Circle Packing: A Mathematical Tale

Kenneth Stephenson

The circle is arguably the most studied object in mathematics, yet I am here to tell the tale of *circle packing*, a topic which is likely to be new to most readers. These packings are configurations of circles satisfying preassigned patterns of tangency, and we will be concerned here with their creation, manipulation, and interpretation. Lest we get off on the wrong foot, I should caution that this is NOT two-dimensional "sphere" packing: rather than being fixed in size, our circles must adjust their radii in tightly choreographed ways if they hope to fit together in a specified pattern.

In posing this as a mathematical *tale*, I am asking the reader for some latitude. From a tale you expect truth without *all* the details; you know that the storyteller will be playing with the plot and timing; you let pictures carry part of the story. We all hope for deep insights, but perhaps sometimes a simple story with a few new twists is enough—may you enjoy this tale in that spirit. Readers who wish to dig into the details can consult the "Reader's Guide" at the end.

Once Upon a Time …

From wagon wheel to mythical symbol, predating history, perfect form to ancient geometers, companion to π, the circle is perhaps the most celebrated object in mathematics.

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The author gratefully acknowledges support of the National Science Foundation, DMS-0101324.

There is indeed a long tradition behind our story. Who can date the most familiar of circle packings, the "penny-packing" seen in the background of Figure 1? Even the "apollonian gasket" (a) has a history stretching across more than two millennia, from the time of Apollonius of Perga to the latest research on limit sets. And circles were never far from the classical solids, as suggested by the sphere caged by a dodecahedron in (b). Equally ancient is the ἄρβηλος or "shoemaker's knife" in (c), and it is amazing that the Greeks had already proved that the nth circle c_n has its center n diameters from the base. This same result can be found, beautifully illustrated, in *sangaku*, wooden temple carvings from seventeenth-century Japan. In comparatively recent times, Descartes established his Circle Theorem for "quads" like that in (d), showing that the *bends* b_j (reciprocal radii) of four mutually tangent circles are related by $(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2)$. Nobel laureate F. Soddy was so taken by this result that he rendered it in verse: *The Kiss Precise* (1936). With such a long and illustrious history, is it surprising or is it inevitable that a new idea about circles should come along?

Birth of an Idea

One can debate whether we see many truly new ideas in mathematics these days. With such a rich history, everything has antecedents—who is to say, for example, what was in the lost books of Apollonius and others? Nonetheless, some topics have fairly well-defined starting points.

Our story traces its origin to William Thurston's famous *Notes.* In constructing 3-manifolds, Thurston proves that associated with any triangulation of a sphere is a "circle packing", that is, a configuration of circles which are tangent with one another in the pattern of the triangulation. Moreover, this packing is unique up to Möbius transformations and inversions of the sphere. This is a remarkable fact, for the pattern of tangencies—which can be arbitrarily intricate—is purely abstract, yet the circle packing superimposes on that pattern a rigid geometry. This is a main theme running through our story, that *circle packing provides a bridge between the combinatoric on the one hand and the geometric on the other.*

Although known in the topological community through the *Notes*, circle packings reached a surprising new audience when Thurston spoke at the 1985 Purdue conference celebrating de Branges's proof of the *Bieberbach Conjecture.* Thurston had recognized in the rigidity of circle packings something like the rigidity shown by analytic functions, and in a talk entitled "A finite Riemann mapping theorem" he illustrated with a scheme for constructing conformal maps based on circle packings. He made an explicit conjecture, in fact, that his "finite" maps would converge, under refinement, to a classical conformal map, the type his Purdue audience knew well. As if that weren't enough, Thurston even threw in an iterative numerical scheme for computing these finite Riemann mappings in practice, with pictures to back it all up.

So this was the situation for your storyteller as he listened to Thurston's Purdue talk: a most surprising theorem and beautiful pictures about patterns of circles, an algorithm for actually computing them, and a conjectured connection to a favorite topic, analytic function theory. This storyteller was hooked!

As for antecedents, Thurston found that his theorem on packings of the sphere followed from prior work by E. Andreev on reflection groups, and some years later Reiner Kuhnau pointed out a 1936 proof by P. Koebe, so I refer to it here as the K-A-T (Koebe-Andreev-Thurston) Theorem. Nonetheless, for our purposes the new idea was born at Purdue in 1985, and our tale can begin.

Internal Development

Once a topic is launched and begins to attract a community, it also begins to develop an internal ecology: special language, key examples and theorems, central themes, and—with luck—a few gems to amaze the uninitiated.

The main players in our story, circles, are well known to us all, and we work in familiar geometric spaces: the sphere $\mathbb{P}$, the euclidean plane $\mathbb{C}$, and the hyperbolic plane as represented by the unit disc $\mathbb{D}$. Working with *configurations* of circles, however, will require a modest bit of bookkeeping, so bear

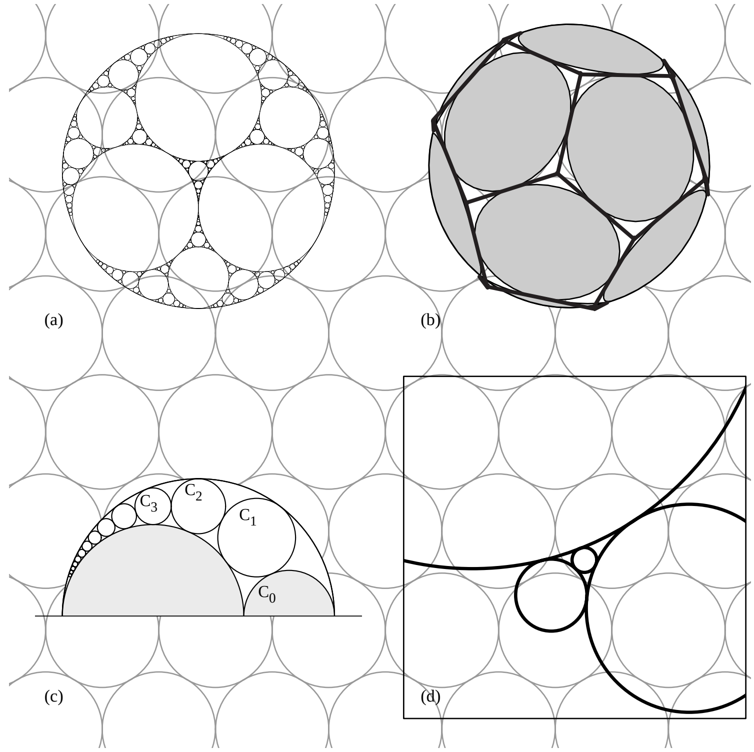


Figure 1. A long tradition.

with me while I introduce the essentials needed to follow the story.

- **Complex:** The tangency patterns for circle packings are encoded as abstract simplicial *2-complexes* K; we assume K is (i.e., triangulates) an oriented topological surface.

- **Packing:** A *circle packing* P for K is a configuration of circles such that for each vertex $v \in K$ there is a corresponding circle c_v, for each edge $\langle v, u\rangle \in K$ the circles c_v and c_u are (externally) tangent, and for each positively oriented face $\langle v, u, w\rangle \in K$ the mutually tangent triple of circles $\langle c_v, c_u, c_w\rangle$ is positively oriented.

- **Label:** A *label* R for K is a collection of putative radii, with $R(v)$ denoting the label for vertex v.

Look to Figure 2 for a very simple first example. Here K is a closed topological disc and P is a euclidean circle packing for K. I show the carrier of the packing in dashed lines to aid in matching circles to their vertices in K; there are 9 interior and 8 boundary circles. Of course the question is how to find such packings, and the key is the label R of radii—knowing K, the tangencies, and R, the sizes, it is a fairly simple matter to lay out the circles themselves. In particular, circle *centers* play a secondary role. *The computational effort in*

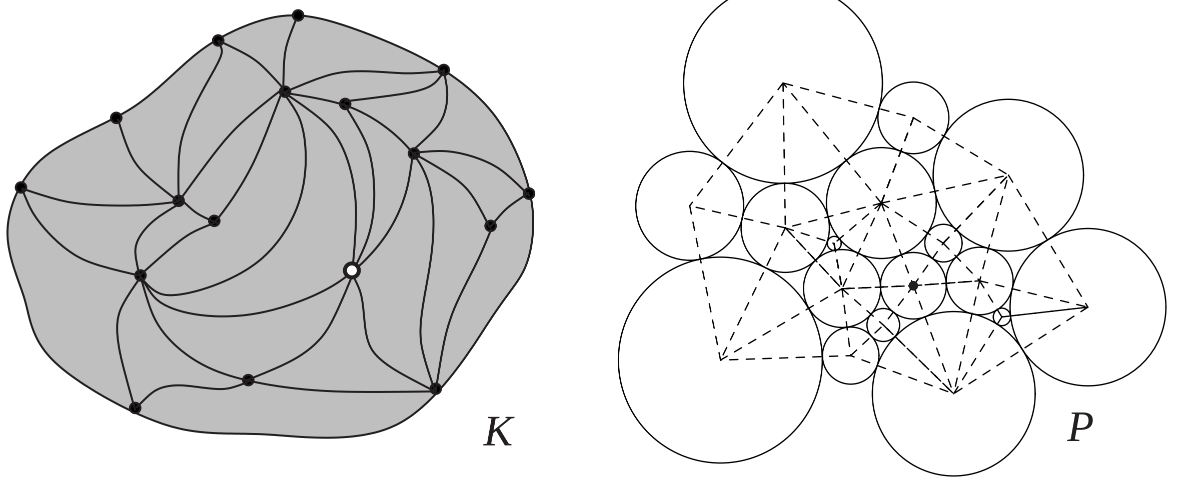


Figure 2. Compare packing P to its complex K.

circle packing lies mainly in computing labels. It is in these computations that circle packing directly confronts geometry and the local-to-global theme plays out. Here, very briefly, is what is involved.

- **Flower:** A circle c_v and the circles tangent to it are called a *flower*. The ordered chain $c_{v_1}, \cdots, c_{v_k}$ of tangent circles, the *petals*, is *closed* when v is an interior vertex of K.

- **Angle Sum:** The *angle sum* $\theta_R(v)$ for vertex v, given label R, is the sum of the angles at c_v in the triangles formed by the triples $\langle c_v, c_{v_j}, c_{v_{j+1}}\rangle$ in its flower. Angle sums are computed via the appropriate law of cosines; in the euclidean case, for example,

$$\theta_R(v) = \sum_{\langle v,u,w\rangle} \arccos\left(\frac{(R(v)+R(u))^2 + (R(v)+R(w))^2 - (R(u)+R(w))^2}{2(R(v)+R(u))(R(v)+R(w))}\right),$$

  where the sum is over all faces containing v.

- **Packing Condition** The flower of an interior vertex v can be realized as an actual geometric flower of circles with radii from R if and only if $\theta_R(v) = 2\pi n$ for some integer $n \geq 1$.

It is clear that circles trying to form a packing for K must tightly choreograph their radii. The packing condition at interior vertices is necessary, so a label R is called a *packing* label if $\theta_R(v)$ is a multiple of 2π for every interior $v \in K$. When K is simply connected, this and a monodromy argument yield a corresponding packing P, and the labels are, in fact, radii. When K is multiply connected, however, the local packing condition alone is not enough, and global obstructions become the focus. Here are a last few pieces of the ecology.

- **Miscellany:** A packing is *univalent* if its circles have mutually disjoint interiors. A *branch* circle c_v in a packing P is an interior circle whose angle sum is $2\pi n$ for integer $n \geq 2$; that is, its petals wrap n times around it. A packing P is *branched* if it has one or more branch circles; otherwise it is *locally univalent.* (Caution: Global univalence is assumed for all circle packings in some parts of the literature, but **not** here.) Möbius transformations map packings to packings; a packing is said to be *essentially* unique with some property if it is unique up to such transformations. In the disc, a *horocycle* is a circle internally tangent to the unit circle and may be treated as a circle of infinite hyperbolic radius.

You are now ready for the internal art of circle packing. *Someone hands you a complex K. Do there exist any circle packings for K? How many? In which geometry? Can they be computed in practice? What are their properties? What do they look like?*

Let's begin by explicating certain extremal packings shown in Figure 3. The spherical packing in (a) illustrates the K-A-T Theorem using the combinatorics of the soccer ball. However, our development really starts in the hyperbolic plane; Figure 3(b) illustrates the key theorem (the outer circle represents the boundary of $\mathbb{D}$).

Key Theorem. *Let K be a closed disc. There exists an essentially unique circle packing $\mathcal{P}_K$ for K in $\mathbb{D}$ that is univalent and whose boundary circles are horocycles.*

The proof involves induction on the number of vertices in K and simple geometric monotonicities, culminating in a result which deserves its own statement. Here $\mathcal{R}_K$ is the hyperbolic packing label for the packing $\mathcal{P}_K$, so this result justifies the adjective "maximal" that I will attach to these extremal packings.

***Discrete Schwarz Lemma** [DSL]. Let K be a closed disc and R any hyperbolic packing label for K. Then $R(v) \le \mathcal{R}_K(v)$ for every vertex v of K; equality for any interior vertex v implies $R \equiv \mathcal{R}_K$.*

Our Key Theorem is easily equivalent to the K-A-T Theorem, but its formulation and proof set the tone for the whole topic. The next step, for example, is to extend the Key Theorem to *open* discs K by exhausting with closed discs $K_j \uparrow K$. When we apply the monotonicity of the DSL to the maximal labels R_j for these nested complexes, a fundamental **dichotomy** emerges:

$$\text{as } j \to \infty \quad \begin{cases} \text{either} & R_j(v) \downarrow r(v) > 0 \quad \forall v \in K \\ \text{or} & R_j(v) \downarrow 0 \quad \forall v \in K. \end{cases}$$

In the former case, a geometric diagonalization argument produces a univalent hyperbolic packing $\mathcal{P}_K$ for K, its label being maximal among hyperbolic labels as in the DSL. In the latter case, the maximal packings of the K_j may be treated as euclidean and rescaled, after which geometric diagonalization produces a euclidean univalent circle packing $\mathcal{P}_K$ for K. Archetypes for the dichotomy are the maximal packings for the constant 6- and 7-degree complexes, the well-known penny-packing, and the heptagonal packing of Figure 3(c), respectively. We can now summarize the simply connected cases.

***Discrete Riemann Mapping Theorem** [DRMT]. If K is a simply connected surface, then there exists an essentially unique, locally finite, univalent circle packing $\mathcal{P}_K$ for K in one and only one of the geometries $\mathbb{P}$, $\mathbb{C}$, or $\mathbb{D}$. The complex K is termed* spherical, parabolic, *or* hyperbolic, *respectively, and $\mathcal{P}_K$ is called its* maximal *packing.*

For complexes K which are not simply connnected, topological arguments provide an infinite universal covering complex $\widetilde{K}$. By the DRMT, $\widetilde{K}$ is parabolic or hyperbolic (the sphere covers only itself) and has a maximal packing $\widetilde{\mathcal{P}}$ in $\mathbb{G}$ (i.e., $\mathbb{C}$ or $\mathbb{D}$, respectively). Essential uniqueness of $\widetilde{\mathcal{P}}$ implies existence of a discrete group Λ of conformal automorphisms of $\mathbb{G}$ under which $\widetilde{\mathcal{P}}$ is invariant. $\mathbb{G}/\Lambda$ defines a Riemann surface S, and the projection $\pi : \mathbb{G} \longrightarrow S$ carries the metric of $\mathbb{G}$ to the *intrinsic* metric of constant curvature on S, euclidean or hyperbolic, as the case may be. Some quiet reflection and arrow-chasing shows that $\pi(\widetilde{\mathcal{P}})$ defines a univalent circle packing $\mathcal{P}_K$ for K in the intrinsic metric on S. In other words, we have found our maximal circle packing $\mathcal{P}_K$ for K.

***Discrete Uniformization Theorem** [DUT]. Let K be a triangulation of an oriented surface S. Then there exists a conformal structure on S such that the resulting Riemann surface supports a circle packing $\mathcal{P}_K$ for K in its intrinsic metric, with $\mathcal{P}_K$ univalent and locally finite. The Riemann surface S is unique up to conformal equivalence, and $\mathcal{P}_K$ is unique up to conformal automorphisms of S.*

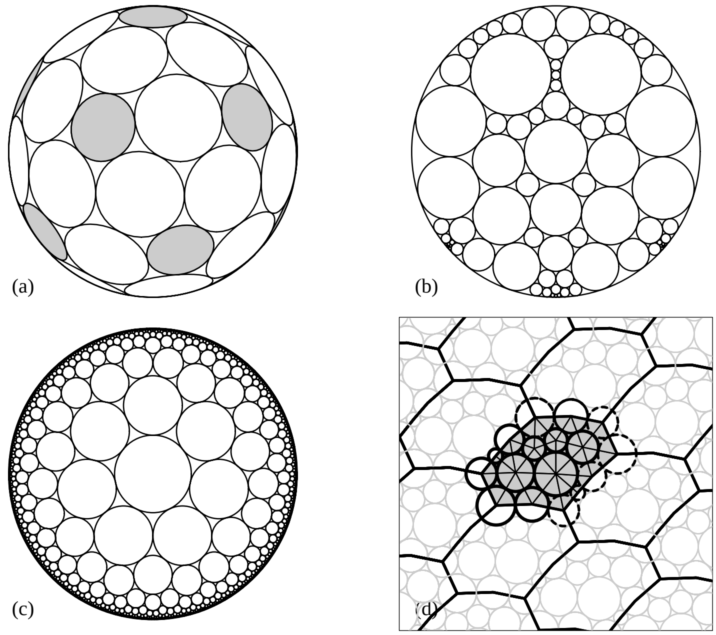


Figure 3. *Maximal* circle packing sampler.

The DUT is illustrated in Figure 3(d) for a torus having just 10 vertices. I have marked a fundamental domain and its images under the covering group. The 10 darkened circles form the torus when you use the dashed circles for side-pairings.

This theorem completes the existence/uniqueness picture for extremal univalent circle packings. It is quite remarkable that *every* complex has circle packings. From the pure *combinatorics* of K one gets not only the circle packing but even the *geometry* in which it must live! This highlights central internal themes of the topic:

combinatorics $\leftrightarrow$ *geometry*
local packing condition $\leftrightarrow$ *global structure*

You can see that the DUT opens a wealth of questions. It is known that the "packable" Riemann surfaces, those supporting some circle packing, are dense in Teichmüller space, but they have yet to be characterized, and the connections between K and the differential geometry of S remain largely unknown.

Extremal packings only scratch the surface; in general a complex K will have a huge variety of additional circle packings if we are allowed to manipulate boundary values and/or branching. When K possesses a boundary, it has been proved that *given any hyperbolic (respectively euclidean) labels for the boundary vertices of K, there exists a unique locally univalent hyperbolic (respectively*

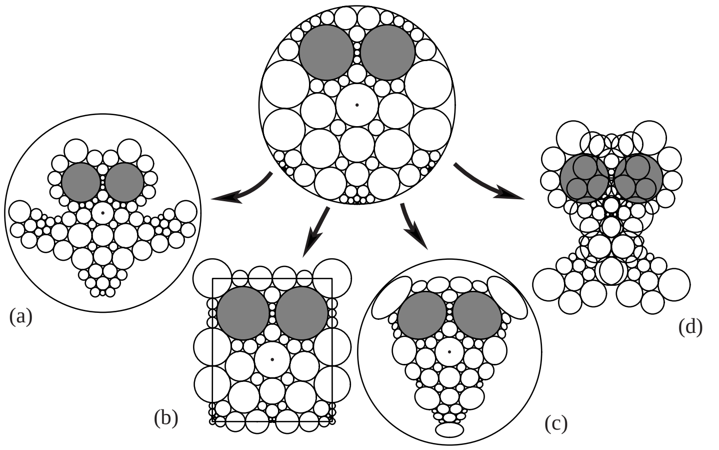


Figure 4. Owl manipulations.

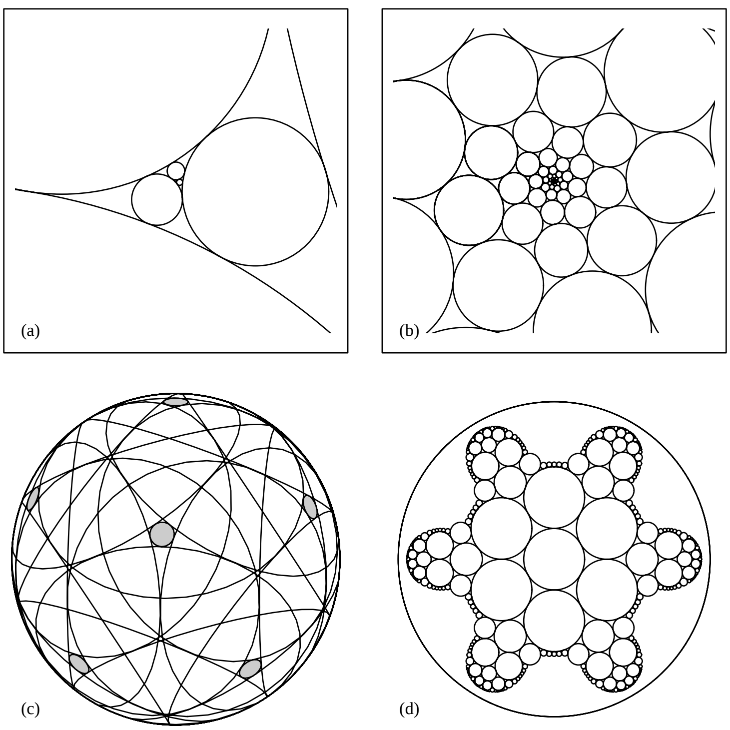


Figure 5. Geometric gems.

euclidean) packing P having those labels as its boundary radii. In a similar vein, necessary and sufficient conditions for finite sets of branch circles have been established in many cases. The most I can do here is illustrate a variety of packings for a single complex. I've done this in Figure 4, using a K with good visual cues. The maximal packing is at the top; then below it, left to right, are a univalent but nonmaximal hyperbolic packing, euclidean and spherical packings with prescribed boundary angle sums, and…the last one? At the heart of this last owl is a single branch circle, one whose petal circles wrap twice around it. We will refer back to these images later.

Pretty as the pictures are, the real gems in this topic are the elementary geometric and monotonicity arguments. Challenge yourself with some of these:

• Distinct circles can intersect in at most two points! Amazingly, Z-X. He and Oded Schramm proved that this is the key to the uniqueness for parabolic maximal packings.

• In hyperbolic geometry the central circle in a flower with n petals has hyperbolic radius no larger than $-\log(\sin(\pi/n))$.

• The important Rodin/Sullivan Ring Lemma: *for $n \geq 3$ there exists a constant $\mathfrak{c}_n > 0$ such that in any closed univalent flower of circles having n petals, no petal can have radius smaller than $\mathfrak{c}_n$ times that of the center.* By Descartes's Circle Theorem, the best constants are all reciprocal integers, beginning with $\mathfrak{c}_3 = 1$, $\mathfrak{c}_4 = 1/4$, $\mathfrak{c}_5 = 1/12$. I'll let the reader compute $\mathfrak{c}_6$.

• Figures 5(a) and (b) show hexagonal spirals. The first, created by Coxeter from the "quad" shown in Figure 1(d), is linked to the golden ratio. The second, along with a whole 2-parameter family of others, results from an observation of Peter Doyle: *for any parameters $a, b > 0$, a chain of six circles with successive radii $\{a, b, b/a, 1/a, 1/b, a/b\}$ will close up precisely around a circle of radius 1 to form a 6-flower.*

• The spherical packing of Figure 5(c), which has the same complex as the packing in Figure 3(a), was generated using $\langle 2, 3, 5\rangle$ "Schwarz" triangles; if you look closely at one of the twelve shaded circles, you will see that its five neighbors wrap twice around it.

• And what of Figure 5(d), the snowflake? Sometimes a pretty picture is just a pretty picture.

The internals of the topic that I have outlined here are wonderfully pure, clean, and accessible, and those who prefer their geometry unadulterated should know that pictures and computers are not necessary for the theory. On the other hand, the pictures certainly add to the topic, and the fact that these packings are essentially computable begs the question of numerical algorithms, which are another source of packing pleasure. There are many, many open questions; let me wrap up with one of my favorites: He and Schramm proved that if K is an open disc which packs $\mathbb{D}$, i.e., is hyperbolic, then it can in fact pack ANY simply connected proper subdomain $\Omega \subset \mathbb{C}$. Consider the combinatorics behind the packing of Figure 3(c) on page 1379, for example. Can you imagine your favorite horribly pathological domain Ω filled—every nook and cranny—with a univalent packing in which every circle has seven neighbors? How does one compute such packings?

Dust Off the Theory

Whatever internal richness a new topic develops, the drive of mathematics is towards the broader view. Where does it fit in the grand scheme? What are the analogies, links, organizing principles, applications? What does this topic need and what can it offer?

The tools we have used so far—basic geometry and trigonometry, surface topology, covering theory—are hardly surprising in the context, but perhaps you wondered about references to the Schwarz Lemma, Riemann Mapping Theorem, and Uniformization. The claim is, quite frankly, that one can look to analytic function theory as a model for organizing circle packings. Let's jump right in.

Definition. *A* discrete analytic function *is a map $f : Q \longrightarrow P$ between circle packings which preserves tangency and orientation.*

The study of circle packings P for K may now be posed as the study of the discrete analytic functions $f : \mathcal{P}_K \longrightarrow P$; namely, for each vertex v of K define $f(C_v) = c_v$, where C_v and c_v are the circles for v in $\mathcal{P}_K$ and P, respectively. For first examples look to Figure 4, where the maximal packing is the common domain for four discrete analytic functions, f_a, f_b, f_c, f_d, mapping to the packings (a), (b), (c), (d), respectively.

With this definition we immediately inherit a wonderful nomenclature: f_a from Figure 4 is a *discrete analytic self-map of* $\mathbb{D}$, and f_b is a *discrete conformal (Riemann) mapping*; the discrete analytic function from the packing of Figure 3(a) to that of Figure 5(c) is a *discrete rational function* with twelve simple branch points. A map from the penny-packing to the Doyle spiral of Figure 5(a) or (b) is a *discrete entire function*, in fact, a *discrete exponential map*. Among my favorite examples are the discrete proper analytic self-mappings of $\mathbb{D}$, the *discrete finite Blaschke products*. And there are many others: discrete disc algebra functions; discrete versions of sine and cosine; a full family of discrete polynomials; and, when K is compact, discrete meromorphic functions, though these are quite challenging and the theory is just in its infancy.

The whole panoply of function-theory machinery also opens to us. In particular, the names attached to the theorems in the last section make perfect sense. One comes to recognize the DSL as the *hyperbolic contraction principle*; analytic self-maps of the hyperbolic plane are hyperbolic contractions. Like its classical counterpart, it plays a central role on the way to DRMT and DUT. The hyperbolic/parabolic dichotomy for infinite complexes is just the classical "type" problem and yields, for example, the Discrete Liouville Theorem: *a parabolic complex K can have no bounded circle packing.* Other notions from the classical theory also enter: discrete versions of extremal length, harmonic measure, random walks, maximum principles, analytic continuation, covering theory, and the list goes on. And if you are looking for a derivative, there is a natural analogue for its modulus in the *sharp function*, defined at circle C_v by $f^\#(C_v) = \text{radius } (f(C_v))/\text{radius } (C_v)$.

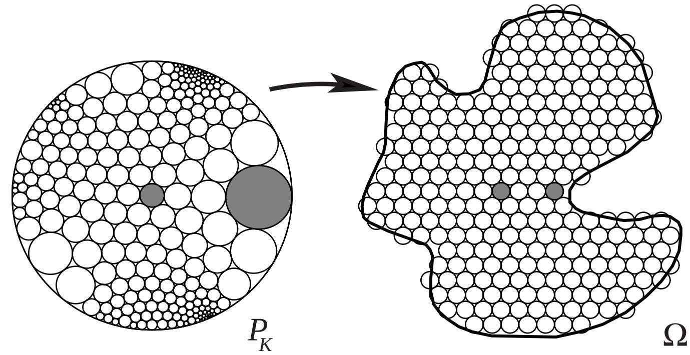


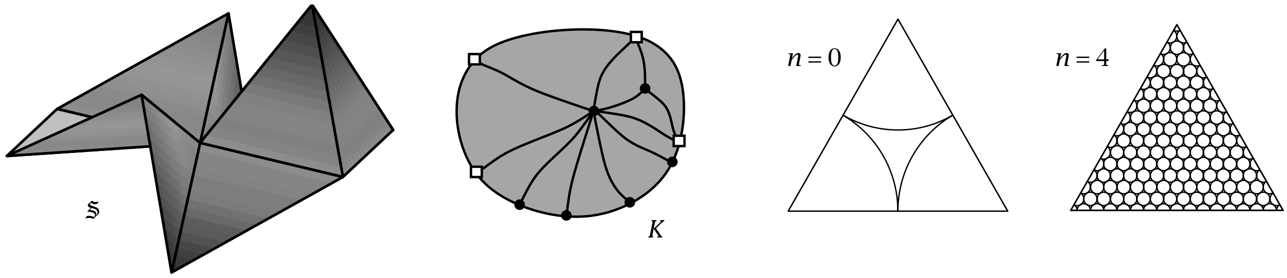
Figure 6. A Thurston "finite" Riemann mapping.

Ultimately, we find in circle packing a remarkably comprehensive analogue of classical analytic functions and the associated theory. And the parallels are not stretches; they almost formulate themselves, just as the Thurston finite Riemann mapping of Figure 6 is so clearly a discrete conformal map. The inevitable question, of course: Does the topic provide more than analogy? more than nomenclature?

This is where Thurston's startling conjecture enters our tale, for he saw in the rigidity of circle packings a direct link to conformality. According to Thurston, if one cookie-cuts a region Ω, as in Figure 6, using increasingly fine hexagonal packings, one obtains discrete mappings which converge to the classical Riemann mapping $F : \mathbb{D} \longrightarrow \Omega$. The conjecture was soon proved by Burt Rodin and Dennis Sullivan in the seminal paper of this topic. Their result has now been extended to more general (i.e., nonhexagonal) and multiply connected complexes, to all three classical geometries, and to nonunivalent and branched packings. The Thurston model holds: given a class of functions, formulate the discrete (i.e., circle packing) analogues, create instances with increasingly fine combinatorics, appropriately normalized, then watch as the discrete versions converge to their classical models. Of course there are details, for example, geometric finiteness conditions on valence and branching, but putting these aside we have this

> • **Metatheorem:** Discrete analytic functions converge under refinement to their classical analytic counterparts.

Thus discrete Blaschke products converge to Blaschke products, discrete polynomials to polynomials, discrete rational functions to rational functions, and so forth. I cannot show these in static pictures, but we can nonetheless capture the intuition quite succinctly: *A classical analytic function is said to "map infinitesimal circles to*

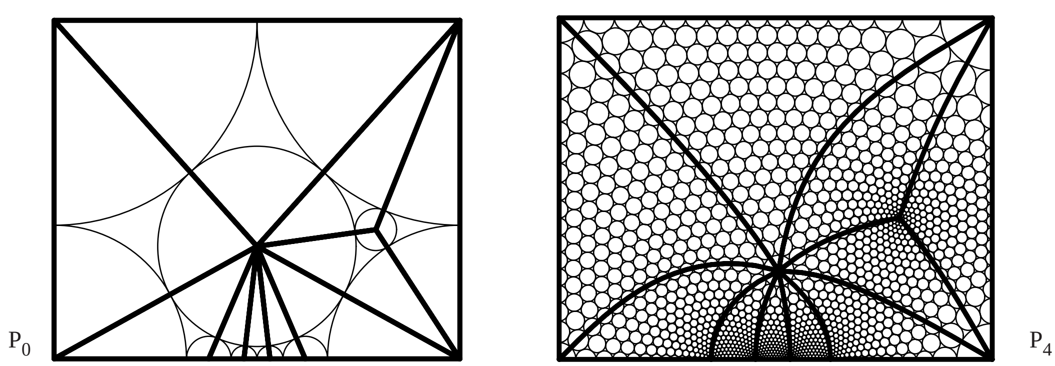


This is a toy problem in discrete conformal geometry; your materials are shown above. $\mathfrak{S}$ is a *piecewise affine* (p.w.a.) surface in 3-space constructed out of ten equilateral triangles; the cartoon K shows how these are pasted together and designates four boundary vertices as "corners". $\mathfrak{S}$ is not "flat"; it has two interior *cone points* with *cone angles* π and 3π. The p.w.a. structure on $\mathfrak{S}$ brings with it a canonical conformal structure, so $\mathfrak{S}$ is in fact a simply connected Riemann surface. By the Riemann Mapping Theorem there exists an essentially unique conformal map $F : \mathfrak{S} \longrightarrow \mathfrak{R}$, where $\mathfrak{R}$ is a plane rectangle and F maps the corner vertices of $\mathfrak{S}$ to the corners of $\mathfrak{R}$. Question: *What are the shapes of the ten faces when they are mapped to* $\mathfrak{R}$*?*

Here is the parallel discrete construction. Mark each equilateral face of $\mathfrak{S}$ with arcs of circles as in the triangle labelled "$n = 0$". These arcs piece together in $\mathfrak{S}$ to define an *in situ* packing Q_0. By using prescribed boundary angle sums, Q_0 can be computationally flattened to the packing P_0 shown below, which has rectangular carrier and the designated vertices as corners. In our terminology, $f_0 : Q_0 \longrightarrow P_0$ is a *discrete conformal mapping*, and it carries the ten faces of $\mathfrak{S}$ to ten triangles in the plane.

Clearly, this "coarse" packing cannot capture the conformal subtleties of $\mathfrak{S}$. Therefore, we use a simple "hex-refine" process which respects the p.w.a. structure of $\mathfrak{S}$: namely, break each equilateral face into four equilateral faces half its size. Applying n such refinement steps gives an *in situ* circle packing Q_n in $\mathfrak{S}$, and flattening Q_n to get a rectangular packing P_n yields a *refined* discrete conformal mapping $f_n : Q_n \longrightarrow P_n$. With four stages of refinement, for example, each face of $\mathfrak{S}$ looks like the triangle labelled "$n = 4$". Its rectangular flat packing P_4 is shown below with the ten faces outlined.

As with Thurston's conjecture for plane regions, it can be proven that the discrete mappings $\{f_n\}$ converge on $\mathfrak{S}$ to the classical mapping F. In other words, as you watch successive image packings P_n, you are seeing the ten faces converge to their true conformal shapes.



infinitesimal circles"; a discrete analytic function does the same, but with real circles.

Fortunately for our pictorial tale we can bring this same intuition to bear in a more directly geometric way. A *Riemann* surface is one having a *conformal structure*, which is, loosely speaking, a consistent way to measure angles. A *conformal* map between Riemann surfaces is one which preserves this measurement (magnitude and orientation). (When things are appropriately formulated, conformal maps are just analytic maps and vice versa.) The discussion in the box above illustrates construction and refinement of *discrete conformal maps* from a Riemann surface to plane rectangles. A formal statement of the limit behavior is somewhat involved, but we have

- **Metatheorem:** Discrete conformal mappings converge under refinement to their classical counterparts.

Any lingering doubts the reader may have about connections with analyticity should be put to rest by the fact that the K-A-T Theorem actually *implies* the Riemann Mapping Theorem for plane domains. The last critical piece involves elementary (though by no means easy) geometric arguments of He and Schramm which replace the original quasiconformal methods in Rodin/Sullivan.

That Special Something

Every topic exhibits, at least to its adherents, some special character that sets it apart, even as it finds its place in the larger scheme. Will it make a lasting imprint on mathematics? Does it represent a paradigm shift? The true believer always holds out hope.

The synergy among mathematics, computation, and visualization that began with Thurston's 1985 talk infuses circle packing with an experimental character that I believe is unique in mathematics. Let me speak in the context of CirclePack, a graphically based software package for creating, manipulating, storing, and displaying packings interactively on the computer screen. CirclePack handles arbitrary triangulated surfaces, simply or multiply connected, with or without boundary, in any of the three geometries. Packings range from 4 to 1,600,000 circles (the current record in one of Bill Floyd's tilings); those up to roughly 10,000 circles now qualify as "routine", since packing times of a few seconds give an interactive feel. Multiply connected packings are manipulated in their intrinsic metrics and are displayed in the standard geometric spaces as fundamental regions with associated side-pairings, as with the torus of Figure 3(d). Nearly all the images in this paper come directly from CirclePack and are typical of what one views on-screen during live experiments.

It is true that nearly every topic has come under the influence of computing in one way or another, even if only in sharing its algorithmic philosophy. What distinguishes circle packing is the depth of the interactions among the mathematics, computations, and visualization—the central results have involved all three.

• **Mathematics:** This is discrete *complex analysis*, so it touches not only function theory but also potential theory and brownian motion, Möbius and conformal geometry, number theory, Fuchsian and Kleinian groups, Riemann surfaces and Teichmüller theory, not to mention applications. This is core mathematics, and the key geometric tools are here: topology, boundary conditions, group actions, branching, and, of course, conformality, in the form of the packing condition. And these tools are not tied to preconceived roles. You want to double a complex across a boundary? slit two surfaces and paste them together? mix boundary conditions? try some fractional branching? puncture a torus or carry out a Dehn twist? Go ahead! You may miss familiar tools—no complex arithmetic, no power series, no functional composition—but much of complex analysis is fundamentally geometric, and you can see it in action.

• **Computation:** Space prevents me from giving the numerics of circle packing its due. Thurston's algorithm works directly on the geometry, manipulating the distribution of curvature among the circles. In the computations of the ten-triangle

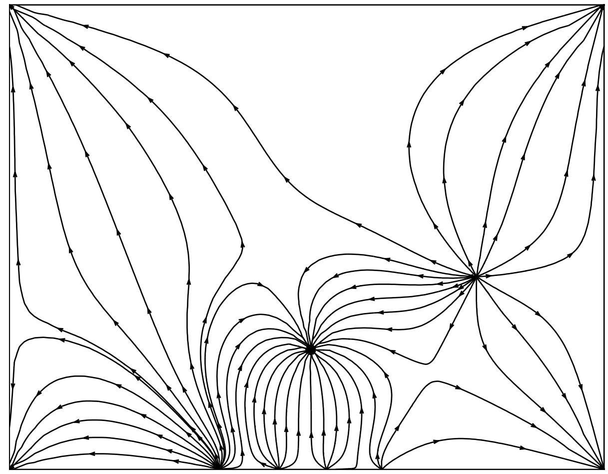


Figure 7. Curvature flow.

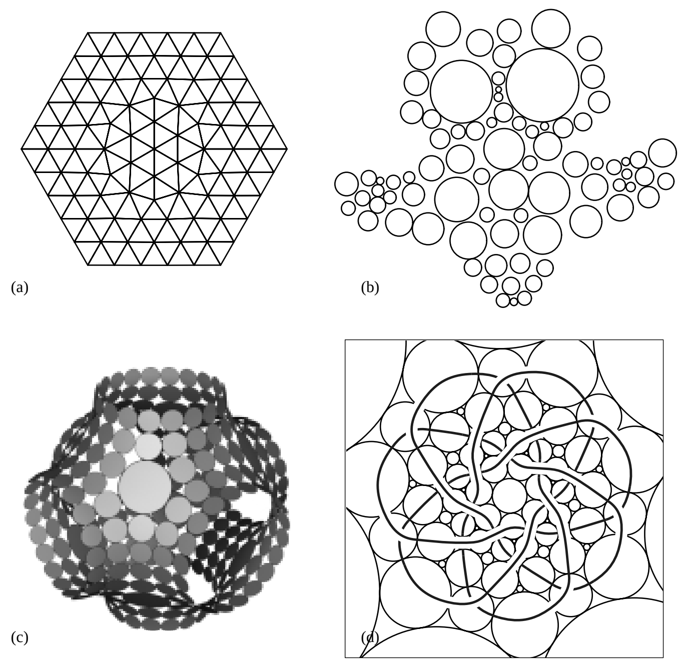


Figure 8. "If it is triangulated, circle pack it!"

pattern in the box on page 1382, for example, there is a remarkably stationary *flow* in the computations: comparing Figure 7 to the packing P_4 there, you can almost see the curvature streaming from points of excess to points of shortage. There is also a markov model of Thurston's algorithm, plus there are alternative algorithms by Colin de Verdière and by Bobenko and Springborn. Every improvement in algorithms seems to be associated with new geometric insights; curvature flow, for example, has a classical interpretation and may aid in parallelizing packing computations. Perhaps the main open question concerns a provable algorithm that works directly in spherical geometry; at this time spherical

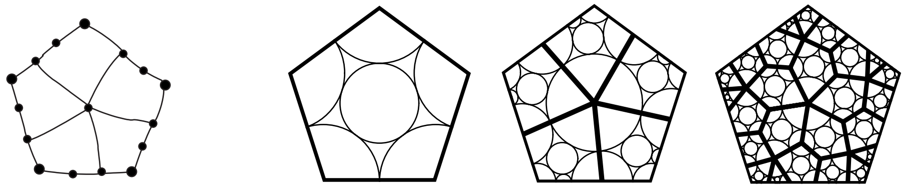


Figure 9. Three stages of subdivision.

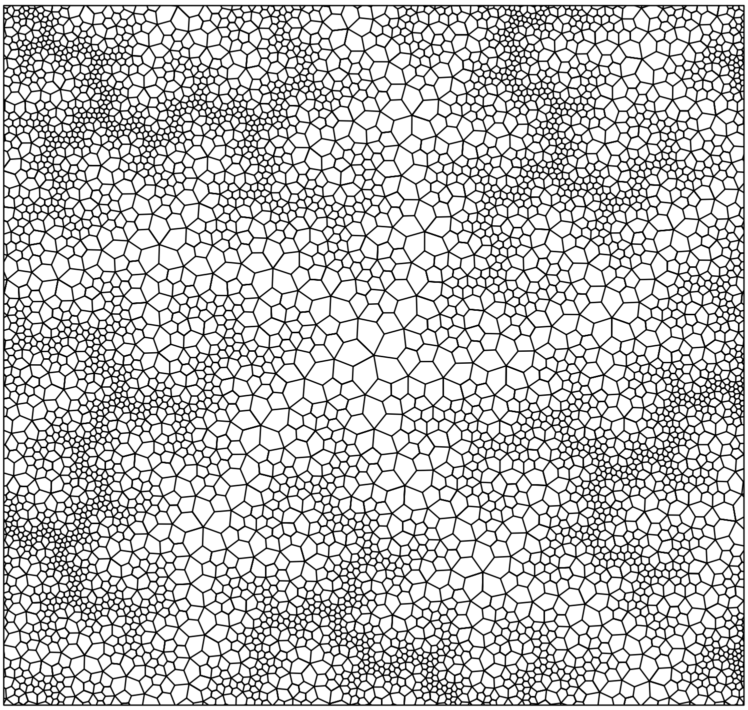


Figure 10. An infinite, subdivision-invariant pattern T.

packing is done in the hyperbolic plane and projected to the sphere.

• **Visualization:** It is the images and interactions with them that really set circle packing apart. You come to anticipate the unexpected in *every* experiment: some surprise symmetry or monotonicity, a classical behavior borne out, a new twist begging explanation, the odd outright mystery. A few examples are shown in Figure 8.

• The "cat's eye" of (a), mere play with edge-flips (Whitehead moves), led to lattice dislocation graphs used by physics colleagues in studies of 2D quenching.

• Thurston's version of K-A-T actually involved circle patterns with prescribed *overlaps* between circles, tangency being just one option. In CirclePack one can specify not only such overlaps but also *inversive distance* packings, raising some challenging new existence and uniqueness questions; (b) is our owl with randomly prescribed inversive distances.

• Use of overlaps in "square grid" packings was initiated by Oded Schramm and picked up by those involved with integrable systems. Alexander Bobenko and his collaborators have extended these ideas into the study of discrete minimal surfaces and related topics; one of their gorgeous images is shown in (c).

• Circle packing powers the visualization of millions of knot and link projections in Thistlethwaite and Hoste's KnotScape program; here the (3, 7) torus knot is shown (with the circles that give it shape). Circle packing has been used to find graph separators, to generate grids, and to study Whitehead moves, hence the motto of Figure 8.

I would argue that CirclePack is to a geometer what a moderately well-equipped laboratory is to an organic chemist (only safer). The potential for open-ended experiments is unique, and yet the machinery is accessible to people at all levels; who knows, a few experiments and you or your students might be hooked!

New Kid on the Block

The new topic has linked itself to a rich classical vein which it has exploited shamelessly: definitions, theorems, examples, philosophy. But former colleagues are beginning to feel used—time for the new kid to step up and contribute.

Much as I would love to tour various applications in discrete function theory, it is probably better to settle on a single, more directly geometric example. 2D tiling is well known for its mixture of combinatorics and geometry, and there is a new theme which grew directly out of circle packing experiments called "conformal" tiling. One reverses tradition by starting with the combinatorics and asking, With what tile shapes and in which geometry can these combinatorics be realized? Let me recount the story of the "twisted pentagonal" tiling. This will necessarily be very brief, but *Notices* readers are known to enjoy a challenge.

The twisted pentagonal subdivision is one of many conceived by Cannon, Floyd, and Parry in ongoing work on Thurston's Geometrization Conjecture; my thanks to Bill Floyd for the combinatorial data.

We begin with the cartoon in Figure 9, which shows a rule for breaking one pentagon into five by adding edges and vertices. Your task, starting with one pentagon, is to repeatedly apply this subdivision rule: at the first stage you get 5 pentagons, then these are subdivided into 25, and these into 125, etc. The combinatorics quickly get out of hand, and circle packing is brought in initially just to get useful embeddings: at each stage, a barycenter added to each pentagon gives a triangulation which can then be circle-packed, giving shape to the pentagons. The first three stages are shown in Figure 9 with their circles.

The pictures are an immediate help, for after a few additional stages you come to realize that the "subdivision" rule can be replaced by a corresponding "expansion" rule. There is, in fact, an essentially unique infinite *combinatorial* tiling T which is subdivision-invariant: i.e., if you simultaneously subdivide all the pentagons, the result is again combinatorially equivalent to T. This T is suggested by Figure 10.

If you caught the spirit of our ten-triangle example in the box on page 1382, you might try the same treatment here. Pasting regular euclidean pentagons together in the pattern of T yields a simply connected p.w.a. Riemann surface $\mathfrak{S}$. Riemann himself would have known that there is a conformal homeomorphism f from $\mathfrak{S}$ to one of $\mathbb{C}$ or $\mathbb{D}$. The images of the faces under f form a so-called *conformal tiling* $\mathcal{T}$ with the combinatorics of T. Is this tiling parabolic or hyperbolic? (i.e., does it lie in $\mathbb{C}$ or in $\mathbb{D}$?) What are the shapes of its tiles? Does the pattern have internal structure? Before circle packing, there was no way to approach such questions, so they weren't asked!

Now we have a method. As you might have suspected, Figure 10 was created using circle packing; it is a rough approximation of $\mathcal{T}$ by "coarse" circle packings like those of Figure 9. Your first instinct might be to improve conformal fidelity via refinement, as we did in the box on page 1382. The key experiments, however, turn out to be of quite a different nature. Stare at $\mathcal{T}$ for a moment—perhaps let your eyes defocus. The longer you look, the more certain you become of some large-scale pattern. Let me help you pull it out. As T is invariant under subdivision, so must it be invariant under *aggregation* (unsubdividing). On the left in Figure 11 are the outlines of the first four stages of aggregation; that is, each of the outlined aggregates is combinatorially equivalent to a subdivision of its predecessor.

Do you see a hint of a pattern now? A little work in PostScript to dilate, rotate, and overlay the outlines leads to the picture on the right in Figure 11. The scale factor turns out to be roughly the same from one stage to the next, suggesting that the tiling is parabolic. More surprising, you find that the corners of the outline from one stage seem to line up with corners of the next: each edge at one stage is replaced by a zig-zag of three edges at the next. Motivated by these very images, Cannon, Floyd, Parry, and Rick Kenyon have confirmed all these observations. In fact, a wealth of mathematics converges in these images: This tiling turns out to be associated with one of Grothendieck's *dessins d'enfants* on the sphere. Hence there exists a rational function with algebraic coefficients whose iterates encode the subdivision rule. That iteration gives an associated Königsfunction k, an entity right out of nineteenth-century function theory, and $\mathcal{T}$ is just the cell decomposition of $\mathbb{C}$ defined by $k^{-1}([0,1])$. And with the scaling confirmed, renormalization (as started on the right in Figure 11) suggests a limit tiling in the pattern of T which would have perfect scaling, *fractal* pentagonal tiles, and a

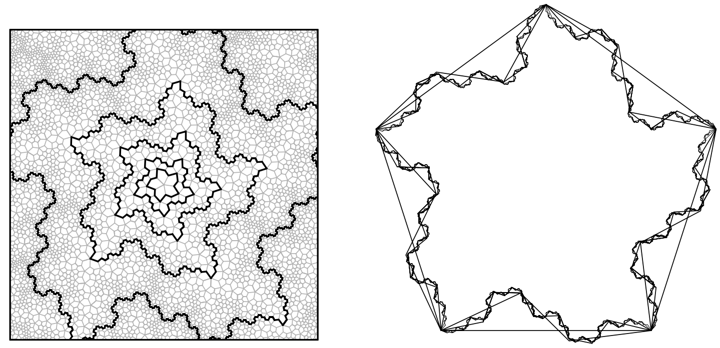


Figure 11. Outline, scale, rotate, and overlay the aggregates.

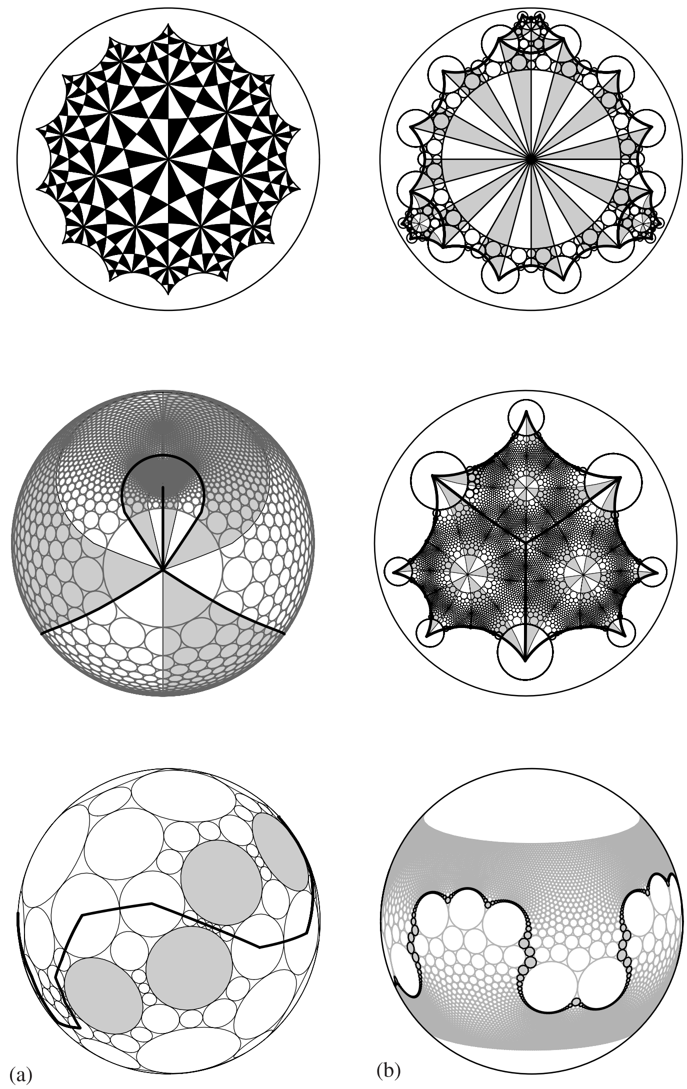


Figure 12. Circle packing for conformal structure.

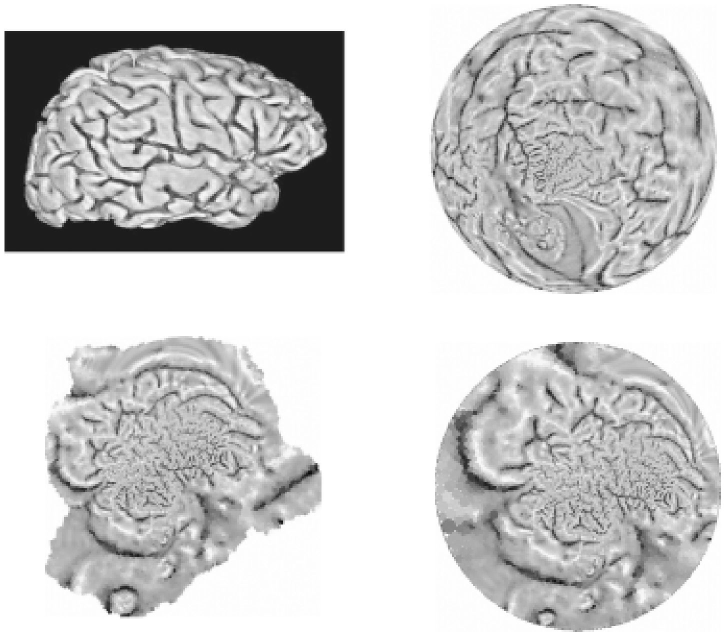


Figure 13. A 3D cortical hemisphere and three flat maps.

fractal subdivision rule. Such is the story of the twisted pentagonal tiling. It is clear that our discrete experiments have faithfully captured some truth in classical conformal geometry.

There are many other combinatoric situations where similar experiments can be run. At the risk of leaving some mysteries for the reader, I have shown a few in Figure 12. The two images at the top are straight out of classical function theory. On the left, circles (not shown) recreate the "Klein" surface in an image made famous over 100 years ago in Fricke/Klein. On the right is the more illusive "Picard" surface; we can now approach this and other surfaces whose triangulations are *not* geodesic. If you are into number theory, the topic known as *dessins d'enfants*, children's drawings, alluded to earlier, tightly binds combinatorics, meromorphic functions, and number fields. Its structures are equilateral, like the ten-triangle example in the box on page 1382, so the topic is a natural for discrete experimentation; the middle images in Figure 12 show genus 0 and genus 2 *dessin* examples, respectively. Likewise, the notion of *conformal welding* used in complex analysis and in the study of 3-manifolds is now an experimental reality. The bottom images in Figure 12 show a toy example, where two owls have been welded together, along with a more serious example produced by George (Brock) Williams.

Note how the tables have turned. We are now starting with combinatorics. Circle packing then imposes a *discrete conformal structure*, that is, a geometry manifesting key characteristics of conformality—notions such as 'type', extremal length, moduli of rings, harmonic measure, and curvature. Then one can search for, perhaps even prove, parallel classical results. In summarizing the intuition, it seems only fair to let the discrete side take the lead: *A discrete conformal structure on a surface is determined by a triangulation; a classical conformal structure is determined in the same way, but with "infinitesimal" triangles.*

I cannot leave this section without mentioning a point of closure in the theory provided by results of He and Schramm. They have made a major advance on a classical conjecture concerned with the conformal mapping of infinitely connected regions, the so-called *Kreisnormierungsproblem*, by applying methods which they developed in circle packing. And exactly whose conjecture was this? None other than P. Koebe himself: he proved the finitely connected case and then applied his classical methods to establish the K-A-T Theorem!

Reaching Out

It is an article of mathematical faith that every topic will find connections to the wider world—eventually. For some, that isn't enough. For some it is real-time exchange between the mathematics and the applications that is the measure of a topic.

The important roles complex analysis traditionally played in the physical sciences—electrostatics, fluid flow, airfoil design, residue computations—are largely gone, replaced by numerical partial differential equations or symbolic packages. But the core of complex analysis is too fundamental to go missing for long. *Surfaces* embedded in three-space are becoming pervasive in new areas of science, image analysis, and computer visualization, and conformal geometry is all about such surfaces. With new tools to (faithfully) access conformality, perhaps complex analysis has new roles to play.

I would like to illustrate briefly with brain-flattening work that has garnered recent exposure outside of mathematics. The work is being carried on by an NSF-sponsored Focused Research Group: Chuck Collins and the author (Tennessee); Phil Bowers, Monica Hurdal, and De Witt Sumners (Florida State); and neuroscientist David Rottenberg (Minnesota).

The first image in Figure 13 shows the type of 3D data which is becoming routinely available through noninvasive techniques such as MRI (magnetic resonance imaging), in this case, one hemisphere of a human cerebrum. Our mental processing occurs largely in the *cortex*, the thin layer of neurons (grey matter) on the brain surface. Neuroscientists wishing to apply surface-based techniques need to map the cortex to a flat domain—hence the topic of "brain-flattening". As you can see, the cortex is an extremely convoluted surface (the shading here reflects the mean curvature), and it is well known that there can exist no flat map which preserves its areas or surface distances. However, by the 150-year-old Riemann Mapping Theorem there does exist a *conformal* flat map. Using standard techniques, one can produce a triangulation which approximates the cortical surface from the volumetric data. Figure 13 illustrates three discrete conformal flat maps based on such a triangulation (180,000 vertices): clockwise from upper right are spherical, hyperbolic, and euclidean flat maps.

It is not our goal to discuss the potential scientific value in these maps (though I will mention that our neuroscience colleagues have a surprising affinity for the hyperbolic maps; perhaps these are reminiscent of the view in a microscope). However, there are some points that do bear on our story.

• First, one does not have to believe that conformality *per se* has any relevance to an application to exploit its amazing richness—existence and uniqueness first, then companion notions such as extremal length and harmonic measure.

• Second, approximation of true conformality may be superfluous if its companion structures appear faithfully at coarse stages, as seems often to be the case with circle packing.

• Finally, the structures take precedence over technique; circle-packing experiments can contribute to a topic even if other methods replace it in practice. What would be nice to hear at the end of a neuro consult is, "You know, we need to hire another conformal geometer."

Conclusion

Of course, a mathematical topic itself never concludes; the tradition is "definition-theorem-proof-publication" as new contributions add to the line. A mathematical tale, on the other hand, must have closure, and the storyteller is allowed to put some personal spin on the story (if not a moral).

I have related this tale in the belief that it has some touch of universality to it. We are drawn to mathematics for a variety of reasons: the clarity of elementary geometry, the discipline of computation, the challenge of richly layered theory and deep questions, the beauty of images, or the pleasures of teaching and applying the results. I feel that I have seen all these in circle packing, and perhaps you have glimpsed parallels with your own favorite topic.

Personally, it has been a pleasure to watch an old friend, complex function theory, emerge in a form with so much appeal: new theory, new applications, stunning visuals, an exciting experimental slant. For me circle packing is *quantum complex analysis, classical in the limit.* The discrete results and their proofs are pure mathematics, the pictures and software being not only unnecessary, but for some, unwanted. Yet the experimentation and visualization, the very programming itself, are at the research frontier here. In this regard, circle packing illustrates the growing challenge mathematics faces to incorporate new modes of research into its practices and literature.

Of course circle packing, like any mathematical topic, has many potential storylines. In that spirit, let me end our mathematical tale in a very traditional way, namely, in the hope that it nourishes others who can pass along their own stories in their own words.

Reader's Guide

Once: [12], [26], [28], [1], [30]. *Birth:* [25], [2], [35], [36], [13]. *Internal:* [27], [5], [4], [9], [21], [3], [22], [10]. *Dust:* [19], [18], [34] (survey), [31]. *Special:* [32], [20] (survey), [15], [7], [17], [29], [6], [23]. *New Kid:* [33] (survey), [21], [19], [11], [8], [37], [14], [16]. *Reach Out:* [24]. (You can download a more complete bibliography and CirclePack from my website: `http://www.math.utk.edu/~kens`.)

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About the Cover

Conformal maps by packing circles

This month's cover illustrates how one of the simplest possible conformal maps can be approximated by the technique of circle packing explained in Kenneth Stephenson's article. In constructing it, the very detailed recipe to be found in "A circle packing algorithm" by Charles Collins and Stephenson (*Computational Gemetry* 25 (2003)) was followed.

Both circle configurations are associated to the same triangulation, which is also shown—vertices correspond to circles, and edges correspond to circles that touch. The corners are left out for technical reasons.

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