



Nonlinear Oscillations of Viscoelastic Rectangular Plates

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Abstract. Nonlinear oscillations of viscoelastic simply supported rectangular plates are studied by assuming the Voigt–Kelvin constitutive model. Using Hamilton’s principle in conjunction with the kinematics associated with Kirchhoff’s plate model, the governing equations of motion including the effect of damping are represented in terms of the transversal deflection and a stress function. Utilizing the Bubnov–Galerkin method, the nonlinear partial differential equations are reduced to an ordinary differential equation which is studied geometrically by approximate construction of the Poincaré maps. Explicit expressions are given for periodic solutions.

Keywords: Nonlinear oscillations, periodic solutions, viscoelastic plates.

1. Introduction

Viscoelastic materials are widely used in vehicle technology, spacecraft attachments and composite material development in order to reduce undesired vibrations. From a mathematical point of view, a dissipation term is included in the differential equations which represents the dynamic behavior of systems composed of such materials. Many researchers include damping effects in their equations of motion through adding a simple linear term, i.e. $\mu \dot{w}$ at which μ is an equivalent damping coefficient and \dot{w} denotes the velocity field in the continuum under consideration. Hadian and Nayfeh [1] used the dynamic analog of the von Kármán equations in polar form to study the symmetric multi-mode response of circular plates. They considered damping effects by introducing a dissipation term which is not a result of known constitutive relations of viscoelastic materials. A similar dissipation term used by Lu et al. [2] for the study of rectangular orthotropic plates by taking transverse shear deformations into account. They augmented the Mathieu–Hill type equations obtained from application of the Bubnov–Galerkin method, with a linear damping term. A systematic approach was only followed by Xia and Lukasiewicz [3, 4]. In this paper we analyze the nonlinear oscillations of viscoelastic rectangular plates by a method similar to Xia and Lukasiewicz [4]. The technique followed in the current research is suitable for prediction and stability analysis of periodic solutions which have great importance in control applications. In the presence of dissipation, by introducing a stress function and applying the Bubnov–Galerkin method, we would be able to utilize the knowledge of dynamical systems for periodically excited rectangular plates. The present formulation provides a better understanding of the influence of energy dissipation. A

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transformation to intermediate action angle variables is also found to be useful when determining the periodic solutions for excitation of a single mode at the 1:1 resonance. Furthermore, analytical expressions are presented for the amplitude and the phase of the periodic motions. It is geometrically shown that two stable and one unstable periodic points exist for very small damping.

2. Governing Equations of Motion

The system under investigation consists of a viscoelastic plate of thickness h , being simply supported at all edges. The x and y axes are located in the middle plane of the undeformed plate as the reference plane. The z axis is normal to the reference plane with upward positive direction. The plate is subjected to a harmonic support motion $Z(t) = A \cos(\omega t)$ in the z direction. Considering the plane stress problem and the Voigt–Kelvin constitutive model of a viscoelastic material [8], the stress-strain relations will be

$$\sigma_z = \sigma_{zx} = \sigma_{zy} = 0, \quad (1a)$$

$$\sigma_x = \frac{E}{1-\nu^2} [(\varepsilon_x + \eta \dot{\varepsilon}_x) + \nu(\varepsilon_y + \eta \dot{\varepsilon}_y)], \quad (1b)$$

$$\sigma_y = \frac{E}{1-\nu^2} [(\varepsilon_y + \eta \dot{\varepsilon}_y) + \nu(\varepsilon_x + \eta \dot{\varepsilon}_x)], \quad (1c)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)} (\varepsilon_{xy} + \eta \dot{\varepsilon}_{xy}), \quad (1d)$$

where ν and E are the Poisson ratio and the modulus of elasticity, respectively. Dot denotes differentiation with respect to the time, t , and η is the dissipation parameter which could be obtained from experimental data. The displacement components u , v and w are assumed to be

$$u(x, y, z; t) = u^\circ(x, y, z; t) - zw_{,x}^\circ, \quad (2a)$$

$$v(x, y, z; t) = v^\circ(x, y, z; t) - zw_{,y}^\circ, \quad (2b)$$

$$w(x, y, z; t) = w^\circ(x, y, z; t), \quad (2c)$$

where u° , v° and w° are the displacement components of the middle plane in the x , y and z directions, respectively. The strain-displacement relations are [5]

$$\begin{aligned} \varepsilon_x &= u_{,x}^\circ - zw_{,xx}^\circ + \frac{1}{2}w_{,x}^{\circ 2}, & \varepsilon_y &= v_{,y}^\circ - zw_{,yy}^\circ + \frac{1}{2}w_{,y}^{\circ 2}, \\ \varepsilon_z &= w_{,z}^\circ + \frac{1}{2}(w_{,x}^{\circ 2} + w_{,y}^{\circ 2}), & \varepsilon_{xy} &= u_{,y}^\circ + v_{,x}^\circ - 2zw_{,xy}^\circ + w_{,x}^\circ w_{,y}^\circ, \end{aligned} \quad (3)$$

and therefore

$$\begin{aligned} \sigma_x &= \frac{E}{1-\nu^2} \left[(u_{,x}^\circ - zw_{,xx}^\circ + \frac{1}{2}w_{,x}^{\circ 2} + \eta(\dot{u}_{,x}^\circ - z\dot{w}_{,xx}^\circ + w_{,x}^\circ \dot{w}_{,x}^\circ)) \right. \\ &\quad \left. + \nu(v_{,y}^\circ - zw_{,yy}^\circ + \frac{1}{2}w_{,y}^{\circ 2} + \eta(\dot{v}_{,y}^\circ - z\dot{w}_{,yy}^\circ + w_{,y}^\circ \dot{w}_{,y}^\circ)) \right], \end{aligned} \quad (4a)$$

$$\sigma_y = \frac{E}{1-\nu^2} \left[(\nu_{,y}^\circ - z w_{,yy} + \frac{1}{2} w_{,y}^2 + \eta(\dot{\nu}_{,y}^\circ - z \dot{w}_{,yy} + w_{,y} \dot{w}_{,y})) \right. \\ \left. + \nu(u_{,x}^\circ - z w_{,xx} + \frac{1}{2} w_{,x}^2 + \eta(\dot{u}_{,x}^\circ - z \dot{w}_{,xx} + w_{,x} \dot{w}_{,x})) \right], \quad (4b)$$

$$\sigma_{xy} = \frac{E}{2(1+\nu)} \left[u_{,y}^\circ + \nu_{,x}^\circ - 2z w_{,xy} + w_{,x} w_{,y} \right. \\ \left. + \eta(\dot{u}_{,y}^\circ + \dot{\nu}_{,x}^\circ - 2z \dot{w}_{,xy} + \dot{w}_{,x} w_{,y} + w_{,x} \dot{w}_{,y}) \right]. \quad (4c)$$

From here on the superscript $^\circ$ will be dropped for the sake of simplified notation. By substituting from Equations (2, 3), and (4) into Hamilton's principle one obtains

$$\int_{t_1}^{t_2} \left[\int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} \rho(\dot{u} \delta \dot{u} + \dot{\nu} \delta \dot{\nu} + \dot{w} \delta \dot{w}) \, dz \, dx \, dy \right. \\ \left. - \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} \int_{-h/2}^{h/2} (\sigma_x \delta \varepsilon_x + \sigma_y \delta \varepsilon_y + \sigma_{xy} \delta \varepsilon_{xy}) \, dz \, dx \, dy \right] dt = 0. \quad (5)$$

Applying the usual variational techniques, the equations of motion are obtained

$$N_{x,x} + N_{xy,y} = \rho h u_{,tt} \approx 0, \quad (6a)$$

$$N_{xy,x} + N_{y,y} = \rho h v_{,tt} \approx 0, \quad (6b)$$

$$-D(w_{,xxxx} + 2\nu w_{,xxyy} + w_{,yyyy}) - D\eta \frac{\partial}{\partial t} (w_{,xxxx} + 2\nu w_{,xxyy} + w_{,yyyy}) \\ + S_1(u_{,xx} w_{,x} + u_{,x} w_{,xx} + \frac{3}{2} w_{,x}^2 w_{,xx} + \nu v_{,xy} w_{,x} + \nu v_{,y} w_{,xx} + \frac{1}{2} \nu w_{,xx} w_{,y}^2 \\ + \nu w_{,x} w_{,y} w_{,xy} + v_{,yy} w_{,y} + v_{,y} w_{,yy} + \nu u_{,xy} w_{,y} + \nu u_{,x} w_{,yy} + \frac{3}{2} w_{,yy} w_{,y}^2 \\ + \frac{1}{2} \nu w_{,yy} w_{,x}^2 + \nu w_{,y} w_{,x} w_{,xy}) - 4S_3 w_{,xxyy} + S_2(w_{,xy} u_{,y} + w_{,y} u_{,xy} + w_{,xy} v_{,x} \\ + w_{,y} v_{,xx} + w_{,xx} w_{,y}^2 + 2w_{,x} w_{,y} w_{,xy} + w_{,xy} u_{,y} + w_{,x} u_{,yy} + w_{,xy} v_{,x} + w_{,x} v_{,xy} \\ + w_{,x}^2 w_{,yy} + 2w_{,x} w_{,xy} w_{,y}) + S_1 \eta (w_{,xx} \dot{u}_{,x} + w_{,x} \dot{u}_{,xx} + w_{,x}^2 \dot{w}_{,xx} + 2w_{,x} w_{,xx} \dot{w}_{,x} \\ + \nu w_{,xx} \dot{v}_{,y} + \nu w_{,x} \dot{v}_{,xy} + \nu w_{,xx} w_{,y} \dot{w}_{,y} + \nu w_{,x} w_{,xy} \dot{w}_{,y} + \nu w_{,x} w_{,y} \dot{w}_{,xy} + w_{,yy} \dot{v}_{,y} \\ + w_{,y} \dot{v}_{,yy} + w_{,y}^2 \dot{w}_{,yy} + 2w_{,y} w_{,yy} \dot{w}_{,y} + \nu w_{,yy} \dot{u}_{,x} + \nu w_{,y} \dot{u}_{,xy} + \nu w_{,yy} w_{,x} \dot{w}_{,x} \\ + \nu w_{,y} w_{,xy} \dot{w}_{,x} + \nu w_{,y} w_{,x} \dot{w}_{,xy}) - 4S_3 \eta \dot{w}_{,xxyy} + S_2 \eta (w_{,xy} \dot{u}_{,y} + w_{,y} \dot{u}_{,xy} \\ + w_{,xy} \dot{v}_{,x} + w_{,y} \dot{v}_{,xx} + \dot{w}_{,xx} w_{,y}^2 + 3\dot{w}_{,x} w_{,y} w_{,xy} + w_{,xx} w_{,y} \dot{w}_{,y} + 3w_{,x} w_{,xy} \dot{w}_{,y} \\ + 2w_{,x} w_{,y} \dot{w}_{,xy} + w_{,xy} \dot{u}_{,y} + w_{,x} \dot{u}_{,yy} + w_{,xy} \dot{v}_{,x} + w_{,x} \dot{v}_{,xy} + w_{,x}^2 \dot{w}_{,yy} \\ + w_{,x} w_{,yy} \dot{w}_{,x}) = \rho h \ddot{w} - \rho h A \omega^2 \cos \omega t, \quad (6c)$$

where

$$S_1 = \frac{Eh}{1 - \nu^2}, \quad S_2 = \frac{Eh}{2(1 + \nu)}, \quad S_3 = \frac{EI}{2(1 + \nu)}, \quad D = \frac{EI}{1 - \nu^2}, \quad I = \frac{h^3}{12},$$

$$N_x = S_1 \left[\left(u_{,x} + \nu v_{,y} + \frac{1}{2}(w_{,x}^2 + \nu w_{,y}^2) \right) + \eta \frac{\partial}{\partial t} \left(u_{,x} + \nu v_{,y} + \frac{1}{2}(w_{,x}^2 + \nu w_{,y}^2) \right) \right], \quad (7a)$$

$$N_y = S_1 \left[\left(v_{,y} + \nu u_{,x} + \frac{1}{2}(w_{,y}^2 + \nu w_{,x}^2) \right) + \eta \frac{\partial}{\partial t} \left(v_{,y} + \nu u_{,x} + \frac{1}{2}(w_{,y}^2 + \nu w_{,x}^2) \right) \right], \quad (7b)$$

$$N_{xy} = S_2 \left[u_{,y} + v_{,x} + w_{,x} w_{,y} + \eta \frac{\partial}{\partial t} (u_{,y} + v_{,x} + w_{,x} w_{,y}) \right]. \quad (7c)$$

The boundary conditions are considered to be simply supported with immovable edges, hence

$$w = 0, \quad \text{at } x = \pm \frac{1}{2} a \quad \text{and} \quad y = \pm \frac{1}{2} b, \quad (8a)$$

$$\begin{cases} u = \dot{u} = N_{xy} = 0 & \text{at } x = \pm \frac{1}{2} a, \\ v = \dot{v} = N_{xy} = 0 & \text{at } y = \pm \frac{1}{2} b, \end{cases} \quad (8b)$$

$$\begin{cases} M_x = (1 + \eta \frac{\partial}{\partial t}) D(w_{,xx} + \nu w_{,yy}) = 0 & \text{at } x = \pm \frac{1}{2} a, \\ M_y = (1 + \eta \frac{\partial}{\partial t}) D(w_{,yy} + \nu w_{,xx}) = 0 & \text{at } y = \pm \frac{1}{2} b, \end{cases} \quad (8c)$$

There is some deficiencies in dealing with Equations (6) in their original form. In finding the solution by harmonic balance technique, they lead to a set of three second order ODEs which have a six dimensional phase space with its own complexities. However, by utilizing the stress function ϕ defined by

$$N_x = \phi_{,yy}, \quad N_y = \phi_{,xx}, \quad N_{xy} = -\phi_{,xy}, \quad (9)$$

one obtains the governing equations of motion in terms of w and ϕ as the transversal equation of motion in the z direction

$$D \left(1 + \eta \frac{\partial}{\partial t} \right) \nabla^2 \nabla^2 w + \rho h w_{,tt} - \phi_{,yy} w_{,xx} - \phi_{,xx} w_{,yy} + 2\phi_{,xy} w_{,xy} = \rho h A \omega^2 \cos \omega t, \quad (10)$$

and the compatibility equation

$$\nabla^2 \nabla^2 \phi = Eh(w_{,xy}^2 - w_{,xx} w_{,yy}). \quad (11)$$

According to Equation (9), Equations (6a) and (6b) will accurately be satisfied. In addition, the dissipation terms are implicitly contributed to Equation (11) due to Equations (7) and (9). Boundary conditions (8) will be satisfied if w is described by the Fourier series

$$w(x, y; t) = \sum_{m=1,3,\dots}^{\infty} \sum_{n=1,3,\dots}^{\infty} \Psi_{mn}(t) \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b}. \quad (12)$$

Taking only the first harmonic into account ($m = 1, n = 1$), and substituting Equation (12) in Equation (11) gives

$$\nabla^2 \nabla^2 \phi = Eh \frac{-\pi^4 \Psi^2}{2a^2 b^2} \left(\cos \frac{2\pi x}{a} + \cos \frac{2\pi y}{b} \right), \quad \Psi = \Psi_{11}(t), \quad (13)$$

from which a particular solution of ϕ is obtained

$$\phi = C_1 \cos \frac{2\pi x}{a} + C_2 \cos \frac{2\pi y}{b} + \frac{1}{2}(C_3 x^2 + C_4 y^2). \quad (14)$$

Coefficients C_1, C_2, C_3 and C_4 are determined according to the prescribed boundary conditions and are found to be

$$\begin{aligned} C_1 &= \frac{-\lambda^2 \Psi^2 Eh}{32}, & C_2 &= \frac{-\Psi^2 Eh}{32\lambda^2}, \\ C_3 &= \frac{\pi^2 Eh(\nu + \lambda^2)}{8a^2(1 - \nu^2)} \Psi^2, & C_4 &= \frac{\pi^2 Eh(1 + \nu\lambda^2)}{8a^2(1 - \nu^2)} \Psi^2, \end{aligned} \quad (15)$$

with $\lambda = a/b$. Substituting for w and ϕ from Equations (12) and (14) into Equation (10) and applying the Bubnov–Galerkin global averaging method [9], an ordinary second-order nonlinear differential equation is obtained for the relative function $\delta(t) = \Psi(t)/h$. Thus,

$$\ddot{\delta} + \zeta \dot{\delta} + \Omega^2 \delta + \gamma \delta^3 = f_0 \cos \omega t, \quad (16)$$

where

$$\begin{aligned} \Omega^2 &= \frac{D\pi^4(1 + \lambda^2)^2}{\rho h a^4}, \\ \gamma &= \frac{\pi^4 Eh^2}{16\rho a^4(1 - \nu^2)} [(5 - \nu^2)(1 + \lambda^4) + 8\nu\lambda^2], \\ f_0 &= \frac{16A\omega^2}{h\pi^2}, \\ \zeta &= \eta\Omega^2. \end{aligned} \quad (17)$$

The state-space representation of Equation (16) may be written as

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -\Omega^2 x_1 + \varepsilon(-\gamma x_1^3 - \zeta x_2 + f_0 \cos \omega t), \end{cases} \quad (18)$$

with $x_1 = \delta$ and $x_2 = \dot{\delta}$; ε is a perturbation parameter which will eventually be set equal to one. All of the constant coefficients in Equation (17) are found to be positive according to the special boundary conditions adopted in this study. It is desired to investigate the system represented by Equation (18) for the periodic behavior. This goal may be attained by various strategies such as the averaging method [7], the method of Poincaré–Lindstedt [6], the fixed-point theorem [10–12] and the multiple scales method [6]. In this study we have chosen the averaging method in order to construct the approximate Poincaré maps. Stationary points in a Poincaré section represent the periodic solutions of the original problem [7]. The averaged system is investigated near the 1:1 resonance where $2\pi/\omega$ -periodic solutions are sought.

3. Averaging the Equations of Motion

Consider the well-known Van der Pol transformation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} \cos \omega t & -\sin \omega t \\ -\omega \sin \omega t & -\omega \cos \omega t \end{bmatrix} \begin{pmatrix} p \\ q \end{pmatrix}. \quad (19)$$

Substitution from Equation (19) into Equation (18) and employing the averaging theorem over one period of the exciting force, $T = 2\pi/\omega$, yields

$$\dot{p} = \varepsilon \left[-\frac{3\gamma}{8\omega}(p^2 + q^2)q - \frac{1}{2}\zeta p - \frac{1}{2}\mu\omega q \right], \quad (20a)$$

$$\dot{q} = \varepsilon \left[\frac{3\gamma}{8\omega}(p^2 + q^2)p - \frac{1}{2}\zeta q + \frac{1}{2}\mu\omega p - \frac{1}{2\omega}f_0 \right], \quad (20b)$$

with

$$\frac{\Omega^2}{\omega^2} = 1 + \varepsilon\mu. \quad (21)$$

Adopting the action and angle variables I and θ as

$$p = \sqrt{2I} \cos \theta, \quad q = \sqrt{2I} \sin \theta, \quad (22)$$

the averaged equations (20) become

$$I' = -2\zeta\omega I - f_0\sqrt{2I} \sin \theta, \quad (23a)$$

$$\theta' = \frac{3}{2}\gamma I + \mu\omega^2 - f_0\frac{1}{\sqrt{2I}} \cos \theta, \quad (23b)$$

where primes denote differentiation with respect to slow the time $\tau = \varepsilon t/(2\omega)$. The stationary points of Equation (23) can be determined through solving the third-order algebraic equation

$$2I \left[\frac{9}{4}\gamma^2 I^2 + 3\gamma\mu\omega^2 I + \omega^2(\zeta^2 + \mu^2\omega^2) \right] - f_0^2 = 0, \quad (24)$$

with the associated angular coordinates, θ_s 's, evaluated by

$$\theta_s = \arctan[-2\zeta\omega/(3\gamma I_s + 2\mu\omega^2)], \quad s = 1, 2 \text{ and } 3, \quad (25)$$

where I_s 's are the real roots of Equation (24).

4. Bifurcations and Periodic Solutions

We are interested in finding periodic solutions of the original system (18). This goal will be achieved by determination of stationary points of the averaged system (23) through solving Equations (24) and (25). Let

$$D = Q^3 + R^2 \quad (26)$$

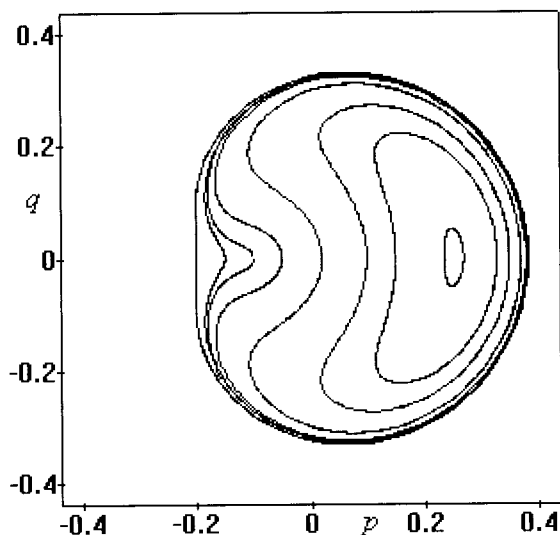


Figure 1a. Phase portrait of the averaged system for $\eta = 0$ and $D > 0$.

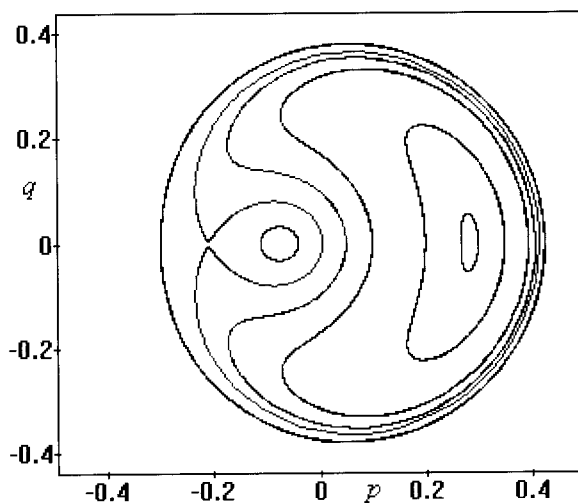


Figure 1b. Phase portrait of the averaged system for $\eta = 0$ and $D < 0$.

be the discriminant of Equation (24) with

$$Q = \frac{4}{27} \frac{\omega^2}{\gamma^2} \left(\zeta^2 - \frac{1}{3} \mu^2 \omega^2 \right), \quad R = \frac{8}{81} \frac{\mu \omega^4}{\gamma^3} \left(\zeta^2 + \frac{1}{9} \mu^2 \omega^2 - \frac{9}{8} \frac{\gamma f_0^2}{\mu \omega^4} \right).$$

Depending on the value of D , four types of flows emerge. These have been shown in Figures 1 and 2. Figures 1a and 1b correspond to the undamped elastic system with $D > 0$ and $D < 0$, respectively. For $D > 0$ the averaged system has only one fixed point indicating a stable periodic solution. For $D < 0$ three fixed points occur. One of these points is a saddle that denotes unstable periodic motion. The other two are centers denoting stable periodic solutions of the original system. Two homoclinic orbits can be distinguished when $D < 0$. Therefore, in the absence of damping effects, chaotic motions can appear if these homoclinic orbits are destroyed. For $\eta \neq 0$ the averaged system loses its symplectic structure and the center type

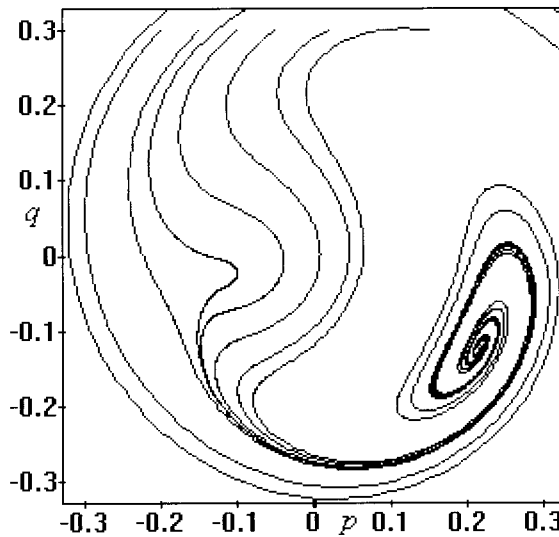


Figure 2a. Phase portrait of the averaged system for $\eta = 0.00001$ and $D > 0$.

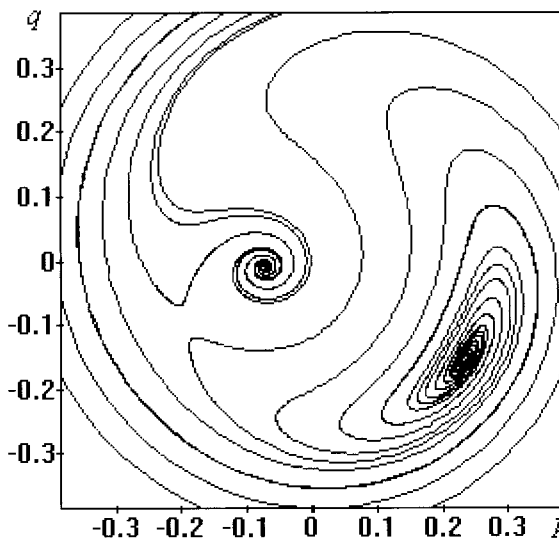


Figure 2b. Phase portrait of the averaged system for $\eta = 0.00001$ and $D < 0$.

fixed points change their position and become attracting periodic motions (see Figures 2a and 2b). Consider the case at which three real roots of Equation (24) exist. According to the experimental data, numerical value of η is small and therefore the rate of energy dissipation will be low. Figure 2b shows the phase plane trajectories of Equation (23) in this case. Using Equations (19) and (22), and the results of Equations (24) and (25), one obtains

$$x_1 = \sqrt{2I_s} \cos(\omega t + \theta_s), \tag{27a}$$

$$x_2 = -\omega \sqrt{2I_s} \sin(\omega t + \theta_s), \quad s = 1, 2, \text{ and } 3. \tag{27b}$$

These are the approximate representations of the periodic solutions.

5. Conclusion

Approximate analytical expressions were obtained for the stable and unstable periodic solutions of viscoelastic rectangular plates subjected to a harmonic support motion. It is geometrically revealed that due to existing small damping, all trajectories of the averaged system are attracted to the stable periodic solutions. Results of this paper may be used for experimental evaluation of the dissipation parameter. By exciting a damped rectangular plate and measuring the state of dynamic equilibrium, one can find damping coefficients through the given analytical expressions.

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