TWISTED INDEX THEORY ON GOOD ORBIFOLDS I: NONCOMMUTATIVE BLOCH THEORY

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ABSTRACT. We study the twisted index theory of elliptic operators on orbifold covering spaces of compact good orbifolds, which are invariant under a projective action of the orbifold fundamental group. We apply these results to obtain qualitative results on real and complex hyperbolic spaces in 2 and 4 dimensions, related to generalizations of the *Bethe-Sommerfeld conjecture* and the *Ten Martini Problem*, on the spectrum of self adjoint elliptic operators which are invariant under a projective action of a discrete cocompact group.

INTRODUCTION

In this paper, we prove a twisted index theorem for elliptic operators on orbifold covering spaces of compact good orbifolds, which are invariant under a projective action of the orbifold fundamental group.

Let Γ be a Fuchsian group of signature $(g, \nu_1, \ldots, \nu_n)$ (cf. section 1 for more details), that is, Γ is the orbifold fundamental group of the 2 dimensional hyperbolic orbifold $\Sigma(g, \nu_1, \ldots, \nu_n)$ of signature $(g, \nu_1, \ldots, \nu_n)$. Using a result of Kasparov [Kas1] on K-amenable groups as well as a calculation by Farsi [Far] of the orbifold K-theory $K^{\bullet}_{orb}(\Sigma(g, \nu_1, \ldots, \nu_n))$ of compact 2-dimensional hyperbolic orbifolds, we are able to compute the K-theory of twisted group C^* algebras, under the assumption that the Dixmier-Douady invariant of the multiplier σ is trivial

$$K_j(C^*(\Gamma, \sigma)) \cong \begin{cases} \mathbb{Z}^{2-n+\sum_{i=1}^{n} \nu_j} & \text{if } j = 0; \\ \\ \mathbb{Z}^{2g} & \text{if } j = 1. \end{cases}$$

Notice that K_0 is much larger in the general Fuchsian group case than in the torsionfree case, where K_0 was determined to be always $\mathbb{Z}^2$, [CHMM]. We also show that the orbifold K-theory of any 2-dimensional orbifold is generated by orbifold line bundles. The result is derived by means of equivariant K-theory and the Baum-Connes [BC] equivariant Chern character with values in the delocalized equivariant cohomology of the smooth surface $\Sigma_{g'}$ that covers the good orbifold $\Sigma(g, \nu_1, \ldots, \nu_n)$. We show that the Seifert invariants, cf. [Sc], correspond to the pairing of the equivariant Chern character of [BC] with a fundamental class in the delocalized equivariant homology of $\Sigma_{g'}$.

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Let tr denote the canonical trace on the twisted group C^*-algebra, $C^*(\Gamma,\sigma)$, which induces a map [tr] on K-theory. Using the results above and our twisted index theorem for orbifolds, we compute in section 2 under the same assumptions as before, the range of the trace on K-theory to be,

$$[\mathrm{tr}](K_0(C^*(\Gamma,\sigma))) = \mathbb{Z}\,\theta + \mathbb{Z} + \sum_{i=1}^{n} \mathbb{Z}(1/\nu_i)$$

where θ denotes the evaluation of the multiplier σ on the fundamental class of Γ. We then apply our calculation of the range of the trace on K-theory to the study of some quantitative aspects of the spectrum of projectively periodic elliptic operators on the hyperbolic plane, what is known as *noncommutative Bloch theory.* Some of the most outstanding open problems about magnetic Schrödinger operators or Hamiltonians on Euclidean space are concerned with the nature of their spectrum, most notably the *Bethe-Sommerfeld conjecture* (BSC) and the *Ten Martini Problem* (TMP) (cf. [Sh]). More precisely, TMP asks whether given a multiplier σ on $\mathbb{Z}^2$, is there an associated Hamiltonian (*i.e.* a Hamiltonian which commutes with the $(\mathbb{Z}^2,\bar{\sigma})$ projective action of $\mathbb{Z}^2$ on $L^2(\mathbb{R}^2)$) possessing a Cantor set type spectrum, in the sense that the intersection of the spectrum of the Hamiltonian with some compact interval in $\mathbb{R}$ is a Cantor set? One can deduce from the range of the trace on K_0 of the twisted group C^*-algebras that when the multiplier takes its values in the roots of unity in $U(1)$ (we say then that it is rational) that such a Hamiltonian cannot exist. However, in the Euclidean case and for almost all irrational numbers, the discrete form of TMP has been settled in the affirmative, cf. [Last]. BSC asserts that if the multiplier is trivial, then the spectrum of any associated Hamiltonian has only a *finite* number of gaps. This was first established in the Euclidean case by Skrigonov [Skri]. In Sections 2 and 3, we are concerned also with generalizations of the TMP and the BSC, which we call the *Generalized Ten Dry Martini Problem* and the *Generalized Bethe-Sommerfeld conjecture.* We prove that the Kadison constant of the twisted group C^*-algebra $C_r^*(\Gamma,\sigma)$ is positive whenever the multiplier is rational, where Γ is now the orbifold fundamental group of a signature $(g,\nu_1,\ldots,\nu_n)$ hyperbolic orbifold. We then use the results of Brüning and Sunada [BrSu] to deduce that when the multiplier is rational, the generalized Ten Dry Martini Problem is answered in the negative, and we leave open the more difficult irrational case. More precisely, we show that the spectrum of such a $(\Gamma,\bar{\sigma})$ projectively periodic elliptic operator is the union of countably many (possibly degenerate) closed intervals which can only accumulate at infinity. This also gives evidence that the generalized Bethe-Sommerfeld conjecture is true, and generalizes earlier results of [CHMM] in the torsion-free case. In order to show that the results are not a purely two-dimensional phenomenon, we present similar results on real and complex hyperbolic four-manifolds, see also [M]. In section 3, we again use the range of the trace theorem above, together with other geometric arguments to give a complete classification upto isomorphism of the twisted group C^* algebras $C_r^*(\Gamma,\sigma)$, where σ is assumed to have trivial Dixmier-Douady invariant as before.

In a forthcoming paper we shall generalize these results by proving a twisted higher index theorem which adapts the index theorems of Atiyah, Connes and Moscovici, and Gromov, to the case of good orbifolds. This will allow us to compute the range of some higher traces on K-theory. More precisely, suppose that c is the area 2-cocycle on the group Γ as given above, and tr_c is the induced cyclic 2-cocycle on a smooth subalgebra of the twisted group C^*-algebra $C^*(\Gamma,\sigma)$, which induces a map $[\mathrm{tr}_c]$ on K-theory. Then we will prove that

$$[\mathrm{tr}_c](K_0(C^*(\Gamma,\sigma))) = \phi\,\mathbb{Z}$$

where $\phi = 2(g-1) + (n-\nu) \in \mathbb{Q}$, $\quad \nu = \sum_{j=1}^n 1/\nu_j$. By relating the hyperbolic Connes-Kubo cyclic 2-cocycle and the area cyclic 2-cocycle, the range of the higher trace on K-theory can be used to compute the values of the Hall conductance: The results will be applied to the study the occurrence of *fractional quantum numbers* in the Quantum Hall Effect on the hyperbolic plane. We will also establish the noncommutative Bloch theory results for discrete Harper type operators, which is the analogue in the discrete case of results proved in the present paper. The results contained in this paper, together with the results on the fractional quantum numbers, were circulated as a preprint in 1998, [MM].

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1. Preliminaries

1.1. **Good orbifolds.** For further details on the fundamental material on orbifolds, see [Sc], [FuSt] and [Bro].

The definition of an orbifold generalises that of a manifold. More precisely, an *orbifold* M of dimension m is a Hausdorff, second countable topological space with a Satake atlas $\mathcal{V} = \{U_i, \phi_i\}$ which covers M, consisting of open sets U_i and homeomorphisms $\phi_i : U_i \to D^m/G_i$, where D^m denotes the unit ball in $\mathbb{R}^m$ and G_i is a finite subgroup of the orthogonal group $O(m)$, satisfying the following compatibility relations; the compositions

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

locally lifts to be a smooth map $\mathbb{R}^m \to \mathbb{R}^m$, whenever the intersection $U_i \cap U_j \neq \emptyset$. The open sets U_i are called local orbifold charts. In general, an orbifold M can be obtained as a quotient $M = X/G$ of an infinitesimally free compact Lie group action on a smooth manifold X. In fact, by Satake [Sat] and Kawasaki [Kaw], X can chosen to be the smooth manifold of orthonormal frames of the orbifold tangent bundle of M (cf. section 1.4) and G can be chosen to be the orthogonal group $O(m)$.

An orbifold covering of M is an orbifold map $f : Y \to M$, where Y is also an orbifold, such that any point on M has a neighbourhood U such that $f^{-1}(U)$ is the disjoint union of open sets U_α, with $f\,|_{U_\alpha} : U_\alpha \to U$ a quotient map between two quotients of $\mathbb{R}^k$ by finite groups $H_1 < H_2$. The generic fibers of the covering map f are isomorphic to a discrete group which acts as deck transformations.

An orbifold M is *good* if it is orbifold-covered by a smooth manifold; it is *bad* otherwise. A good orbifold is said to be *orientable* if it is orbifold covered by an oriented manifold and the deck transformations act via orientation preserving diffeomorphisms on the orbifold cover. Equivalently, as shown in [Sat] and [Kaw], an orbifold is orientable if it has an oriented frame bundle X such that $M = X/SO(m)$.

We next recall briefly some basic notions on Euclidean and hyperbolic orbifolds, which are by fiat orbifolds whose universal orbifold covering space is Euclidean space and hyperbolic space respectively. We are mainly interested in the case of 2 dimensions, and we will assume that the orbifolds in this paper are orientable.

A 2-dimensional compact orbifold has singularities that are cone points or reflector lines. Up to passing to $\mathbb{Z}_2$-orbifold covers, it is always possible to reduce to the case with only isolated cone points.

Let $\mathbb{H}$ denote the hyperbolic plane and Γ a Fuchsian group of signature $(g, \nu_1, \ldots, \nu_n)$, that is, Γ is a discrete cocompact subgroup of $PSL(2, \mathbb{R})$ of genus g and with n elliptic elements of order $\nu_1, \ldots, \nu_n$ respectively. Explicitly,

$$\Gamma = \Big\{ A_i, B_i, C_j \in PSL(2, \mathbb{R}) | i = 1, \ldots g, \quad j = 1, \ldots n,$$

$$\prod_{i=1}^{g} [A_i, B_i] C_1 \ldots C_n = 1, \quad C_j^{\nu_j} = 1, \quad j = 1, \ldots n \Big\}$$

Then the corresponding compact oriented hyperbolic 2-orbifold of signature $(g, \nu_1, \ldots, \nu_n)$ is defined as the quotient space

$$\Sigma(g, \nu_1, \ldots, \nu_n) = \Gamma \backslash \mathbb{H}.$$

A compact oriented 2-dimensional Euclidean orbifold is obtained in a similar manner, but with $\mathbb{H}$ replaced by $\mathbb{R}^2$, and a complete list of these can be found in [Sc]. Then $\Sigma(g, \nu_1, \ldots, \nu_n)$ is a compact surface of genus g with n elliptic points $\{p_j\}_{j=1}^n$ such that each p_j has a small coordinate neighborhood $U_{p_j} \cong D^2/\mathbb{Z}_{\nu_j}$, where D^2 denotes the unit disk in $\mathbb{R}^2$ and $\mathbb{Z}_{\nu_j}$ is the cyclic group of order ν_j, $\;j = 1, \ldots n$. Observe that the complement $\Sigma(g, \nu_1, \ldots, \nu_n) \setminus \cup_{j=1}^n U_{p_j}$ is a compact Riemann surface of genus g and with n boundary components. The group Γ is the orbifold fundamental group of $\Sigma(g, \nu_1, \ldots, \nu_n)$, where the generators C_j can be represented by the n boundary components of the surface $\Sigma(g, \nu_1, \ldots, \nu_n) \setminus \cup_{j=1}^n U_{p_j}$.

All Euclidean and hyperbolic 2-dimensional orbifolds $\Sigma(g, \nu_1, \ldots, \nu_n)$ are good, being in fact orbifold covered by a smooth surface $\Sigma_{g'}$ cf. [Sc], i.e. there is a finite group G acting on $\Sigma_{g'}$ with quotient $\Sigma(g, \nu_1, \ldots, \nu_n)$, where $g' = 1 + \frac{\#(G)}{2}(2(g-1) + (n - \nu))$ and where $\nu = \sum_{j=1}^n 1/\nu_j$. According to the classification of 2-dimensional orbifolds given in [Sc], the only bad 2-orbifolds are the "teardrop", with underlying surface S^2 and one cone point of cone angle $2\pi/p$, and the "double teardrop", with underlying surface S^2 and two cone points with angles $2\pi/p$ and $2\pi/q$, $p \neq q$.

In this paper we restrict our attention to good orbifolds. It should be pointed out that the techniques used in this paper cannot be extended directly to the case of bad orbifolds. It is reasonable to expect that the index theory on bad orbifolds will involve analytical techniques for more general conic type singularities.

1.2. **The twisted C^* algebra.** Let M be a good, compact orbifold, and $\mathcal{E} \to M$ be an orbifold vector bundle over M, and $\widetilde{\mathcal{E}} \to \widetilde{M}$ be its lift to the universal orbifold covering space $\Gamma \to \widetilde{M} \to M$, which is by assumption a simply-connected smooth manifold. We will now briefly review how to construct a $(\Gamma, \bar\sigma)$-action (where σ is a multiplier on Γ and $\bar\sigma$ denotes its complex conjugate) on $L^2(\widetilde{M})$. Let $\omega = d\eta$ be an exact 2-form on $\widetilde{M}$ such that ω is also Γ-invariant, although η is *not* assumed to be Γ-invariant. Define a Hermitian connection on the trivial line bundle over $\widetilde{M}$ as

$$\nabla = d + i\eta$$

Its curvature is $\nabla^2 = i\omega$. Then ∇ defines a $(\Gamma, \bar\sigma)$ action on E $L^2(\widetilde{M}, \widetilde{\mathcal{S}^\pm \otimes E})$ as follows:

Since ω is Γ invariant, one has $\forall \gamma \in \Gamma$

$$0 = \gamma^*\omega - \omega = d(\gamma^*\eta - \eta),$$

so that $\gamma^*\eta - \eta$ is a closed 1-form on a simply-connected manifold $\widetilde{M} \Rightarrow \gamma^*\eta - \eta = d\psi_\gamma$ for some smooth function ψ_γ on $\widetilde{M}$ satisfying

- $\psi_\gamma(x) + \psi_{\gamma'}(\gamma x) - \psi_{\gamma'\gamma}(x)$ is independent of x $\forall x \in \widetilde{M}$
- $\psi_\gamma(x_0) = 0$ for some $x_0 \in \widetilde{M}$ $\forall \gamma \in \Gamma$

Then $\bar\sigma(\gamma, \gamma') = \exp(i\psi_\gamma(\gamma' x_0))$ defines a multiplier on Γ. Now define the $(\Gamma, \bar\sigma)$ action as follows: For $u \in L^2(\widetilde{M}, \widetilde{\mathcal{S}^\pm \otimes E})$, $U_\gamma\, u = \gamma^* u$, $S_\gamma u = \exp(i\,\psi_\gamma)\, u$, define $T_\gamma = U_\gamma \circ S_\gamma$. Then it satisfies $T_{\gamma_1} T_{\gamma_2} = \bar\sigma(\gamma_1, \gamma_2)\, T_{\gamma_1\gamma_2}$, and so it defines a $(\Gamma, \bar\sigma)$-action. It can be shown that only multipliers $\bar\sigma$ such that the Dixmier-Douady invariant $\delta(\bar\sigma) = 0$ can give rise to $(\Gamma, \bar\sigma)$-actions in this way cf. section 2.2 for a further discussion.

Let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \to L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ be a self adjoint elliptic differential operator that commutes with a $(\Gamma, \bar\sigma)$-action T_γ $\forall \gamma \in \Gamma$ on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$. Then by the functional calculus, all the spectral projections of D, $E_\lambda = \chi_{[0,\lambda]}(D)$ are bounded self adjoint operators on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ that commute with T_γ $\forall \gamma \in \Gamma$. Now the commutant of the $(\Gamma, \bar\sigma)$-action on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ is a von Neumann algebra

$$\mathcal{U}(\sigma) = \left\{ Q \in B(L^2(\widetilde{M}, \widetilde{\mathcal{E}})) \middle| \quad T_\gamma Q = Q T_\gamma \quad \forall \gamma \in \Gamma \right\}.$$

Since $T_\gamma Q = Q T_\gamma$, one sees that

$$e^{i\phi_\gamma(x)} k_Q(\gamma x, \gamma y) e^{-i\phi_\gamma(y)} = k_Q(x, y)$$

$\forall x, y \in \widetilde{M}$ $\forall \gamma \in \Gamma$, where k_Q denotes the Schwartz kernel of Q. In particular, observe that $\mathrm{tr}(k_Q(x,x))$ is a Γ-invariant function on $\widetilde{M}$. Using this, one sees that there is a semifinite trace on this von Neumann algebra

$$\mathrm{tr} : \mathcal{U}(\sigma) \to \mathbb{C}$$

defined as in the untwisted case due to Atiyah [At],

$$Q \to \int_M \mathrm{tr}(k_Q(x,x))dx$$

where k_Q denotes the Schwartz kernel of Q. Note that this trace is finite whenever k_Q is continuous in a neighborhood of the diagonal in $\widetilde{M} \times \widetilde{M}$.

By elliptic regularity, the spectral projection E_λ has a smooth Schwartz kernel, so that in particular, the spectral density function, $N_Q(\lambda) = \mathrm{tr}(E_\lambda) < \infty$ $\forall \lambda$, is well defined.

If $\mathcal{F}$ is a fundamental domain for the action of Γ on $\widetilde{M}$, one sees that

$$L^2(\widetilde{M},\widetilde{\mathcal{E}}) \cong L^2(\Gamma)\otimes L^2(\mathcal{F},\widetilde{\mathcal{E}}|_{\mathcal{F}})$$

which can be proved by choosing a bounded measurable almost everywhere smooth section of the orbifold covering $\widetilde{M}\to M$. Therefore it follows easily that

$$\mathcal{U}(\sigma)\cong\mathcal{U}(\Gamma,\sigma)\otimes B(L^2(\mathcal{F},\widetilde{\mathcal{E}}|_{\mathcal{F}}))$$

where the twisted group von Neumann algebra $\mathcal{U}(\Gamma,\sigma)$ is the weak closure of the twisted group algebra $\mathbb{C}(\Gamma,\sigma)$ and $B(L^2(\mathcal{F},\widetilde{\mathcal{E}}|_{\mathcal{F}}))$ denotes the algebra of bounded operators on the Hilbert space $L^2(\mathcal{F},\widetilde{\mathcal{E}}|_{\mathcal{F}})$. There is a natural subalgebra $C^*(\sigma)$ of $\mathcal{U}(\sigma)$ which is defined as follows. Let

$$\begin{aligned} {C_c}^\infty(\sigma) = &\{Q\in\mathcal{U}(\sigma) | k_Q \text{ is smooth and supported in a compact} \\ &\text{neighborhood of the diagonal}\} \end{aligned}$$

Then $C^*(\sigma)$ is defined to be the *norm closure* of ${C_c}^\infty(\sigma)$. It can also be shown to be the norm closure of

$$\left\{Q\in\mathcal{U}_{\widetilde{M}}(\Gamma,\sigma) | k_Q \text{ is smooth and } k_Q(x,y) \text{ is } L^1 \text{ in both the } x \text{ and } y \text{ variables separately}\right\}$$

The elements of $C^*(\sigma)$ have the additional property of some off-diagonal decay. It can be shown that

$$C^*(\sigma)\cong C^*(\Gamma,\sigma)\otimes\mathcal{K}(L^2(\mathcal{F},\widetilde{\mathcal{E}}|_{\mathcal{F}}))$$

where the twisted group C^* algebra $C^*(\Gamma,\sigma)$ is the norm closure of the twisted group algebra $\mathbb{C}(\Gamma,\sigma)$ and $\mathcal{K}(L^2(\mathcal{F},\widetilde{\mathcal{E}}|_{\mathcal{F}}))$ denotes the algebra of compact operators on the Hilbert space $L^2(\mathcal{F},\widetilde{\mathcal{E}}|_{\mathcal{F}})$.

1.3. **The C^* algebra of an orbifold.** Let M be an oriented orbifold of dimension m, that is $M = P/SO(m)$, where P is the bundle of oriented frames on the orbifold tangent bundle (cf. section 1.4). Then the C^* algebra of the orbifold M is by fiat the crossed product $C^*(M) = C(P)\rtimes SO(m)$, where $C(P)$ denotes the C^* algebra of continuous functions on P. We will now study some Morita equivalent descriptions of $C^*(M)$ that will be useful for us later. The following is one such, and is due to [Far].

Proposition 1.1. *Let M be a good orbifold, which is orbifold covered by the smooth manifold X, i.e. $M = X/G$. Then the C^* algebras $C_0(X)\rtimes G$ and $C^*(M)$ are strongly Morita equivalent.*

In the two dimensional case, there is yet another C^* algebra that is strongly Morita equivalent to the C^* algebra of the orbifold. Let Γ be as before. Then Γ acts freely on $PSL(2,\mathbb{R})$, and therefore the quotient space $\Gamma\backslash PSL(2,\mathbb{R}) = P(g,\nu_1,\dots,\nu_n)$ is a smooth compact manifold, with a right action of $SO(2)$ that is only infinitesimally free. The C^* algebra of the hyperbolic orbifold $\Sigma(g,\nu_1,\dots,\nu_n)$ is by fiat the crossed product C^* algebra

$$C^*(\Sigma(g,\nu_1,\dots,\nu_n)) = C(P(g,\nu_1,\dots,\nu_n))\rtimes SO(2)$$

cf. [Co]. If $SO(2)$ did act freely on $P(g,\nu_1,\dots,\nu_n)$ (which is the case when $\nu_1=\ldots=\nu_n=1$), then it is known that $C^*(\Sigma(g,\nu_1,\dots,\nu_n))$ and $C(\Sigma(g,\nu_1,\dots,\nu_n))$ are strongly Morita equivalent as C^* algebras.

We shall next describe a natural algebra which is Morita equivalent to the C^* algebra of the orbifold $\Sigma(g,\nu_1,\dots,\nu_n)$. Now Γ has a torsionfree subgroup $\Gamma_{g'}$ of finite index, such that the quotient $\Gamma_{g'}\backslash\mathbb{H} = \Gamma_{g'}\backslash PSL(2,\mathbb{R})/SO(2) = \Sigma_{g'}$ is a compact Riemann surface of genus $g' = 1+\frac{\#(G)}{2}(2(g-1)+(n-\nu))$ where $\nu=\sum_{j=1}^n 1/\nu_j$, cf. Theorem 2.5 [Sc], and the orbifold Euler characteristic calculations in there. Then $G\to\Sigma_{g'}\to\Sigma(g,\nu_1,\dots,\nu_n)$ is a finite orbifold cover, i.e. a ramified covering space, where $G=\Gamma_{g'}\backslash\Gamma$.

Proposition 1.2. *The C^* algebras $C(\Sigma_{g'})\rtimes G$, $C^*(\Sigma(g,\nu_1,\dots,\nu_n))$ and $C_0(\mathbb{H})\rtimes\Gamma$ are strongly Morita equivalent to each other.*

Proof. The strong Morita equivalence of the last two C^* algebras is contained in the previous Proposition. Since strong Morita equivalence is an equivalence relation, it suffices to prove that the first two C^* algebras are strongly Morita equivalent. Let $\hat{P}=\Gamma_{g'}\backslash PSL(2,\mathbb{R})$ where $SO(2)$ acts on $\hat{P}$ the right, and therefore commutes with the left G action on $\hat{P}$. Moreover, the actions of G and $SO(2)$ on $\hat{P}$ are free, and therefore one can apply a theorem of Green, [Green] which implies in particular that $C_o(G\backslash\hat{P})\rtimes SO(2)$ and $C_o(\hat{P}/SO(2))\rtimes G$ are strongly Morita equivalent, i.e. $C_o(P(g,\nu_1,\dots,\nu_n)\rtimes SO(2)$ and $C_o(\Sigma_{g'})\rtimes G$ are strongly Morita equivalent, proving the proposition. □

1.4. **Orbifold vector bundles and K-theory.** Because of the Morita equivalences of the last section 1.3, we can give several alternate and equivalent descriptions of orbifold vector bundles over orbifolds. Firstly, there is the description using transition functions cf. [Sat], [Kaw]. Equivalently, one can view an orbifold vector bundle over an m dimensional orbifold M as being an $SO(m)$ equivariant vector bundle over the bundle P of oriented frames of the orbifold tangent bundle. In the case of a good orbifold M, which is orbifold covered by a smooth manifold X. Let G be the discrete group acting on X, $G\to X\to M=X/G$. Then an orbifold vector bundle on M is the quotient $\mathcal{V}_M=G\backslash\mathcal{V}_X$ of a vector bundle over X by the G action. Notice that an orbifold vector bundle is not a vector bundle over M: in fact, the fibre at a singular point is isomorphic to a quotient of a vector space by a finite group action.

The Grothendieck group of isomorphism classes of orbifold vector bundles on the orbifold M is called the *orbifold K-theory* of M and is denoted by $K^0_{orb}(M)$, which by a result of [BC], [Far] is canonically isomorphic to $K_0(C^*(M))$. By the Morita equivalence of section 1.3, one then has $K^0_{orb}(M)\cong K^0_{SO(m)}(P)$, and by the Julg-Green theorem [Ju], [Green], the second group is somorphic to $K_0(C(P)\rtimes SO(m))$. In the case when M is a good orbifold, by Proposition 1.1, one sees that $K^0_{orb}(M)\cong K^0(C_0(X)\rtimes G)=K^0_G(X)$.

We will now be mainly interested in orbifold line bundles over the hyperbolic 2-orbifolds. Let G be the finite group determined by the exact sequence $1\to\Gamma_{g'}\to\Gamma\to G\to 1$. Then G acts on $\Sigma_{g'}$ with quotient the orbifold $\Sigma(g,\nu_1,\dots,\nu_n)$.

An orbifold line bundle $\mathcal{L}$ on $\Sigma(g,\nu_1,\dots,\nu_n)$ is given by

$$\mathcal{L}=G\backslash(P\times_{SO(2)}\mathbb{C}),$$

where P is a principal $SO(2)$-bundle on the smooth surface $\Sigma_{g'}$. Notice that the $SO(2)$ and the G actions commute, and are free on the total space P. An orbifold line bundle has an associated

Seifert fibred space $G\backslash P$. A more explicit local geometric construction of $\mathcal{L}$ is given in [Sc]. An orbifold line bundle $\mathcal{L}$ over a hyperbolic orbifold $\Sigma(g,\nu_1,\dots,\nu_n)$ is specified by the Chern class of the pullback line bundle on the smooth surface $\Sigma_{g'}$, together with the Seifert data. That is the pairs of numbers (β_j,ν_j), where β_j satisfies the following condition. Given the exact sequence

$$1\to\mathbb{Z}\to\pi_1(P)\to\pi_1^{orb}(\Sigma(g,\nu_1,\dots,\nu_n))\to 1,$$

let $\tilde{C}_j$ be an element of $\pi_1(P)$ that maps to the generator C_j of the orbifold fundamental group. Let C be the generator of the fundamental group of the fibre. Then we have $C_j^{\nu_j}=1$ and $C^{\beta_j}=\tilde{C}_j^{\nu_j}$. The choice of β_j can be normalised so that $0<\beta_j<\nu_j$.

More geometrically, let $\Sigma(g,\nu_1,\dots,\nu_n)$ be a hyperbolic orbifold with cone points $p_1,\dots,p_n$. Let Σ' be the complement of the union of small disks around the cone points. The orbifold line bundle induces a line bundle $\mathcal{L}'$ over the smooth surface with boundary Σ', trivialized over the boundary components of Σ'. Moreover, the restriction of the orbifold line bundle $\mathcal{L}$ over the small disks D_{p_i} around each cone point p_i is obtained by considering a surgery on the trivial product $\mathbb{C}\times D_{p_i}$ obtained by cutting open along a radius in $\mathbb{C}$ and gluing back after performing a rotation on D_{p_i} by an angle $2\pi q/\nu_i$. With this notation the Seifert invariants are (q_i,ν_i) with $\beta_i q_i\equiv 1$ $(mod \nu_i)$.

Thus, an orbifold line bundle has a finite set of singular fibres at the cone points. The orbifold line bundle $\mathcal{L}$ pulls back to a G-equivariant line bundle $\tilde{\mathcal{L}}$ over the smooth surface $\Sigma_{g'}$ that orbifold covers $\Sigma(g,\nu_1,\dots,\nu_n)$. All the orbifold line bundles with trivial orbifold Euler class, as defined in [Sc], lift to the trivial line bundle on $\Sigma_{g'}$.

In [Sc] the classification of Seifert-fibred spaces is derived using the Seifert invariants, namely the Chern class of the line bundle $\tilde{\mathcal{L}}$, together with the Seifert data (β_j,ν_j) of the singular fibres at the cone points p_j. We show in the following that the Seifert invariants can be recovered from the image of the Baum-Connes equivariant Chern character [BC].

1.5. **Baum-Connes Chern character.** We have seen that the algebra $C^*(\Sigma(g,\nu_1,\dots,\nu_n))$ is strongly Morita equivalent to the cross product $C(\Sigma_{g'})\rtimes G$. Therefore the relevant K-theory is

$$K_0(C(\Sigma_{g'})\rtimes G)=K^0_{SO(2)}(G\backslash\hat{P})=K^0_G(\Sigma_{g'}),$$

where $\hat{P}=\Gamma_{g'}\backslash PSL(2,\mathbb{R})$.

We recall briefly the definition of delocalised equivariant cohomology for a finite group action on a smooth manifold [BC]. Let G be a finite group acting smoothly and properly on a compact smooth manifold X. Let M be the good orbifold $M=G\backslash X$. Given any $\gamma\in G$, the subset X^γ of X given by

$$X^\gamma=\{(x,\gamma)\in X\times G\mid \gamma x=x\}$$

is a smooth compact submanifold. Let $\hat{X}$ be the disjoint union of the X^γ for $\gamma\in G$. The complex $\Omega_G(\hat{X})$ of G-invariant de Rham forms on $\hat{X}$ with coefficients in $\mathbb{C}$ computes the delocalised equivariant cohomology $H^\bullet(X,G)$, which is $\mathbb{Z}_2$ graded by forms of even and odd degree. The dual complex that computes delocalised homology is obtained by considering G-invariant de Rham currents on $\hat{X}$. Thus we have

$$H^\bullet(X,G) = H^\bullet(\Omega_G(\hat{X}),d) = H^\bullet(\hat{X}/G,\mathbb{C})$$
$$= H^\bullet(\hat{X},\mathbb{C})^G = \bigoplus_{\gamma\in G} H^\bullet(X^\gamma,\mathbb{C}).$$

According to [BC], Theorem 7.14, the delocalised equivariant cohomology is isomorphic to the cyclic cohomology of the algebra $C^\infty(X)\rtimes G$,

$$H^0(X,G) \cong HC^{ev}(C^\infty(X)\rtimes G),$$
$$H^1(X,G) \cong HC^{odd}(C^\infty(X)\rtimes G).$$

The Baum-Connes equivariant Chern character

$$ch_G : K_G^0(X) \to H^0(X,G)$$

is an isomorphism over the complex numbers. Equivalently, the Baum-Connes equivariant Chern character can be viewed as

$$ch_G : K^0_{orb}(M) \to H^0_{orb}(M)$$

where the *orbifold cohomology* is by definition $H^j_{orb}(M) = H^j(X,G)$ for $j=0,1$.

In our case the delocalised equivariant cohomology and the Baum-Connes Chern character have a simple expression. Let $\Sigma_{g'}$ be the smooth surface that orbifold covers $\Sigma(g,\nu_1,\dots,\nu_n)$. Let G be the finite group $1\to\Gamma_{g'}\to\Gamma\to G\to 1$. Let $G_{p_j}\cong\mathbb{Z}_{\nu_j}$ be the stabilizer of the cone point p_j in $\Sigma(g,\nu_1,\dots,\nu_n)$. Then we have

$$\Sigma_{g'}^\gamma = \begin{cases} \Sigma_{g'} & \text{if} \quad \gamma = 1; \\ \{p_j\} & \text{if} \quad \gamma\in G_{p_j}\backslash\{1\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Thus the delocalised equivariant cohomology and orbifold cohomology is given by

$$H^0_{orb}(\Sigma(g,\nu_1,\dots,\nu_n)) = H^0(\Sigma_{g'},G) = H^0(\Sigma_{g'})\oplus H^2(\Sigma_{g'})\oplus\mathbb{C}^{\sum_j(\nu_j-1)},$$

where each $\mathbb{C}^{\nu_j-1}$ is given by ν_j-1 copies of $H^0(p_j)$, and

$$H^1_{orb}(\Sigma(g,\nu_1,\dots,\nu_n) = H^1(\Sigma_{g'},G) = H^1(\Sigma_{g'}).$$

Let $\mathcal{L}$ be an orbifold line bundle in $K_0(C(\Sigma_g)\rtimes G) = K^0_G(\Sigma_g)$, and let $\tilde{\mathcal{L}}$ be the corresponding line bundle over the surface $\Sigma_{g'}$. An element γ in the stabiliser G_{p_j} acts on the restriction of $\mathcal{L}|_{\Sigma_g^\gamma} = \mathcal{L}|_{p_j} = \mathbb{C}$ as multiplication by $\lambda(\gamma) = e^{2\pi i\beta_j/\nu_j}$.

Thus, the Baum-Connes Chern character of $\mathcal{L}$ is given by

$$ch_G(\mathcal{L}) = (1, c_1(\tilde{\mathcal{L}}), e^{2\pi i\beta_1/\nu_1},\dots,e^{2\pi i(\nu_1-1)\beta_1/\nu_1},\dots e^{2\pi i\beta_n/\nu_n},\dots,e^{2\pi i(\nu_n-1)\beta_n/\nu_n}).$$

Proposition 1.3. *The Baum-Connes Chern character classifies orbifold line bundles over the orbifold* $\Sigma(g,\nu_1,\dots,\nu_n)$.

Proof. According to [Sc] the orbifold line bundles are classified by the orbifold Euler number

$$e(\Sigma(g,\nu_1,\dots,\nu_n)) = < c_1(\tilde{\mathcal{L}}), [\Sigma_{g'}] > + \sum_j \beta_j/\nu_j,$$

given in terms of the Chern number $< c_1(\tilde{\mathcal{L}}), [\Sigma_{g'}] >$ and the Seifert invariants (β_j, ν_j). $\square$

Notice that we have the isomorphism in K-theory, $K^0_G(\Sigma_{g'}) = K^0_{SO(2)}(G\backslash\hat{P})$ and the Chern character isomorphisms (with $\mathbb{C}$ coefficients)

$$ch_G : K^0_G(\Sigma_{g'}) \to H^0(\Sigma_{g'}, G) \cong HC^{ev}(C^\infty(\Sigma_{g'}) \rtimes G)$$

and

$$ch_{SO(2)} : K^0_{SO(2)}(\Gamma\backslash PSL(2,\mathbb{R})) \to HC^{ev}(C^\infty(\Gamma\backslash PSL(2,\mathbb{R})) \rtimes SO(2)).$$

Moreover, we have an isomorphism

$$HC^\bullet(C^\infty(\Gamma\backslash PSL(2,\mathbb{R})) \rtimes SO(2)) \cong H^\bullet_{SO(2)}(\Gamma\backslash PSL(2,\mathbb{R})).$$

Thus, we obtain

$$HC^{ev}(C^\infty(\Gamma\backslash PSL(2,\mathbb{R})) \rtimes SO(2)) \cong HC^{ev}(C^\infty(\Sigma_{g'}) \rtimes G)$$

with $\mathbb{C}$ coefficients, via the Chern character.

Thus orbifold line bundles on $\Sigma(g,\nu_1,\dots,\nu_n)$ can be described equivalently as G-equivariant line bundles over the covering smooth surface $\Sigma_{g'}$, and again as $SO(2)$-equivariant line bundles on $G\backslash\hat{P}$.

Remarks. With the notation used in the previous section, let G be a finite group acting smoothly and properly on a smooth compact oriented manifold X. There is a natural choice of a fundamental class $[X]_G \in H_0(X,G)$ in the delocalized equivariant homology of X, given by the fundamental classes of each compact oriented smooth submanifold X^γ, $[X]_G = \oplus_{\gamma\in G}[X^\gamma]$. In the case of hyperbolic 2-orbifolds, the equivariant fundamental class $[\Sigma_{g'}]_G$ is given by

$$[\Sigma_{g'}]_G = [\Sigma_{g'}] \oplus_j [p_j]^{\nu_j - 1} \in H_2(\Sigma_{g'}, \mathbb{C}) \oplus_j (H_0(p_j,\mathbb{C}))^{\nu_j-1}.$$

The corresponding equivariant Euler number $< ch_G(\mathcal{L}), [\Sigma_{g'}]_G >$ is obtained by evaluating

$$< ch_G(\mathcal{L}), [\Sigma_{g'}]_G > = < c_1(\tilde{\mathcal{L}}), [\Sigma_{g'}] > + \sum_{j=1}^n \sum_{\gamma\in G_{p_j}\backslash\{1\}} \lambda(\gamma).$$

1.6. **Classifying space of the orbifold fundamental group.** Here we find it convenient to follow Baum, Connes and Higson [BC], [BCH]. Let M be a good orbifold, that is its orbifold universal cover $\widetilde{M}$ is a smooth manifold which has a *proper* Γ-action, where Γ denotes the orbifold fundamental group of M. That is, the map

$$\widetilde{M} \times \Gamma \to \widetilde{M} \times \widetilde{M}$$
$$(x,\gamma) \to (x, \gamma x)$$

is a proper map. The universal example for such a proper action is denoted in [BC], [BCH] by $\underline{E}\Gamma$. It is universal in the sense that there is a continuous Γ-map

$$f : \widetilde{M} \to \underline{E}\Gamma$$

which is unique upto Γ-homotopy, and moreover $\underline{E}\Gamma$ itself is unique upto Γ-homotopy. The quotient $\underline{B}\Gamma = \Gamma\backslash\underline{E}\Gamma$ is an orbifold. Just as $B\Gamma$ classifies isomorphism classes of Γ-covering spaces, it can be shown that $\underline{B}\Gamma$ classifies isomorphism classes of orbifold Γ-covering spaces.

Examples 1. It turns out that if Γ is a discrete subgroup of a connected Lie group G, then $\underline{E}\Gamma = G/K$, where K is a maximal compact subgroup.

Examples 2. The orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$, viewed as the quotient space $\Sigma(g, \nu_1, \dots, \nu_n) = \Gamma\backslash\mathbb{H}$ is an example of the above construction.

This is the main class of examples that we are concerned with in this paper.

Let $S\Gamma$ denote the set of all elements of Γ which are of finite order. Then $S\Gamma$ is not empty, since $1 \in S\Gamma$. Γ acts on $S\Gamma$ by conjugation, and let $F\Gamma$ denote the associated permutation module over $\mathbb{C}$, i.e.

$$F\Gamma = \left\{ \sum_{\alpha\in S\Gamma} \lambda_\alpha[\alpha] \,\middle|\, \lambda_\alpha \in \mathbb{C} \quad \text{and} \quad \lambda_\alpha = 0 \quad \text{except for a finite number of} \quad \alpha \right\}$$

1.7. **Twisting an elliptic operator.** We will discuss elliptic operators only on good orbifolds, and refer to [Kaw] for the general case. Let M be a good orbifold, that is the universal orbifold cover $\widetilde{M}$ of M is a smooth manifold. Let $\widetilde{W} \to \widetilde{M}$ be a Γ-invariant Hermitian vector bundle over $\widetilde{M}$. Let D be a 1st order elliptic differential operator on M,

$$D : L^2(M, \mathcal{E}) \to L^2(M, \mathcal{F})$$

acting on L^2 orbifold sections of the orbifold vector bundles $\mathcal{E}, \mathcal{F}$ over M. By fiat, D is a Γ-equivariant 1st order elliptic differential operator $\widetilde{D}$ on the smooth manifold $\widetilde{M}$,

$$\widetilde{D} : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \to L^2(\widetilde{M}, \widetilde{\mathcal{F}}).$$

Given any connection $\nabla^{\widetilde{W}}$ on $\widetilde{W}$ which is compatible with the Γ action and the Hermitian metric, we wish to define an extension of the elliptic operator $\widetilde{D}$, to act on sections of $\widetilde{\mathcal{E}} \otimes \widetilde{W}$, $\widetilde{\mathcal{F}} \otimes \widetilde{W}$.

$$\widetilde{D} \otimes \nabla^{\widetilde{W}} : \Gamma(M, \widetilde{\mathcal{E}} \otimes \widetilde{W}) \to \Gamma(M, \widetilde{\mathcal{F}} \otimes \widetilde{W})$$

and we want it to satisfy the following property: If $\widetilde{W}$ is a trivial bundle, and ∇^0 is the trivial connection on $\widetilde{W}$, then for $u \in \Gamma(\widetilde{M}, \widetilde{\mathcal{E}})$, $h \in \Gamma(\widetilde{M}, \widetilde{W})$ such that $\nabla^0 h = 0$,

$$(\widetilde{D} \otimes \nabla^0)(u \otimes h) = (\widetilde{D}u) \otimes h$$

To do this, define a morphism

$$S = S_D : \widetilde{\mathcal{E}} \otimes T^*\widetilde{M} \to \widetilde{\mathcal{F}}$$

$$S(u \otimes df) = \widetilde{D}(fu) - f\widetilde{D}u$$

for $f \in C^\infty(\widetilde{M})$ and $u \in \Gamma(M, \widetilde{\mathcal{E}})$. Then S is a tensorial. Consider $S = S \otimes 1 : \widetilde{\mathcal{E}} \otimes T^*\widetilde{M} \otimes \widetilde{W} \to \widetilde{\mathcal{F}} \otimes \widetilde{W}$ defined by

$$S(u \otimes df \otimes e) = S(u \otimes df) \otimes e$$

for u, f as before and $e \in \Gamma(M, \widetilde{W})$.
Recall that a connection $\nabla^{\widetilde{W}}$ on $\widetilde{W}$ is a derivation

$$\nabla^{\widetilde{W}} : \Gamma(\widetilde{M}, \widetilde{W}) \to \Gamma(\widetilde{M}, T^*\widetilde{M} \otimes \widetilde{W})$$

Define $\widetilde{D} \otimes \nabla^{\widetilde{W}}$ as

$$(\widetilde{D} \otimes \nabla^{\widetilde{W}})(u \otimes e) = (Du) \otimes e + S(u \otimes \nabla^{\widetilde{W}} e)$$

Then $\widetilde{D} \otimes \nabla^{\widetilde{W}}$ is a 1^{st} order elliptic operator.

1.8. **Twisted index theorem for orbifolds.** Let M be a compact orbifold of dimension $n = 4\ell$. Let $\Gamma \to \widetilde{M} \xrightarrow{p} M$ be the universal orbifold cover of M and the orbifold fundamental group is Γ. Let D be an elliptic 1st order operator on M, that is a $\widetilde{D}$ on $\widetilde{M}$,

$$\widetilde{D} \,:\, L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \to L^2(\widetilde{M}, \widetilde{\mathcal{F}}),$$

such that $\widetilde{D}$ commutes with the Γ-action on $\widetilde{M}$.

Now let $\tilde{\omega}$ be a Γ-invariant closed 2-form on $\tilde{M}$, $\tilde{\omega} = d\eta$. Define $\nabla = d + i\eta$. Then ∇ is a Hermitian connection on the trivial line bundle over $\widetilde{M}$, and the curvature of ∇, $(\nabla)^2 = i\,\tilde{\omega}$. (Here $s \in \mathbb{R}$.) Then ∇ defines a projective (Γ, σ)-action on L^2 spinors as in section 1.2.

Consider the *twisted elliptic operator* on $\widetilde{M}$,

$$\widetilde{D} \otimes \nabla \,:\, L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \to L^2(\widetilde{M}, \widetilde{\mathcal{F}})$$

Then $\widetilde{D} \otimes \nabla$ no longer commutes with Γ, but it does commute with the projective (Γ, σ) action. Let P_+, P_- be the orthogonal projections onto the nullspace of $\widetilde{D} \otimes \nabla$ and $(\widetilde{D} \otimes \nabla)^*$ respectively since

$$(\widetilde{D} \otimes \nabla)\; P_+ = 0 \qquad \text{and} \qquad (\widetilde{D} \otimes \nabla)^*\; P_- = 0$$

By elliptic regularily, it follows that the Schwartz (or integral) kernels of $P_\pm$ are smooth. Since $\widetilde{D} \otimes \nabla$ and its adjoint commutes with the (Γ, σ) action, one has

$$e^{i\phi_\gamma(x)} P_\pm(\gamma x, \gamma y)\, e^{-i\phi_\gamma(y)} = P_\pm(x, y) \quad \forall \gamma \in \Gamma.$$

In particular, $P_\pm(x, x)$ is a Γ-invariant function on $\widetilde{M}$. One can define the *von Neumann trace* as Atiyah did in the untwisted case

$$\operatorname{tr}(P_\pm) = \int_M \operatorname{tr}(P_\pm(x, x))\; dx.$$

The L^2-index is by definition

$$\operatorname{index}_{L^2}(\widetilde{D} \otimes \nabla) = \operatorname{tr}(P_+) - \operatorname{tr}(P_-).$$

To describe the next theorem, we will briefly review some material on characteristic classes for orbifold vector bundles. Let M be a good orbifold, that is the universal orbifold cover $\Gamma \to \widetilde{M} \to M$ of M is a smooth manifold. Then the orbifold tangent bundle TM of M, can be viewed as the Γ-equivariant bundle $T\widetilde{M}$ on $\widetilde{M}$. similar comments apply to the orbifold cotangent bundle T^*M and more generally, any orbifold vector bundle on M. It is then clear that choosing Γ-invariant connections on the Γ-invariant vector bundles on $\widetilde{M}$, one can define the Chern-Weil representatives of the characteristic classes of the Γ-invariant vector bundles on $\widetilde{M}$. These characteristic classes are Γ-invariant and so define cohomology classes on M. For further details, see [Kaw].

Theorem 1.4. *Let M be a compact, even dimensional, good orbifold, Γ be its orbifold fundamental group, $\widetilde{D}$ be a Γ-invariant twisted Dirac operator on $\widetilde{M}$, where $\Gamma \to \widetilde{M} \to M$ is the universal orbifold cover of M. Then one has*

$$\operatorname{Ind}_{\Gamma,\sigma}(\widetilde{D} \otimes \nabla) = \frac{q!}{(2\pi i)^q (2q!)} \langle Td(M) \cup ch(symb(D)) \cup e^{\omega}, [T^*M] \rangle$$

where $Td(M)$ denotes the Todd characteristic class of the complexified orbifold tangent bundle of M which is pulled back to the orbifold cotangent bundle T^*M, $ch(symb(D))$ is the Chern character of the symbol of the operator D,

Proof. The proof is similar to the case of Atiyah's L^2 index theorem for covering spaces. An important conceptual difference lies in the fact that $\tilde{M}$ is an orbifold cover, and not an actual cover of M. We have $\tilde{D} = \widetilde{\not{\partial}}^{\pm} \otimes \nabla^{\mathcal{E}} = \widetilde{\not{\partial}}^{\pm}_{\mathcal{E}}$. Let $k^{\pm}(t,x,y)$ denote the heat kernel of the Γ-invariant Dirac operators $(\widetilde{\not{\partial}}^{\pm}_{\mathcal{E}} \otimes \nabla)^2$ on the universal orbifold cover of M, and $P^{\pm}(x,y)$ the smooth Schwartz kernels o f the orthogonal projections $P^{\pm}$ onto the nullspace of $\widetilde{\not{\partial}}^{\pm}_{\mathcal{E}} \otimes \nabla^s$. By a general result of Cheeger-Gromov-Taylor (see also Roe), the heat kernel $k^{\pm}(t,x,y)$ converges uniformly over compact subsets of $\widetilde{M} \times \widetilde{M}$ to $P^{\pm}(x,y)$, as $t \to \infty$. Therefore one has

$$\operatorname{Ind}_{\Gamma,\sigma}(\widetilde{\not{\partial}}^{+}_{\mathcal{E}} \otimes \nabla) = \operatorname{tr}_s(e^{-t(\widetilde{\not{\partial}}_{\mathcal{E}} \otimes \nabla)^2})$$

$$\begin{aligned} \lim_{t\to\infty} \operatorname{tr}(e^{-t(\widetilde{\not{\partial}}^{\pm}_{\mathcal{E}} \otimes \nabla)^2}) &= \lim_{t\to\infty} \int_M \operatorname{tr}(k^{\pm}(t,x,x))dx \\ &= \int_M \operatorname{tr}(P^{\pm}(x,x))dx \\ &= \operatorname{tr}(P^{\pm}) \end{aligned} \tag{1}$$

Next observe that

$$\begin{aligned} \frac{\partial}{\partial t} \operatorname{tr}_s(e^{-t(\widetilde{\not{\partial}}_{\mathcal{E}} \otimes \nabla)^2}) &= -\operatorname{tr}_s((\widetilde{\not{\partial}}_{\mathcal{E}} \otimes \nabla)^2 e^{-t(\widetilde{\not{\partial}}_{\mathcal{E}} \otimes \nabla)^2}) \\ &= -\operatorname{tr}_s([\widetilde{\not{\partial}}_{\mathcal{E}} \otimes \nabla, (\widetilde{\not{\partial}}_{\mathcal{E}} \otimes \nabla) e^{-t(\widetilde{\not{\partial}}_{\mathcal{E}} \otimes \nabla)^2}]) \\ &= 0 \end{aligned}$$

since $\widetilde{\partial\!\!\!/}_{\mathcal{E}} \otimes \nabla$ is an odd operator. Here tr_s denotes the graded trace, i.e. the composition of the trace tr and the grading operator. Therefore we deduce that

$$\begin{aligned}
\mathrm{tr}_s(e^{-t(\widetilde{\partial\!\!\!/}_{\mathcal{E}}\otimes\nabla)^2}) &= \lim_{t\to\infty} \mathrm{tr}_s(e^{-t(\widetilde{\partial\!\!\!/}_{\mathcal{E}}\otimes\nabla)^2}) \\
&= \mathrm{tr}_s(P) \qquad (2)\\
&= \mathrm{index}_{L^2}(\widetilde{\partial\!\!\!/}_{\mathcal{E}}^{+} \otimes \nabla).
\end{aligned}$$

By the local index theorem of Atiyah-Bott-Patodi [ABP], Getzler [Get], one has

$$\lim_{t\to 0}\big(\mathrm{tr}(k^+(t,x,x)) - \mathrm{tr}(k^-(t,x,x))\big) = [\hat{A}(\Omega)\ \mathrm{tr}(e^{R^{\tilde{\mathcal{E}}}})e^{\omega}]_n \qquad (3)$$

where $[\ ]_n$ denotes the component of degree $n = \dim M$, Ω is the curvature of the metric on $\widetilde{M}$, $R^{\tilde{\mathcal{E}}}$ is the curvature of the connection on $\tilde{\mathcal{E}}$. Combining equations (1), (2) and (3), one has

$$\begin{aligned}
\mathrm{Ind}_{\Gamma,\sigma}(\widetilde{\partial\!\!\!/}_{\mathcal{E}}^{+} \otimes \nabla) &= \mathrm{index}_{L^2}(\widetilde{\partial\!\!\!/}_{\mathcal{E}}^{+} \otimes \nabla) \\
&= \int_M \hat{A}(\Omega)\ \mathrm{tr}(e^{R^{\tilde{\mathcal{E}}}})e^{\omega}.
\end{aligned}$$

□

Remarks. A particular case of Theorem 1.4 highlights a key new phenomenon in the case of orbifolds, viz. in the special case when the multiplier $\sigma = 1$ is trivial, then $\mathrm{Ind}_{\Gamma,1}(\widetilde{D})$ is the L^2 index of $\widetilde{D}$ as defined Atiyah. By comparing with the cohomological formula due to Kawasaki [Kaw] for the Fredholm index of the operator D on the orbifold M, we see that in general these are *not* equal, and the error term is a rational number which can be expressed explicitly as a cohomological formula on the lower dimensional strata of the orbifold M. Notice that, in the smooth case, Atiyah's L^2 index theorem can be viewed as an integrality statement for the L^2 index $\mathrm{Ind}_{\Gamma,1}(\widetilde{D})$. Thus, we see that for general orbifolds the L^2 index of $\widetilde{D}$, $\mathrm{Ind}_{\Gamma,1}(\widetilde{D})$ is only a *rational* number. This was also observed by [Far].

2. Range of the trace and the Kadison constant

In this section, we will first calculate the range of the canonical trace map on K_0 of the twisted group C^*-algebras for Fuchsian groups Γ of signature $(g, \nu_1, \dots, \nu_n)$. We use in an essential way some of the results of the previous section such as the twisted version of the L^2-index theorem of Atiyah [At], which is due to Gromov [Gr2], and which is proved in Theorem 1.4. This enables us to deduce information about projections in the twisted group C^*-algebras. In the case of no twisting, this follows because the Baum-Connes conjecture is known to be true while these results are also well known for the case of the irrational rotation algebras, and for the twisted groups C^* algebras of the fundamental groups of closed Riemann surfaces of positive genus [CHMM]. Our theorem generalises most of these results. Moreover, we prove analogous results in the case of compact, real and complex hyperbolic four-manifolds. We will apply the results of this section in the next section to study some quantitative aspects of the spectrum of projectively periodic elliptic operators, mainly on orbifold covering spaces of hyperbolic orbifolds.

2.1. **The isomorphism classes of algebras $C^*(\Gamma, \sigma)$.** Let $\sigma \in Z^2(\Gamma, U(1))$ be a multiplier on Γ, where Γ is a Fuchsian group of signature $(g, \nu_1, \ldots, \nu_n)$. If $\sigma' \in Z^2(\Gamma, U(1))$ is another multiplier on Γ such that $[\sigma] = [\sigma'] \in H^2(\Gamma, U(1))$, then it can be easily shown that $C^*(\Gamma, \sigma) \cong C^*(\Gamma, \sigma')$. That is, the isomorphism classes of the C^*-algebras $C^*(\Gamma_{g'}, \sigma)$ are naturally parametrized by $H^2(\Gamma, U(1))$. In particular, if we consider only multipliers σ such that $\delta(\sigma) = 0$, we see that these are parametrised by $\ker(\delta) \subset H^2(\Gamma, U(1))$. It follows from the discussion at the beginning of the next subsection that $\ker(\delta) \cong U(1)$. We summarize this below.

Lemma 2.1. *Let Γ be a Fuchsian group of signature $(g, \nu_1, \ldots, \nu_n)$. Then the isomorphism classes of twisted group C^*-algebras $C^*(\Gamma, \sigma)$ such that $\delta(\sigma) = 0$ are naturally parametrized by $U(1)$.*

2.2. **K-theory of twisted group C^* algebras.** We begin by computing the K-theory of twisted group C^*-algebras for Fuchsian groups Γ of signature $(g, \nu_1, \ldots, \nu_n)$. Let σ be a multiplier on Γ. It defines a cohomology class $[\sigma] \in H^2(\Gamma, U(1))$. Consider now the short exact sequence of coefficient groups

$$1 \to \mathbb{Z} \xrightarrow{i} \mathbb{R} \xrightarrow{e^{2\pi\sqrt{-1}}} U(1) \to 1,$$

which gives rise to a long exact sequence of cohomology groups (the change of coefficient groups sequence)

$$\cdots \to H^2(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^2(\Gamma, \mathbb{R}) \xrightarrow{e^{2\pi\sqrt{-1}}{}_*} H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^3(\Gamma, \mathbb{R}). \tag{4}$$

We first show that the map

$$H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z})$$

is a a surjection.

In fact, it is enought to show that $H^3(\Gamma, \mathbb{R}) = \{0\}$. In order to see this it is enough to notice that we have a G action on $B\Gamma_{g'}$ with quotient $B\Gamma$,

$$G \to B\Gamma_{g'} \xrightarrow{\lambda} B\Gamma \tag{5}$$

and therefore the Leray-Serre spectral sequence, we have

$$E^2 = Tor^{H_*(G,\mathbb{R})}(\mathbb{R}, H_*(B\Gamma_{g'}, \mathbb{R}))$$

that converges to $H_*(B\Gamma, \mathbb{R})$. Moreover, we have

$$E^2 = Tor^{H_*(G,\mathbb{R})}(\mathbb{R}, \mathbb{R})$$

converging to $H_*(BG, \mathbb{R})$, [McCl] 7.16. Notice also that, with $\mathbb{R}$ coefficients, we have $H_q(BG, \mathbb{R}) = \{0\}$ for $q > 0$. Thus we obtain that, with $\mathbb{R}$ coefficients, $H_q(B\Gamma, \mathbb{R}) \cong H^q(B\Gamma, \mathbb{R})$ is $\mathbb{R}$ in degrees $q = 0$ and $q = 2$, $\mathbb{R}^{2g}$ in degree $q = 1$, and trivial in degrees $q > 2$. In particular, (4) now becomes

$$\cdots \to H^2(\Gamma, \mathbb{Z}) \xrightarrow{i_*} H^2(\Gamma, \mathbb{R}) \xrightarrow{e^{2\pi\sqrt{-1}}{}_*} H^2(\Gamma, U(1)) \xrightarrow{\delta} H^3(\Gamma, \mathbb{Z}) \xrightarrow{i_*} 0. \tag{6}$$

In the following $[\Gamma]$ will denote a choice of a generator in $H_2(B\Gamma,\mathbb{R}) \cong \mathbb{R} \cong H^2(B\Gamma,\mathbb{R})$. Using equation (5) and the previous argument, we see that $\lambda_*[\Sigma_{g'}] = \#(G)[\Gamma]$, since $B\Gamma_{g'}$ and $\Sigma_{g'}$ are homotopy equivalent, and where $\#(G)$ denotes the order of the finite group G.

In particular, for any multiplier σ of Γ with $[\sigma] \in H^2(\Gamma_{g'}, U(1))$ and with $\delta(\sigma) = 0$, there is a $\mathbb{R}$-valued 2-cocycle ζ on Γ with $[\zeta] \in H^2(\Gamma,\mathbb{R})$ such that $[e^{2\pi\sqrt{-1}\zeta}] = [\sigma]$. Define a homotopy $[\sigma_t] = [e^{2\pi\sqrt{-1}t\zeta}] \quad \forall t \in [0,1]$ which is a homotopy of multipliers σ_t that connects the multiplier σ and the trivial multiplier. Note also that this homotopy is *canonical* and not dependent on the particular choice of ζ. Therefore one obtains a homotopy of isomorphism classes of twisted group C^*-algebras $C^*(\Gamma,\sigma_t)$ connecting $C^*(\Gamma,\sigma)$ and $C^*(\Gamma)$. It is this homotopy which will essentially be used to show that $C^*(\Gamma,\sigma)$ and $C^*(\Gamma)$ have the same K-theory.

Let $\Gamma \subset G$ be a discrete cocompact subgroup of G and A be an algebra admitting an action of Γ by automorphisms. Then the cross product algebra $[A \otimes C_0(G)] \rtimes \Gamma$, is Morita equivalent to the algebra of continuous sections vanishing at infinity $C_0(\Gamma\backslash G, \mathcal{E})$, where $\mathcal{E} \to \Gamma\backslash G$ is the flat A-bundle defined as the quotient

$$\mathcal{E} = (A \times G)/\Gamma \to \Gamma\backslash G.$$

Here we consider the diagonal action of Γ on $A \times G$. We refer the reader to [Kas1] for the technical defintion of a K-amenable group. However we mention that any solvable Lie group, and in fact any amenable Lie group is K-amenable, and in fact it is shown in [Kas1], [JuKas] that the non-amenable groups $\mathbf{SO}_0(n,1)$, $\mathbf{SU}(n,1)$ are K-amenable Lie groups. Also, Cuntz [Cu] has shown that the class of K-amenable groups is closed under the operations of taking subgroups, under free products and under direct products.

Theorem 2.2 ([Kas1],[Kas2])**.** *If G is K-amenable, then $(A \rtimes \Gamma) \otimes C_0(G)$ and $[A \otimes C_0(G)] \rtimes \Gamma$ have the same K-equivariant K-theory, where K acts in the standard way on G and trivially on the other factors.*

Combining Theorem 2.2 with the remarks above, one gets the following important corollary.

Corollary 2.3. *If G is K-amenable, then $(A \rtimes \Gamma) \otimes C_0(G)$ and $C_0(\Gamma\backslash G, \mathcal{E})$ have the same K-equivariant K-theory. Equivalently, one has for $j = 0,1$,*

$$K_{Kj}(C_0(\Gamma\backslash G, \mathcal{E})) \cong K_{Kj+\dim(G/K)}(A \rtimes \Gamma).$$

We now come to the main theorem of this section, which generalizes theorems of [CHMM], [PR], [PR2].

Theorem 2.4. *Suppose that Γ is a discrete cocompact subgroup in a K-amenable Lie group G and that K is a maximal compact subgroup of G. Then*

$$K_\bullet(C^*(\Gamma,\sigma)) \cong {K_K}^{\bullet+\dim(G/K)}(\Gamma\backslash G, \delta(B_\sigma)),$$

where $\sigma \in H^2(\Gamma, U(1))$ is any multiplier on Γ, ${K_K}^\bullet(\Gamma\backslash G, \delta(B_\sigma))$ is the twisted K-equivariant K-theory of a continuous trace C^*-algebra B_σ with spectrum $\Gamma\backslash G$, and $\delta(B_\sigma)$ denotes the Dixmier-Douady invariant of B_σ.

Proof. Let $\sigma \in H^2(\Gamma, U(1))$, then the *twisted* cross product algebra $A \rtimes_\sigma \Gamma$ is stably equivalent to the cross product $(A \otimes \mathcal{K}) \rtimes \Gamma$ where $\mathcal{K}$ denotes compact operators. This is the Packer-Raeburn stabilization trick [PR], which we now describe in more detail. Let $V : \Gamma \to U(\ell^2(\Gamma))$ denote the left regular $(\Gamma, \bar{\sigma})$ representation on $\ell^2(\Gamma)$, i.e. for $\gamma, \gamma_1 \in \Gamma$ and $f \in \ell^2(\Gamma)$

$$(V(\gamma_1)f)(\gamma) = \bar{\sigma}(\gamma_1, \gamma_1^{-1}\gamma)f(\gamma_1^{-1}\gamma).$$

Then for $\gamma_1, \gamma_2 \in \ell^2(\Gamma)$, V satisfies $V(\gamma_1)V(\gamma_2) = \bar{\sigma}(\gamma_1, \gamma_2)V(\gamma_1\gamma_2)$. That is, V is a projective representation of Γ. Since Ad is trivial on $U(1)$, it follows that $\alpha(\gamma) = Ad(V(\gamma))$ is a representation of Γ on $\mathcal{K}$. This is easily generalised to the case when $\mathbb{C}$ is replaced by the $*$ algebra A.

Using Corollary 2.3, one sees that $A \rtimes_\sigma \Gamma \otimes C_0(G)$ and $C_0(\Gamma\backslash G, \mathcal{E}_\sigma)$ have the same K-equivariant K-theory, whenever G is K-amenable, where

$$\mathcal{E}_\sigma = (A \otimes \mathcal{K} \times G)/\Gamma \to \Gamma\backslash G$$

is a flat $A \otimes \mathcal{K}$-bundle over $\Gamma\backslash G$ and K is a maximal compact subgroup of G. In the particular case when $A = \mathbb{C}$, one sees that $C_r^*(\Gamma, \sigma) \otimes C_0(G)$ and $C_0(\Gamma\backslash G, \mathcal{E}_\sigma)$ have the same K-equivariant K-theory whenever G is K-amenable, where

$$\mathcal{E}_\sigma = (\mathcal{K} \times G)/\Gamma \to \Gamma\backslash G.$$

But the twisted K-equivariant K-theory ${K_K}^*(\Gamma\backslash G, \delta(B_\sigma))$ is by definition the K-equivariant K-theory of the continuous trace C^*-algebra $B_\sigma = C_0(\Gamma\backslash G, \mathcal{E}_\sigma)$ with spectrum $\Gamma\backslash G$. Therefore

$$K_\bullet(C^*(\Gamma, \sigma)) \cong {K_K}^{\bullet + \dim(G/K)}(\Gamma\backslash G, \delta(B_\sigma)).$$

□

Our next main result says that for discrete cocompact subgroups in K-amenable Lie groups, the reduced and unreduced twisted group C^*-algebras have canonically isomorphic K-theories. Therefore all the results that we prove regarding the K-theory of these reduced twisted group C^*-algebras are also valid for the unreduced twisted group C^*-algebras.

Theorem 2.5. *Let $\sigma \in H^2(\Gamma, U(1))$ be a multiplier on Γ and Γ be a discrete cocompact subgroup in a K-amenable Lie group. Then the canonical morphism $C^*(\Gamma, \sigma) \to C_r^*(\Gamma, \sigma)$ induces an isomorphism*

$$K_*(C^*(\Gamma, \sigma)) \cong K_*(C_r^*(\Gamma, \sigma)).$$

Proof. We note that by the Packer-Raeburn trick, one has

$$C^*(\Gamma, \sigma) \otimes \mathcal{K} \cong \mathcal{K} \rtimes \Gamma$$

and

$$C_r^*(\Gamma, \sigma) \otimes \mathcal{K} \cong \mathcal{K} \rtimes_r \Gamma,$$

where $\rtimes_r$ denotes the reduced crossed product. Since Γ is a lattice in a K-amenable Lie group, the canonical morphism $\mathcal{K} \rtimes \Gamma \to \mathcal{K} \rtimes_r \Gamma$ induces an isomorphism (cf. [Cu])

$$K_*(\mathcal{K} \rtimes \Gamma) \cong K_*(\mathcal{K} \rtimes_r \Gamma),$$

which proves the result. □

We now specialize to the case when $G = \mathbf{SO}_0(2,1)$, $K = \mathbf{SO}(2)$ and $\Gamma = \Gamma(g, \nu_1, \dots, \nu_n)$ is a Fuchsian group, i.e. the orbifold fundamental group of a hyperbolic orbifold of signature $(g, \nu_1, \dots, \nu_n)$, $\Sigma(g, \nu_1, \dots, \nu_n)$, where $\Gamma \subset G$ (note that G is K-amenable), or when $G = \mathbb{R}^2$, $K = \{e\}$ and $g = 1$, with Γ being a cocompact crystallographic group.

Proposition 2.6. *Let σ be a multiplier on the Fuchsian group Γ of signature $(g, \nu_1, \dots, \nu_n)$ such that $\delta(\sigma) = 0$. Then one has*

1. $K_0(C^*(\Gamma, \sigma)) \cong K_0(C^*(\Gamma)) \cong K^0_{orb}(\Sigma(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2-n+\sum_{j=1}^n \nu_j}$
2. $K_1(C^*(\Gamma, \sigma)) \cong K_1(C^*(\Gamma)) \cong K^1_{orb}(\Sigma(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2g}$.

Proof. Now by a result due to Kasparov [Kas1], which he proves by connecting the regular representation to the trivial one via the complementary series, one has

$$K_\bullet(C^*(\Gamma)) \cong K^\bullet_{SO(2)}(P(g, \nu_1, \dots, \nu_n)) = K^\bullet_{orb}(\Sigma(g, \nu_1, \dots, \nu_n)).$$

We recall next the calculation of Farsi [Far] for the orbifold K-theory of the hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$

$$K^0_{orb}(\Sigma(g, \nu_1, \dots, \nu_n)) \equiv K_0(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) = K^0_{SO(2)}(P(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2-n+\sum_{j=1}^n \nu_j}$$

and

$$K^1_{orb}(\Sigma(g, \nu_1, \dots, \nu_n)) \equiv K_1(C^*(\Sigma(g, \nu_1, \dots, \nu_n))) = K^1_{SO(2)}(P(g, \nu_1, \dots, \nu_n)) \cong \mathbb{Z}^{2g}$$

By Theorem 2.4 we have

$$K_j(C^*(\Gamma)) \cong K^j_{SO(2)}(P(g, \nu_1, \dots, \nu_n)) \quad \text{for } j = 0, 1,$$

and more generally

$$K_j(C^*(\Gamma, \sigma)) \cong K^j_{SO(2)}(P(g, \nu_1, \dots, \nu_n), \delta(B_\sigma)), \quad j = 0, 1,$$

where $B_\sigma = C(P(g, \nu_1, \dots, \nu_n), \mathcal{E}_\sigma)$. Finally, because $\mathcal{E}_\sigma$ is a locally trivial bundle of C^*-algebras over $P(g, \nu_1, \dots, \nu_n)$, with fibre $\mathcal{K}$ (= compact operators), it has a Dixmier-Douady invariant $\delta(B_\sigma)$ which can be viewed as the obstruction to B_σ being Morita equivalent to $C(\Sigma_g)$. But by assumption $\delta(B_\sigma) = \delta(\sigma) = 0$. Therefore B_σ is Morita equivalent to $C(P(g, \nu_1, \dots, \nu_n))$ and we conclude that

$$K_j(C^*(\Gamma, \sigma)) \cong K^j_{SO(2)}(P(g, \nu_1, \dots, \nu_n)) \cong K^j_{orb}(\Sigma(g, \nu_1, \dots, \nu_n)) \quad j = 0, 1.$$

□

2.3. **Twisted Kasparov map.** Let Γ be as before, that is, Γ is the orbifold fundamental group of the hyperbolic orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$. Then for any multiplier σ on Γ, the *twisted Kasparov isomorphism*,

$$\mu_\sigma : K^\bullet_{orb}(\Sigma(g, \nu_1, \dots, \nu_n)) \to K_\bullet(C^*_r(\Gamma, \sigma)) \tag{7}$$

is defined as follows. Let $\mathcal{E} \to \Sigma(g, \nu_1, \dots, \nu_n)$ be an orbifold vector bundle over $\Sigma(g, \nu_1, \dots, \nu_n)$ defining an element $[\mathcal{E}]$ in $K^0(\Sigma(g, \nu_1, \dots, \nu_n))$. As in [Kaw], one can form the twisted Dirac operator $\not{\partial}^+_{\mathcal{E}} : L^2(\Sigma(g, \nu_1, \dots, \nu_n), \mathcal{S}^+ \otimes \mathcal{E}) \to L^2(\Sigma(g, \nu_1, \dots, \nu_n), \mathcal{S}^- \otimes \mathcal{E})$ where $\mathcal{S}^\pm$ denote the

$\frac{1}{2}$ spinor bundles over $\Sigma(g,\nu_1,\ldots,\nu_n)$. By Proposition 2.6 of the previous subsection, there is a canonical isomorphism

$$K_\bullet(C_r^*(\Gamma,\sigma)) \cong K^\bullet_{orb}(\Sigma(g,\nu_1,\ldots,\nu_n)).$$

Both of these maps are assembled to yield the twisted Kasparov map as in (7). Observe that $\Sigma(g,\nu_1,\ldots,\nu_n) = \underline{B}\Gamma$, and that the twisted Kasparov map has a natural generalisation, which will be studied elsewhere.

We next describe this map more explicitly. One can lift the twisted Dirac operator $\not{\partial}_{\mathcal{E}}^+$ as above, to a Γ-invariant operator $\widetilde{\not{\partial}_{\mathcal{E}}^+}$ on $\mathbb{H} = \widetilde{\Sigma}(g,\nu_1,\ldots,\nu_n)$, which is the universal orbifold cover of $\Sigma(g,\nu_1,\ldots,\nu_n)$,

$$\widetilde{\not{\partial}_{\mathcal{E}}^+} : L^2(\mathbb{H},\widetilde{\mathcal{S}^+\otimes\mathcal{E}}) \to L^2(\mathbb{H},\widetilde{\mathcal{S}^-\otimes\mathcal{E}})$$

Therefore as before in (6), for any multiplier σ of Γ with $\delta([\sigma]) = 1$, there is a $\mathbb{R}$-valued 2-cocycle ζ on Γ with $[\zeta] \in H^2(\Gamma,\mathbb{R})$ such that $[e^{2\pi\sqrt{-1}\zeta}] = [\sigma]$. By the earlier argument using spectral sequences and the fibration as in equation (5), we see that the map λ induces an isomorphism $H^2(\Gamma,\mathbb{R}) \cong H^2(\Gamma_{g'},\mathbb{R})$, and therefore there is a 2-form ω on $\Sigma_{g'}$ such that $[e^{2\pi\sqrt{-1}\omega}] = [\sigma]$. Of course, the choice of ω is not unique, but this will not affect the results that we are concerned with. Let $\widetilde{\omega}$ denote the lift of ω to the universal cover $\mathbb{H}$. Since the hyperbolic plane $\mathbb{H}$ is contractible, it follows that $\widetilde{\omega} = d\eta$ where η is a 1-form on $\mathbb{H}$ which is not in general Γ invariant. Now $\nabla = d - i\eta$ is a Hermitian connection on the trivial complex line bundle on $\mathbb{H}$. Note that the curvature of ∇ is $\nabla^2 = i\widetilde{\omega}$. Consider now the twisted Dirac operator $\widetilde{\not{\partial}}_{\mathcal{E}}^+$ which is twisted again by the connection ∇,

$$\widetilde{\not{\partial}_{\mathcal{E}}^+}\otimes\nabla : L^2(\mathbb{H},\widetilde{\mathcal{S}^+\otimes\mathcal{E}}) \to L^2(\mathbb{H},\widetilde{\mathcal{S}^-\otimes\mathcal{E}}).$$

It does not commute with the Γ action, but it does commute with the projective (Γ,σ)-action which is defined by the multiplier σ, and it has an (Γ,σ)-L^2-index

$$\mathrm{ind}_{(\Gamma,\sigma)}(\widetilde{\not{\partial}_{\mathcal{E}}^+}\otimes\nabla) \in K_0(C_r^*(\Gamma,\sigma)).$$

Formally, $\mathrm{ind}_{(\Gamma,\sigma)}(\widetilde{\not{\partial}_{\mathcal{E}}^+}\otimes\nabla) = [P^+] - [P^-]$, where $P^\pm$ denotes the projection to the L^2 kernel of $\widetilde{\not{\partial}_{\mathcal{E}}^\pm}\otimes\nabla$. The problem is that in general, $P^\pm$ only lies in the twisted von Neumann algebra, and therefore one has to add a compact perturbation in $C_r^*(\Gamma,\sigma)$ to the operator, in order to properly define the index. This is essentially the C^* index of Mishchenko-Fomenko [MiFo], see also [Ros1]. Then observe that the *twisted Kasparov map* is

$$\mu_\sigma([\mathcal{E}]) = \mathrm{ind}_{(\Gamma,\sigma)}(\widetilde{\not{\partial}_{\mathcal{E}}^+}\otimes\nabla)) \in K_0(C^*(\Gamma,\sigma)).$$

The canonical trace on $C_r^*(\Gamma,\sigma))$ induces a linear map

$$[\mathrm{tr}] : K_0(C_r^*(\Gamma,\sigma)) \to \mathbb{R}$$

which is called the *trace map* in K-theory. Explicitly, we first extend tr to matrices with entries in $C^*(\Gamma,\sigma)$ as (with Trace denoting matrix trace):

$$\mathrm{tr}(f\otimes r) = \mathrm{Trace}(r)\,\mathrm{tr}(f).$$

Then the extension of tr to K_0 is given by $[\mathrm{tr}]([e]-[f]) = \mathrm{tr}(e)-\mathrm{tr}(f)$, where e, f are idempotent matrices with entries in $C^*(\Gamma,\sigma))$.

2.4. **Range of the trace map on K_0: the case of two dimensional orbifolds.** We can now state the first major theorem of this section.

Theorem 2.7. *Let Γ be a Fuchsian group of signature $(g,\nu_1,\dots,\nu_n)$, and σ be a multiplier of Γ such that $\delta(\sigma)=0$. Then the range of the trace map is*

$$[\mathrm{tr}](K_0(C_r^*(\Gamma,\sigma))) = \mathbb{Z}\,\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i),$$

where $2\pi\theta = \langle[\sigma],[\Gamma]\rangle \in (0,1]$ is the result of pairing the multiplier σ with the fundamental class of Γ (cf. subsection 2.1).

Proof. We first observe that by the results of the previous subsection the twisted Kasparov map is an isomorphism. Therefore to compute the range of the trace map on K_0, it suffices to compute the range of the trace map on elements of the form

$$\mu_\sigma([\mathcal{E}^0]-[\mathcal{E}^1])$$

for any element

$$[\mathcal{E}^0]-[\mathcal{E}^1] \in K^0_{orb}(\Sigma(g,\nu_1,\dots,\nu_n)).$$

where $\mathcal{E}^0,\mathcal{E}^1$ are orbifold vector bundles over the orbifold $\Sigma(g,\nu_1,\dots,\nu_n)$, which as in section 1, can be viewed as G-equivariant vector bundles over the Riemann surface $\Sigma_{g'}$ which is an orbifold G covering of the orbifold $\Sigma(g,\nu_1,\dots,\nu_n)$.

By the twisted L^2 index theorem for orbifolds from the previous section, one has

$$[\mathrm{tr}](\mathrm{Ind}_{(\Gamma,\sigma)}(\widetilde{\not{\partial}^+_{\mathcal{E}}\otimes\nabla})) = \frac{1}{2\pi}\int_{\Sigma(g,\nu_1,\dots,\nu_n)} \hat{A}(\Omega)\,\mathrm{tr}(e^{R^{\mathcal{E}}})e^{\omega}. \tag{8}$$

We next simplify the right hand side of equation (8) using

$$\begin{aligned}\hat{A}(\Omega) &= 1\\ \mathrm{tr}(e^{R^{\mathcal{E}}}) &= \mathrm{rank}\,\mathcal{E} + \mathrm{tr}(R^{\mathcal{E}})\\ e^{\omega} &= 1+\omega.\end{aligned}$$

Therefore one has

$$[\mathrm{tr}](\mathrm{Ind}_{(\Gamma,\sigma)}(\widetilde{\not{\partial}^+_{\mathcal{E}}\otimes\nabla})) = \mathrm{rank}\,\mathcal{E}\,\frac{\int_{\Sigma(g,\nu_1,\dots,\nu_n)}\omega}{2\pi} + \frac{\int_{\Sigma(g,\nu_1,\dots,\nu_n)}\mathrm{tr}(R^{\mathcal{E}})}{2\pi},$$

Now by the index theorem for orbifolds, due to Kawasaki [Kaw], we see that

$$\frac{\int_{\Sigma(g,\nu_1,\dots,\nu_n)}\mathrm{tr}(R^{\mathcal{E}})}{2\pi} + \frac{\sum_{i=1}^n \beta_i/\nu_i}{2\pi} = \mathrm{index}(\not{\partial}^+_{\mathcal{E}}) \in \mathbb{Z},$$

Therefore we see that

$$\frac{\int_{\Sigma(g,\nu_1,\dots,\nu_n)}\mathrm{tr}(R^{\mathcal{E}})}{2\pi} \in \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$$

Observe that

$$\int_{\Sigma(g,\nu_1,\dots,\nu_n)} \omega = \frac{1}{\#(G)} \int_{\Sigma_{g'}} \omega = \langle [\omega], [\Gamma] \rangle$$

since $\Sigma_{g'}$ is an orbifold G covering of the orbifold $\Sigma(g, \nu_1, \dots, \nu_n)$ and $[\Gamma]$ is equal to $\frac{[\Sigma_{g'}]}{\#(G)}$, cf. section 2.1 and that by assumption,

$$\frac{\langle [\omega], [\Gamma] \rangle}{2\pi} - \theta \in \mathbb{Z}.$$

It follows that the range of the trace map on K_0 is

$$\mathbb{Z}\frac{\langle [\omega], [\Gamma] \rangle}{2\pi} + \mathbb{Z} + \sum_{i=1}^{n} \mathbb{Z}(1/\nu_i) = \mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^{n} \mathbb{Z}(1/\nu_i).$$

□

We will now discuss one application of this result, leaving further applications to the next section.The application studies the number of projections in the twisted group C^*-algebra, which is a problem of independent interest.

Proposition 2.8. *Let Γ be as in Theorem 2.7. Let σ be a multiplier on Γ such that $\delta(\sigma) = 0$, and $2\pi\theta = \langle [\sigma], [\Gamma] \rangle \in (0, 1]$ be the result of pairing the cohomology class of σ with the fundamental class of Γ. If θ is rational, then there are at most a finite number of unitary equivalence classes of projections, other than 0 and 1, in the reduced twisted group C^*-algebra $C^*_r(\Gamma, \sigma)$.*

Proof. By assumption, $\theta = p/q$. Let P be a projection in $C^*_r(\Gamma, \sigma)$. Then $1-P$ is also a projection in $C^*_r(\Gamma, \sigma)$ and one has

$$1 = \mathrm{tr}(1) = \mathrm{tr}(P) + \mathrm{tr}(1 - P).$$

Each term in the above equation is non-negative. Since σ is rational and by Theorem 2.7, it follows that the Kadison constant $C_\sigma(\Gamma) > 0$ (see section 3 for the definition) and $\mathrm{tr}(P) \in \{0, C_\sigma(\Gamma), 2C_\sigma(\Gamma), \dots 1\}$. By faithfulness and normality of the trace tr, it follows that there are at most a finite number of unitary equivalence classes of projections, other than those of 0 and 1 in $C^*_r(\Gamma, \sigma)$. □

2.5. **Range of the trace map on K_0: the case of 4 dimensional real and complex hyperbolic manifolds.** We prove here the analogue of the results of the previous subsection, for the case of compact, real and complex hyperbolic four dimensional manifolds.

We now set some notation for the theorem below. Let Γ be a discrete, torsion-free cocompact subgroup of $G = \mathrm{SO}_0(1, 4)$ or $\mathbf{SU}(1, 2)$. We will assume that $\delta(\sigma) = 0$, therefore there is a closed Γ-invariant two form ω on G/K, where K is a maximal compact subgroup of G, such that $[e^{2i\pi\omega}] = [\sigma]$. Let $Q(a, b) = \langle a \cup b, [\Gamma] \rangle \quad a, b \in H^2(\Gamma, \mathbb{R})$ be the intersection form on $\Gamma\backslash G/K$. Define the linear functional $T_\omega : H^2(\Gamma, \mathbb{Z}) \to \mathbb{R}$ as $T_\omega(a) = Q(\omega, a)$. Then we have:

Theorem 2.9. *Let Γ be a discrete, torsion-free cocompact subgroup of $\mathrm{SO}_0(1, 4)$ or of $\mathbf{SU}(1, 2)$, and σ be a multiplier of Γ such that $\delta(\sigma) = 0$. We assume also that $\Gamma\backslash G/K$ is a spin manifold. Then the range of the trace map is*

$$[\mathrm{tr}](K_0(C^*_r(\Gamma, \sigma))) = \mathbb{Z}\theta + \mathbb{Z} + B,$$

where $2(2\pi)^2\theta = \langle[\omega\cup\omega],[\Gamma]\rangle$ *is the result of pairing the cup product of multipliers* $[\omega\cup\omega]$ *with the fundamental class of* Γ*, and* $B = \mathrm{range}(T_\omega)$.

Proof. We observe again that by the results of the previous subsection the twisted Kasparov map is an isomorphism. Therefore to compute the range of the trace map on K_0, it suffices to compute the range of the trace map on elements of the form

$$\mu_\sigma([\mathcal{E}^0]-[\mathcal{E}^1])$$

for any element

$$[\mathcal{E}^0]-[\mathcal{E}^1]\in K^0(\Gamma\backslash G/K).$$

where $\mathcal{E}^0,\mathcal{E}^1$ are vector bundles over the compact manifold $\Gamma\backslash G/K$.

By the twisted L^2 index theorem from the previous section, one has

$$[\mathrm{tr}](\mathrm{Ind}_{(\Gamma,\sigma)}(\widetilde{\not{\partial}^+_{\mathcal{E}}\otimes\nabla})) = \frac{1}{(2\pi)^2}\int_{\Gamma\backslash G/K}\hat{A}(\Omega)\,\mathrm{tr}(e^{R^{\mathcal{E}}})e^{\omega}. \tag{9}$$

We next simplify the right hand side of equation (9) using

$$\hat{A}(\Omega) = 1-\frac{1}{24}p_1(\Omega)$$
$$\mathrm{tr}(e^{R^{\mathcal{E}}}) = \mathrm{rank}\,\mathcal{E}+\mathrm{tr}(R^{\mathcal{E}})+\frac{1}{2}\mathrm{tr}({R^{\mathcal{E}}}^2)$$
$$e^{\omega} = 1+\omega+\frac{1}{2}\omega^2.$$

Therefore one has

$$[\mathrm{tr}](\mathrm{Ind}_{(\Gamma,\sigma)}(\widetilde{\not{\partial}^+_{\mathcal{E}}\otimes\nabla})) = \mathrm{rank}\,\mathcal{E}\,\frac{\int_{\Gamma\backslash G/K}\omega^2}{2(2\pi)^2}$$

$$+\frac{\int_{\Gamma\backslash G/K}\mathrm{tr}({R^{\mathcal{E}}}^2)}{2(2\pi)^2}-\frac{\int_{\Gamma\backslash G/K}p_1(\Omega)}{24(2\pi)^2}+\frac{\int_{\Gamma\backslash G/K}\mathrm{tr}(R^{\mathcal{E}})\wedge\omega}{(2\pi)^2},$$

Now by the Atiyah-Singer index theorem [AS], we see that

$$-\frac{\int_{\Gamma\backslash G/K}p_1(\Omega)}{24(2\pi)^2}+\frac{\int_{\Gamma\backslash G/K}\mathrm{tr}({R^{\mathcal{E}}}^2)}{2(2\pi)^2} = \mathrm{index}(\not{\partial}^+_{\mathcal{E}})\in\mathbb{Z},$$

Therefore we see that

$$[\mathrm{tr}](K_0(C_r^*(\Gamma,\sigma))) = \mathbb{Z}\theta+\mathbb{Z}+B,$$

where $2(2\pi)^2\theta = \int_{\Gamma\backslash G/K}\omega^2$ and $B = \mathrm{range}(T_\omega)$. $\square$

Remarks. The spin hypothesis on $\Gamma\backslash G/K$ can be easily replaced by spin$^{\mathbb{C}}$, without much alteration in the proof.

The proof of the following proposition is similar to that of Proposition 2.8 and we omit it.

Proposition 2.10. *Let Γ be as in Theorem 2.9. Let σ be a multiplier on Γ such that $\delta(\sigma) = 0$. If σ defines a rational cohomology class, then there are at most a finite number of unitary equivalence classes of projections, other than 0 and 1, in the reduced twisted group C^*-algebra $C_r^*(\Gamma, \sigma)$.*

3. Applications to the spectral theory of projectively periodic elliptic operators and the classification of twisted group C^* algebras

In this section, we apply the range of the trace theorem, to prove some qualitative results on the spectrum of projectively periodic self adjoint elliptic operators on the universal covering of a good orbifold, or what is now best known as non-commutative Bloch theory. In particular, we study generalizations of the hyperbolic analogues of the Ten Martini Problem in [CHMM] and the Bethe-Sommerfeld conjecture. We also classify upto isomorphism, the twisted group C^* algebras for a cocompact Fuchsian group.

Let M be a compact, good orbifold, that is, the universal cover $\Gamma \to \widetilde{M} \to M$ is a smooth manifold and we will assume as before that there is a $(\Gamma, \bar\sigma)$-action on $L^2(\widetilde{M})$ given by $T_\gamma = U_\gamma \circ S_\gamma \, \forall \gamma \in \Gamma$. Let $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{F}}$ be Hermitian vector bundles on M and let $\widetilde{\mathcal{E}}$, $\widetilde{\mathcal{F}}$ be the corresponding lifts to Γ-invariants Hermitian vector bundles on $\widetilde{M}$. Then there are (Γ, σ)-actions on $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ and $L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ which are also given by $T_\gamma = U_\gamma \circ S_\gamma \, \forall \gamma \in \Gamma$.

Now let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \to L^2(\widetilde{M}, \widetilde{\mathcal{F}})$ be a self adjoint elliptic differential operator that commutes with the $(\Gamma, \bar\sigma)$-action that was defined earlier. We begin with some basic facts about the spectrum of such an operator. Recall that the *discrete spectrum* of D, $spec_{disc}(D)$ consists of all the eigenvalues of D that have finite multiplicity, and the *essential spectrum* of D, $spec_{ess}(D)$ consists of the complement $spec(D) \setminus spec_{disc}(D)$. That is, $spec_{ess}(D)$ consists of the set of accumulation points of the spectrum of D, $spec(D)$. Our first goal is to prove that the essential spectrum is unbounded. Our proof will be a modification of an argument in [Sar].

Lemma 3.1. *Let $D : L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \to L^2(\widetilde{M}, \widetilde{\mathcal{E}})$ be a self adjoint elliptic differential operator that commutes with the $(\Gamma, \bar\sigma)$-action. Then the* discrete spectrum *of D is empty.*

Proof. Let λ be an eigenvalue of D nd V denote the corresponding eigenspace. Then V is a (Γ, σ)-invariant subspace of $L^2(\widetilde{M}, \widetilde{\mathcal{E}})$. If $\mathcal{F}$ is a fundamental domain for the action of Γ on $\widetilde{M}$, one sees that

$$L^2(\widetilde{M}, \widetilde{\mathcal{E}}) \cong L^2(\Gamma) \otimes L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}})$$

which can be proved by choosing a bounded measurable almost everywhere smooth section of the orbifold covering $\widetilde{M} \to M$. Here $(\Gamma, \bar\sigma)$-action on $L^2(\mathcal{F}, \widetilde{\mathcal{E}}|_{\mathcal{F}})$ is trivial, and is the left regular $(\Gamma, \bar\sigma)$-action on $L^2(\Gamma)$. Therefore it suffices to show that the dimension of *any* $(\Gamma, \bar\sigma)$- invariant subspace V of $L^2(\Gamma)$ is infinite dimensional. Let $\{v_1, \ldots, v_N\}$ be an orthonormal basis for V. Then one has

$$T_\gamma v_i(\gamma') = \sum_{j=1}^{N} U_{ij}(\gamma) v_j(\gamma') \qquad \forall \gamma, \gamma' \in \Gamma$$

where $U = (U_{ij}(\gamma))$ is some $N \times N$ unitary matrix. Therefore

$$\begin{aligned} N = \sum_{j=1}^{N} ||v_i||^2 &= \sum_{j=1}^{N} \sum_{\gamma \in \Gamma} |v_i(\gamma\gamma')|^2 \\ &= \sum_{\gamma\in\Gamma} \sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=1}^{N} U_{ij}(\gamma) \overline{U_{ik}(\gamma)} v_j(\gamma') \overline{v_k(\gamma')} \\ &= \sum_{\gamma\in\Gamma} \sum_{j=1}^{N} |v_j(\gamma')|^2 \\ &= \#(\Gamma) \sum_{j=1}^{N} |v_j(\gamma')|^2. \end{aligned}$$

Since $\#(\Gamma) = \infty$, it follows that either $N = 0$ or $N = \infty$. □

Corollary 3.2. *Let D be as in Lemma 3.1 above. Then the* essential spectrum *of D coincides with the spectrum of D, and so it is unbounded.*

Proof. By the Lemma above, we conclude that $spec_{ess}(D)$ and $spec(D)$ coincide. Since D is an unbounded self-adjoint operator, it is a standard fact that $spec(D)$ is unbounded cf. [Gi], yielding the result. □

Note that in general the spectral projections of D, $E_\lambda \notin C^*(\mathcal{G}, \sigma)$. However one has

Proposition 3.3 (Sunada, Bruning-Sunada). *Let D be as in Lemma 3.1 above. If $\lambda_0 \notin spec(D)$, then $E_{\lambda_0} \in C^*(\mathcal{G}, \sigma)$.*

Proof. Firstly, there is a standard reduction to the case when D is positive and of even order $d \geq 2$ cf. [BrSu], so we will assume this without loss of generality. By a result of Greiner, see also Bruning-Sunada [BrSu], there are off diagonal estimates for the Schwartz kernel of the heat operator e^{-tD}

$$|k_t(x,y)| \leq C_1 t^{-n/d} exp\left(-C_2 d(x,y)^{d/(d-1)} t^{-1/(d-1)}\right)$$

for some positive contants C_1, C_2 and for $t > 0$ in any compact interval. Since the volume growth of a orbifold covering space is at most exponential, we see in particular that $|k_t(x,y)|$ is L^1 in both variable separately, so that

$$e^{-tD} \in C^*(\mathcal{G}, \sigma).$$

Note that $\chi_{[0,e^{-t\lambda}]}(D) = \chi_{[0,\lambda]}(e^{-tD})$ Let $t = 1$ and $\lambda_1 = -\log \lambda_0$. Then $\lambda_1 \notin \mathrm{spec}(e^{-D})$ and

$$\chi_{[0,\lambda_1]}(e^{-D}) = \phi(e^{-D})$$

where ϕ is a compactly supported smooth function, $\phi \cong 1$ on $[0, \lambda_1]$ and $\phi \cong 0$ on the remainder of the spectrum. Since $C^*(\mathcal{G}, \sigma)$ is closed under the continuous functional calculus, it follows that $\phi(e^{-D}) \in C^*(\mathcal{G}, \sigma)$, that is $E_{\lambda_0} \in C^*(\mathcal{G}, \sigma)$. □

We will now recall the definition of the Kadison constant of a twisted group C^*-algebra. The *Kadison constant* of $C_r^*(\Gamma, \sigma)$ is defined by:

$$C_\sigma(\Gamma) = \inf\{\mathrm{tr}(P) : P \text{ is a non-zero projection in } C_r^*(\Gamma, \sigma) \otimes \mathcal{K}\}.$$

Let D be any $(\Gamma, \bar{\sigma})$-invariant self-adjoint elliptic differential operator D on $\widetilde{M}$. Since the spectrum $spec(D)$ is a closed subset of $\mathbb{R}$, its complement $\mathbb{R} \backslash spec(D)$ is a countable union of *disjoint* open intervals. Each such interval is called a *gap* in the spectrum.

Proposition 3.4. *Let Γ be a Fuchsian group of signature $(g, \nu_1, \ldots, \nu_n)$. Let σ be a multiplier on Γ such that $\delta(\sigma) = 0$, and $2\pi\theta = \langle [\sigma], [\Gamma] \rangle \in (0, 1]$ be the result of pairing the cohomology class of σ with the fundamental class of Γ. If θ is rational, then the spectrum of any $(\Gamma, \bar{\sigma})$-invariant self-adjoint elliptic differential operator D on $\widetilde{M}$ has only a finite number of gaps in the spectrum in every half line $(-\infty, \lambda]$. Here $\Gamma \to \widetilde{M} \to M$ is the universal orbifold covering of a compact good orbifold M with orbifold fundamental group Γ. In particular, the intersection of spec(D) with any compact interval in $\mathbb{R}$ is never a Cantor set.*

Proof. By Proposition 2.8 and Theorem 2.7, it follows that one has the estimate $C_\sigma(\Gamma) \geq 1/q > 0$. Then one applies the main result in Brüning-Sunada [BrSu] to deduce the proposition. □

In words, we have shown that whenever the multiplier is rational, then the spectrum of a projectively periodic elliptic operator is the union of countably many (possibly degenerate) closed intervals, which can only accumulate at infinity.

Recall the important Γ-invariant elliptic differential operator, which is called the Schrödinger operator

$$H_V = \Delta + V$$

where Δ denotes the Laplacian on functions on $\widetilde{M}$ and V is a Γ-invariant function on $\widetilde{M}$. It is known that the Baum-Connes conjecture is true for all amenable discrete subgroups of a connected Lie group and also for discrete subgroups of $SO(n, 1)$ [Kas1] and $SU(n, 1)$ [JuKas]. For all these groups Γ, it follows that the Kadison constant $C_1(\Gamma)$ is positive. Therefore we see by the arguments above that the spectrum of the periodic elliptic operator H_V is the union of countably many (possibly degenerate) closed intervals, which can only accumulate at infinity. This gives evidence for the following

Conjecture (The Generalized Bethe-Sommerfeld conjecture). *The spectrum of any Γ-invariant Schrödinger operator H_V has only a* finite *number of bands, in the sense that the intersection of the resolvent set with $\mathbb{R}$ has only a finite number of components.*

We remark that the Bethe-Sommerfeld conjecture has been proved completely by Skriganov [Skri] in the Euclidean case.

This leaves open the question of whether there are $(\Gamma, \bar{\sigma})$-invariant elliptic differential operators D on $\mathbb{H}$ with Cantor spectrum when θ is irrational. In the Euclidean case, this is usually known as the *Ten Martini Problem*, and is to date, not completely solved, though much progress has been made (cf. [Sh]). We pose a generalization of this problem to the hyperbolic case (which also includes the Euclidean case):

Conjecture (The Generalized Ten Dry Martini Problem). *Let σ be a multiplier on Γ such that $\delta(\sigma) = 0$, and $2\pi\theta = \langle[\sigma], [\Gamma]\rangle \in (0, 1]$ be the result of pairing the cohomology class of σ with the fundamental class of Γ. If θ is irrational, then there is a $(\Gamma, \bar{\sigma})$-invariant elliptic differential operator D on $\mathbb{H}$ which has a Cantor set type spectrum, in the sense that the intersection of spec(D) with some compact interval in $\mathbb{R}$ is a Cantor set.*

3.1. **The four-dimensional case.** The spectral properties studied in this section do not represent a purely two-dimensional phenomenon. In fact, it is possible to derive similar results in higher dimensions, as the following example shows.

Proposition 3.5. *Let Γ be a discrete, torsion-free cocompact subgroup of $\mathrm{SO}_0(1,4)$ or of $\mathbf{SU}(1,2)$, and σ be a multiplier of Γ such that $\delta(\sigma) = 0$. We assume also that $\Gamma\backslash G/K$ is a spin manifold. If $[\omega] \in H^2(M, \mathbb{R})$ is a rational cohomology class, then the spectrum of any $(\Gamma, \bar{\sigma})$-invariant self-adjoint elliptic differential operator D on M has only a finite number of gaps in the spectrum in every half line $(-\infty, \lambda]$. In particular, the intersection of spec(D) with any compact interval in $\mathbb{R}$ is never a Cantor set.*

Proof. By Proposition 2.10 and Theorem 2.9, it follows that one has the estimate $C_\sigma(\Gamma) \geq 1/q > 0$. Then one applies the main result in Brüning-Sunada [BrSu] to deduce the proposition. □

3.2. **On the classification of twisted group C^*-algebras.** We will now use the range of the trace Theorem 2.7, to give a complete classification, up to isomorphism, of the twisted group C^*-algebras $C^*(\Gamma, \sigma)$, where Let Γ be a Fuchsian group of signature $(g, \nu_1, \dots, \nu_n)$ and we assume as before that $\delta(\sigma) = 0$.

Proposition 3.6 (Isomorphism classification of twisted group C^*-algebras). *Let $\sigma, \sigma' \in H^2(\Gamma, \mathbb{R}/\mathbb{Z})$ be multipliers on Γ satisfying $\delta(\sigma) = 0 = \delta(\sigma')$, and $2\pi\theta = \langle\sigma, [\Gamma]\rangle \in (0, 1]$, $2\pi\theta' = \langle\sigma', [\Gamma]\rangle \in (0, 1]$ be the result of pairing σ, σ' with the fundamental class of Γ. Then $C^*(\Gamma, \sigma) \cong C^*(\Gamma, \sigma')$ if and only if $\theta' \in \{(\theta + \sum_{i=1}^n \beta_i/\nu_i) \quad mod 1, (1 - \theta + \sum_{i=1}^n \beta_i/\nu_i) \quad mod 1\}$, where $0 \leq \beta_i \leq \nu_i - 1 \quad \forall i = 1, \dots, n$.*

Proof. Let tr and tr$'$ denote the canonical traces on $C^*(\Gamma, \sigma)$ and $C^*(\Gamma, \sigma')$ respectively. Let

$$\phi : C^*(\Gamma, \sigma) \cong C^*(\Gamma, \sigma')$$

be an isomorphism, and let

$$\phi_* : K_0(C^*(\Gamma, \sigma)) \cong K_0(C^*(\Gamma, \sigma'))$$

denote the induced map on K_0. By Theorem 2.7, the range of the trace map on K_0 is

$$[\mathrm{tr}](K_0(C^*(\Gamma, \sigma))) = \mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$$

and

$$[\mathrm{tr}'](K_0(C^*(\Gamma, \sigma'))) = \mathbb{Z}\theta' + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i).$$

Therefore if θ is irrational, then $\mathbb{Z}\theta + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i) = \mathbb{Z}\theta' + \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$ implies that θ' is also irrational and that $\theta \pm \theta' \in \mathbb{Z} + \sum_{i=1}^n \mathbb{Z}(1/\nu_i)$. Since $\theta, \theta' \in (0, 1]$, one deduces that

$\theta' \in \{(\theta + \sum_{i=1}^n \beta_i/\nu_i)\mathrm{mod}1, (1-\theta+\sum_{i=1}^n \beta_i/\nu_i)\mathrm{mod}1\}$, where $0 \le \beta_i \le \nu_i - 1 \quad \forall i = 1,\dots,n$. Virtually the same argument holds when θ is rational, but one argues in K-theory first, and applies the trace only at the final step.

First observe that a diffeomorphism $C : \Sigma_{g'} \to \Sigma_{g'}$ lifts to a diffeomorphism C' of $\mathbb{H}$ such that $C'\Gamma C'^{-1} = \Gamma$, i.e. it defines an automorphism of Γ. Recall that the finite group

$$G = \{C_i : \quad C_i^{\nu_i} = 1 \quad \forall i = 1,\dots,n\}$$

acts on $\Sigma_{g'}$ with quotient $\Sigma(g, \nu_1, \dots, \nu_n)$. By the observation above, we see that G also acts as automorphisms of Γ. Now $C_i[\Gamma] = \lambda_i[\Gamma]$, where $\lambda_i \in \mathbb{C}$. Since $C_iC_j[\Gamma] = \lambda_i\lambda_j[\Gamma]$ and $C_i^{\nu_i} = 1$, it follows that λ_i is an ν_i^{th} root of unity, i.e. $\lambda_i = e^{2\pi\sqrt{-1}(1/\nu_i)}$. Let $C \in G$, i.e. $C = \prod_{i=1}^n C_i^{\beta_i}$. We evaluate $< C^*[\sigma], [\Gamma] >=< [\sigma], C_*[\Gamma] >= < [\prod_{i=1}^n \lambda_i^{\beta_i}\sigma], [\Gamma] > = \theta + \sum_{i=1}^n \beta_i/\nu_i$. As in subsection 2.1 we see that $C^*[\sigma] = [\prod_{i=1}^n \lambda_i^{\beta_i}\sigma] \in \ker\delta \subset H^2(\Gamma, U(1))$. Therefore the automorphism C_* of Γ induces an isomorphism of twisted group C^*-algebras

$$C^*(\Gamma,\sigma) \cong C^*(\Gamma, C^*\sigma) \cong C^*(\Gamma, \lambda\sigma).$$

where $\lambda = \prod_{i=1}^n \lambda_i^{\beta_i}$.

Now let $\psi : \Sigma_{g'} \to \Sigma_{g'}$ be an orientation reversing diffeomorphism. Then as observed earlier, ψ induces an automorphism $\psi_* : \Gamma \to \Gamma$ of Γ. We evaluate $< \psi^*[\sigma], [\Gamma] >=< [\sigma], \psi_*[\Gamma] >= \overline{< [\sigma], [\Gamma] >} =< [\bar\sigma], [\Gamma] >$, since ψ is orientation reversing. As in subsection 2.1 we see that $\psi^*[\sigma] = [\bar\sigma] \in \ker\delta \subset H^2(\Gamma, U(1))$. Therefore the automorphism ψ_* of Γ induces an isomorphism of twisted group C^*-algebras

$$C^*(\Gamma,\sigma) \cong C^*(\Gamma, \psi^*\sigma) \cong C^*(\Gamma, \bar\sigma).$$

Therefore if $\theta' \in \{(\theta + \sum_{i=1}^n \beta_i/\nu_i) \quad \mathrm{mod}1, (1-\theta+\sum_{i=1}^n \beta_i/\nu_i) \quad \mathrm{mod}1\}$, where $0 \le \beta_i \le \nu_i - 1 \quad \forall i = 1,\dots,n$, one has $C^*(\Gamma,\sigma) \cong C^*(\Gamma,\sigma')$, completing the proof of the proposition. □

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