D}_\lambda^+ \\ \mathcal{D}_\lambda^- & 0 \end{pmatrix} := \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}$$

$$\mathcal{D}_\lambda^2 = \Box_\lambda$$

$$\Box_\lambda = \begin{pmatrix} N(\lambda) & 0 \\ 0 & N(\lambda) \end{pmatrix}$$

On $\mathcal{H} = \ell^2(\Lambda) \oplus \ell^2(\Lambda)$ operator

$$\mathcal{D}e_{\lambda,\pm} = \mathcal{D}_\lambda e_{\lambda,\pm}$$

$\mathbb{Z}/2\mathbb{Z}$-grading

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$C^*(\Lambda, \sigma)$ acting diagonally on $\mathcal{H}$
Bounded commutators:

$$[\mathcal{D}, R_\eta^\sigma]e_{\lambda,\pm} = \sigma(\lambda, \eta)\,\eta_\mp\,e_{\eta+\lambda,\pm}$$

$\eta_+ = \eta_1$ and $\eta_- = \eta_2$

Problems:
- $\mathcal{D}$ not self-adjoint
- $\mathcal{D}$ has infinite multiplicities in the spectrum: noncompact symmetry group $V = \epsilon^{\mathbb{Z}}$

Two issues:
- work over $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ instead of $\mathbb{C}$
- use indefinite inner product spaces (Krein)

$c : \mathbb{K} \to \mathbb{K}$ Galois involution $c : x \mapsto x'$

$\mathcal{V}$ a $\mathbb{K}$-vector space, $T : \mathcal{V} \to \mathcal{V}$ is c-linear if

$$T(av + bw) = c(a)T(v) + c(b)T(w)$$

• Lorentzian pairing:

$$(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$$

non-degenerate, c-linear in first variable, linear in the second

• $\mathbb{K}$-Krein space: $(\mathcal{V}, (\cdot, \cdot))$ with c-linear involution $\kappa : \mathcal{V} \to \mathcal{V}$

$- (\kappa\cdot, \cdot) = c(\cdot, \kappa\cdot)$

$-$ For all $v \neq 0$ in $\mathcal{V}$, the elements $(\kappa v, v) \in \mathbb{K}$ are totally positive

• Krein adjoint $T^{\dagger}$ of $\mathbb{K}$-linear T

$$(v, Tw) = (T^{\dagger}v, w)$$

- Associated real Hilbert spaces

$$\mathcal{V}_{\mathbb{R},i} := \mathcal{V} \otimes_{\iota_i(\mathbb{K})} \mathbb{R}$$

two embeddings $\iota_i : \mathbb{K} \hookrightarrow \mathbb{R}$

$$\begin{aligned} \langle v, w \rangle &= \tfrac{1}{2}\iota_1\left((\kappa v, w) + (v, \kappa w)\right) \\ &= \tfrac{1}{2}\iota_2\left((\kappa v, w) + (v, \kappa w)\right) \end{aligned}$$

- For $\mathbb{K}$-linear operator T on $\mathbb{K}$-Krein space $\mathcal{V}$

$$\mathbb{M}_i(T) := \inf_{(v,v)=1} \iota_i(Tv, Tv), \quad i = 1, 2$$

Krein-bounded: $\mathbb{M}_i(T) > -\infty$

(Note: need not be bounded in associated Hilbert space)

Lorentzian $\mathbb{K}$-spectral triples: $(\mathcal{A}, \mathcal{V}, \mathcal{D})$

$-$ $\mathcal{A}$ involutive $\mathbb{K}$-algebra

$-$ $\mathcal{V}$ a $\mathbb{K}$-Krein space with action

$$\pi : \mathcal{A} \to \mathsf{End}_{\mathbb{K}}(\mathcal{V})$$

by Krein-bounded

$$\mathbb{M}_i(\pi(a)) > -\infty$$

$\pi(a^*) = \pi(a)^{\dagger}$

$-$ Densely defined $\mathbb{K}$-linear $\mathcal{D}$ with $\mathcal{D}^{\dagger} = \mathcal{D}$

$-$ Krein-bounded commutators $C_a := [\mathcal{D}, \pi(a)]$

$$\mathbb{M}_i(C_a) > -\infty, \quad \forall a \in \mathcal{A}$$

Continue...

...continue

– ∃ densely def $\mathbb{K}$-linear $U : \mathcal{V} \to \mathcal{V}$

$$(Uv, Uv) = (v, v), \quad v \in Dom(U)$$

$U^\dagger = U^{-1}$ and $U^\dagger \mathcal{D} U = \mathcal{D}$

– Commutators $C_{a,U} := [\mathcal{D}, \pi_U(a)]$ Krein-bounded

$$\mathbb{M}_i(C_{a,U}) > -\infty, \quad \forall a \in \mathcal{A}$$

$\pi_U(a) = U^\dagger \pi(a) U$

– U unbounded $U = U^*$ in assoc Hilbert space

– p-summability

$$\sum_n \left| \langle U e_n, |\mathcal{D}^2| U e_n \rangle \right|^{-s/2} < \infty, \quad \forall s \geq p$$

e_n = o.n.basis of complement of zero modes
of $|\mathcal{D}^2|$ on Hilbert space

Lorentzian $\mathbb{K}$-spectral triple for NC tori

- $\mathbb{K}$-Krein space $\mathcal{V}_\Lambda$ spanned by e_λ, $\lambda \in \Lambda$

$$(v, w) := \sum_\lambda c(a_\lambda) b_\lambda$$

$v = \sum_\lambda a_\lambda e_\lambda$ and $w = \sum_\lambda b_\lambda e_\lambda$

- Action of group ring $\mathbb{K}[\Lambda]$

$$R_\lambda e_\eta = e_{\lambda+\eta}$$

$$\mathbb{M}_i(R_\lambda) > -\infty$$

In this case: bounded in assoc Hilbert space

- Twisted group ring $\omega \in \mathbb{K}^*$ with $N(\omega) = \omega\omega' = 1$

$$\varpi(\lambda, \eta) = \omega^{(n,m)\wedge(r,k)}$$

$\lambda = (n + m\theta, n + m\theta')$ and $\eta = (r + k\theta, r + k\theta')$
$\mathbb{K}^*$-valued 2-cocycle ϖ on Λ

$$R_\lambda^{\varpi} R_\eta^{\varpi} = \varpi(\lambda, \eta) R_{\lambda+\eta}^{\varpi}$$

twisted $\mathbb{K}(\Lambda, \varpi)$

- Action of $\mathbb{K}(\Lambda, \varpi)$ on $\mathcal{V}_\Lambda$

$$R_\lambda^\varpi e_\eta = \varpi(\eta, \lambda) e_{\lambda + \eta}$$

Krein-bounded $\mathbb{M}_i(R_\lambda^\varpi) > -\infty$ not-bounded in assoc Hilbert

- Preserves Lorentzian pairing

$$(R_\lambda^\varpi e_\eta, R_\lambda^\varpi e_\zeta) = c(\omega(\eta, \lambda)) \omega(\zeta, \lambda) \delta_{\eta, \zeta} =$$

$$N(\omega(\eta, \lambda)) \delta_{\eta, \zeta} = (e_\eta, e_\zeta)$$

- Lorentzian Dirac operator $\mathcal{D}_\mathbb{K}$

$$\mathcal{D}_{\mathbb{K}, \lambda} e_{\lambda, \pm} = \begin{pmatrix} 0 & \mathcal{D}_\lambda^+ \\ \mathcal{D}_\lambda^- & 0 \end{pmatrix} e_{\lambda, \pm} := \begin{pmatrix} 0 & \ell \\ c(\ell) & 0 \end{pmatrix} e_{\lambda, \pm}$$

Krein self-adjoint $\mathcal{D}_\mathbb{K}^\dagger = \mathcal{D}_\mathbb{K}$

- On associated real Hilbert spaces

$$\mathcal{D}_\lambda = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} \quad \text{and} \quad c(\mathcal{D}_\lambda) = \begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix}$$

- Commutators Krein-bounded

$$[\mathcal{D}_{\mathbb{K}}, R_\lambda^\varpi] e_{\eta,\pm} = \varpi(\eta, \lambda) \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} e_{\lambda+\eta,\pm}$$

$$([\mathcal{D}_{\mathbb{K}}, R_\lambda^\varpi]v, [\mathcal{D}_{\mathbb{K}}, R_\lambda^\varpi]v) = N(\lambda)(v, v)$$

- Fundamental domain $\mathcal{F}_V$ in Λ
$\rho(\lambda) \in \mathbb{Z}$ unique s.t. $\lambda = A_\epsilon^{\rho(\lambda)}(\mu)$, with $\mu \in \mathcal{F}_V$

$$T_\epsilon e_{\lambda,\pm} := \begin{pmatrix} \epsilon^{\rho(\lambda)} & 0 \\ 0 & \epsilon^{-\rho(\lambda)} \end{pmatrix} e_{\lambda,\pm}$$

$$J e_{\lambda,\pm} = e_{J(\lambda),\pm}, \quad J(A_\epsilon^k(\mu)) = A_\epsilon^{-k}(\mu)$$

$U_\epsilon = T_\epsilon J$

$$U_\epsilon e_{\lambda,\pm} = \begin{pmatrix} \epsilon^{-\rho(\lambda)} & 0 \\ 0 & \epsilon^{\rho(\lambda)} \end{pmatrix} e_{J(\lambda),\pm}$$

- $U_\epsilon^\dagger = U_\epsilon^{-1}$ and $(U_\epsilon v, U_\epsilon v) = (v, v)$. Symmetry

$$U_\epsilon^\dagger \mathcal{D}_\mathbb{K} U_\epsilon = \mathcal{D}_\mathbb{K}$$

- Krein-bounded commutators

$$\mathbb{M}_i([\mathcal{D}_\mathbb{K}, U_\epsilon^\dagger R_\lambda^\varpi U_\epsilon]) > -\infty$$

$$(U_\epsilon^\dagger [\mathcal{D}_\mathbb{K}, R_\lambda^\varpi] U_\epsilon v, U_\epsilon^\dagger [\mathcal{D}_\mathbb{K}, R_\lambda^\varpi] U_\epsilon v) = N(\lambda)(v, v)$$

- Hilbert space adjoint $U_\epsilon^* = U_\epsilon$

$$U_\epsilon^* e_{\lambda,\pm} = (\kappa U_\epsilon^\dagger \kappa) e_{\lambda,\pm} = c(U_\epsilon^\dagger) e_{\lambda,\pm}$$

$$= \begin{pmatrix} \epsilon^{-\rho(\lambda)} & 0 \\ 0 & \epsilon^{\rho(\lambda)} \end{pmatrix} e_{J(\lambda),\pm} = U_\epsilon e_{\lambda,\pm}$$

unbounded

- Finite summability:

$$|\mathcal{D}_{\mathbb{K}}^2|e_{\lambda,\pm} = \begin{pmatrix} |N(\lambda)| & 0 \\ 0 & |N(\lambda)| \end{pmatrix} e_{\lambda,\pm}$$

- On complement of zero modes

$$\sum_{\lambda \neq 0} \left| \langle U_\epsilon e_{\lambda,\pm}, |\mathcal{D}_{\mathbb{K}}^2| U_\epsilon e_{\lambda,\pm} \rangle \right|^{-s/2} =$$

$$\sum_{\lambda \neq 0} (\epsilon^{2\rho(\lambda)} + \epsilon^{-2\rho(\lambda)})^{-s/2} |N(\lambda)|^{-s/2} =$$

$$\sum_{k \in \mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s/2} \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} |N(\mu)|^{-s/2}$$

Eta function

$$\eta_{\mathcal{D}}(s) := \sum_n \operatorname{sign}(\langle Ue_n, \mathcal{D}^2 Ue_n\rangle) \left|\langle Ue_n, |\mathcal{D}^2| Ue_n\rangle\right|^{-s/2}$$

Shimizu L-function

$$\eta_{\mathcal{D}_{\mathbb{K}}}(s) = L\left(\Lambda, V, \frac{s}{2}\right) Z_\epsilon\left(\frac{s}{2}\right)$$

$L(\Lambda, V, s)$ Shimizu L-function and

$$Z_\epsilon(s/2) = \sum_{k\in\mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s/2}$$

3-dim geometry adiabatic $\tilde{\slashed{D}}_{\theta,\theta'}$

$$\zeta_{\tilde{\slashed{D}}_{\theta,\theta'}}(s) = 2Z_\epsilon(s/2) \sum_{\mu\in(\Lambda\smallsetminus\{0\})/V} |N(\mu)|^{-s/2}$$

Eta vanishes: symmetry

Restriction $\tilde{\slashed{D}}^+_{\theta,\theta'}$ to positive modes of B_θ

$$\eta_{\tilde{\slashed{D}}^+_{\theta,\theta'}}(s) = L(\Lambda, V, s/2) Z_\epsilon(s/2) = \eta_{\mathcal{D}_{\mathbb{K}}}(s)$$

Residue:

$$\mathrm{Res}_{s=0}\, \eta_{\widetilde{\not{D}}^+_{\theta,\theta'}}(s) = \frac{L(\Lambda, V, 0)}{\log \epsilon}$$

$$Z_\epsilon(s) := \sum_{k \in \mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s}$$

$$\Gamma(s) Z_\epsilon(s) = \int_0^\infty g_\epsilon(t)\, t^{s-1} dt$$

$$g_\epsilon(t) = \left(\sum_{k \in \mathbb{Z}} e^{-(\epsilon^{2k} + \epsilon^{-2k})t} \right)$$

$$g_\epsilon(t) = -e^{-2t} + 2h_\epsilon(t) - 2 \sum_{k=0}^\infty e^{-\epsilon^{2k} t}(1 - e^{-\epsilon^{-2k} t})$$

$$h_\epsilon(t) = \sum_{k=0}^\infty e^{-\epsilon^{2k} t}$$

$$h_\epsilon(t) - h_\epsilon(\epsilon^2 t) = e^{-t} = \sum_{r=0}^\infty \frac{(-1)^r}{r!} t^r$$

$$h_\epsilon(t) = \frac{1}{2 \log \epsilon} \log(1/t) + C - \sum_{r=0}^\infty \frac{(-1)^r}{r!(\epsilon^{2r} - 1)} t^r$$

$\Gamma(s) Z_\epsilon(s)$ double pole $s = 0$, simple poles $s \in \mathbb{Z}_{<0}$

$$\mathrm{Res}_{s=0} Z_\epsilon(s) = \frac{1}{\log \epsilon}$$

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