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Periodicity and chaos in a flexible crank-rocker mechanism

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Abstract

In this paper we introduce a crank-rocker mechanism at which the rocker is flexible. Using Hamilton's principle we obtain the governing equations of motion for the elastic mode of the rocker. By applying the Bubnov–Galerkin global averaging method, we reduce the governing equations of motion to an ordinary differential equation which is Duffing's oscillator with time varying coefficients. Through the application of Banach's fixed-point theorem we predict the periodic solutions. Then we study the geometrical features of the motion near the 1 : 1, 1 : 2 and 2 : 1 commensurabilities. It is also shown that homoclinic and heteroclinic orbits can exist for the system.

Keywords: Chaos; Non-linear oscillations; Periodic solutions; Resonances

1. Introduction

Duffing's equation has played an important role in the understanding of non-linear dynamical systems. It is one of the simplest systems that demonstrates periodic, quasi-periodic and chaotic behaviors. The classical problem is represented by the equation

$$\ddot{y} + \omega^2 y + \gamma y^3 = A + B\cos\omega t, \tag{1}$$

with ω, γ, A and B as constants. This problem was formulated by Duffing in 1918 and since then has been investigated by many researchers. Most of the motion characteristics of this system can be found in Berdichevsky et al. [1] and references therein. In this paper we study Duffing's oscillator with time varying coefficients given by

$$\ddot{y} + P(t)y + Q(t)y^3 = F(t), \tag{2}$$

where $P(t)$, $Q(t)$ and $F(t)$ are single periodic functions of time. A qualitative study of Eq. (2) can be found in the work of Alekseev [2–4] where he applied the methods of symbolic dynamics. For $F(t) = 0$ it has been shown that the system approximates Sitnikov's problem (Hagel [5], Jalali and Pourtakdoust [6]). Hagel [5] utilized a perturbation technique based on the expansion of Poisson brackets and studied certain regular solutions. Jalali and Pourtakdoust [6] showed that in the neighborhood of certain resonances one could integrate the equations of motion in terms of Jacobi's

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elliptic functions. In this study we first construct a mechanical system that its behavior is characterized by Eq. (2). Then, following the prospects of Mehri and Emami-Rad [7], we prove the existence of periodic solutions using Banach's fixed point theorem. By expanding $P(t)$, $Q(t)$ and $F(t)$ in Taylor series and averaging the Hamiltonian function, we study the behavior of the system near resonances. Finally, through the numerical construction of the Poincaré maps, we search for those regions of the phase space that become chaotic.

2. The problem statement

Consider the system shown in Fig. 1. That is a crank-rocker mechanism at which the rocker AE slides in the guide B while B is hinged to the crank CD. The rocker is assumed to be flexible. We impose a tensile force on the rocker by a spring installed between the guide B and the end shoulder of the rocker. We consider the free length of the spring to be equal to l. Thus, the rocker will be under tension for all times. The crank rotates clockwise with the constant angular velocity ω. When the motion starts, the rocker is deflected due to transversal inertia forces and begins to oscillate along with its rigid-body rotation. The kinematics of the rigid-body motion is completely known for this mechanism and the following relations hold:

$$L = L(t) = d(1 + \delta^2 - 2\,\delta\cos\omega t)^{1/2}, \quad \delta = r/d, \tag{3a}$$

$$\ddot{\theta} = (-1 + \delta^2)\omega^2\delta\sin\omega t(1 + 2\delta^2 - 4\delta\cos\omega t + \delta^4 - 4\delta^3\cos\omega t + 4\delta^2\cos^2\omega t)^{-1}. \tag{3b}$$

The crank is considered to be rigid and it is desired to investigate the oscillations of the flexible rocker around its rigid-body motion. In our analysis we can neglect the effect of the portion CE if δ is small.

3. Equations of motion

We assume the following strain–displacement relation:

$$\varepsilon_x = \frac{\partial u}{\partial x} + z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{4}$$

where $u(x,t)$ and $w(x,t)$ are the longitudinal and transversal elastic displacements of the rocker in the rotating x–y coordinate system, respectively. The variation of the strain energy function is

$$\delta U = \int_D \sigma_x\,\delta\varepsilon_x\,\mathrm{d}D, \tag{5}$$

with $\sigma_x = E\varepsilon_x$. E is the modulus of elasticity, and D is the domain of extension of the beam AC. At point x of the beam, the kinetic energy per unit volume is given by

$$\frac{1}{2}\rho(\dot{u}^2 + (\dot{w} + x\dot{\theta})^2), \tag{6}$$

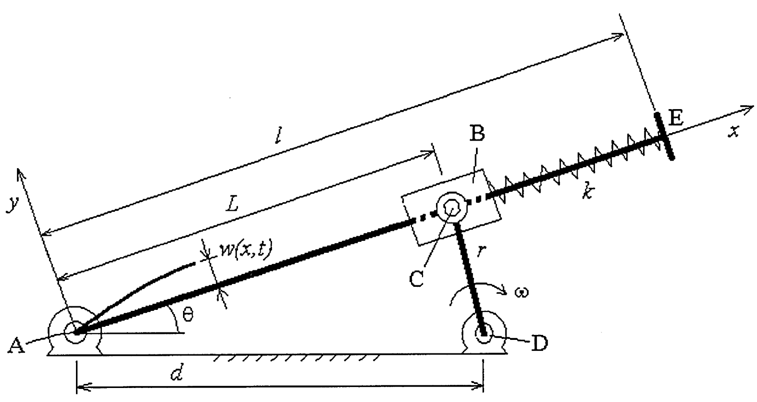

Fig. 1. Crank-rocker mechanism with flexible rocker.

which can be used to express the variation of the kinetic energy function as follows:

$$\delta T = \int_D \rho (u_{,t}\delta u_{,t} + w_{,t}\delta w_{,t} + x\dot{\theta}\delta w_{,t})\,\mathrm{d}D, \tag{7}$$

$$_{,t} \equiv \frac{\partial}{\partial t}, \quad _{,x} \equiv \frac{\partial}{\partial x}, \quad \cdot \equiv \frac{\mathrm{d}}{\mathrm{d}t},$$

where ρ is the mass density of the rocker. Since θ, $\dot{\theta}$ and $\ddot{\theta}$ are specified from the kinematics of the motion, they have no variation. By applying Eqs. (5) and (7) in Hamilton's principle,

$$\int_{t_1}^{t_2} (\delta T - \delta U)\,\mathrm{d}t = 0, \tag{8}$$

the governing equations of motion are obtained as

$$EAu_{,xx} + EAw_{,x}w_{,xx} = -\rho u_{,tt}, \tag{9a}$$

$$EIw_{,xxxx} + \rho w_{,tt} + \rho x\ddot{\theta} - EA(\tfrac{3}{2}w_{,xx}w_{,x}^2 + u_{,xx}w_{,x} + u_{,x}w_{,xx}) = 0, \tag{9b}$$

where A and I are the cross-sectional area and the area moment of inertia of the beam, respectively. The associated boundary conditions are (for the beam AC after neglecting the effect of the portion CE)

$$u(0,t) = 0,$$

$$\sigma_x(L,t)|_{\text{mid-plane}} = E\left(u_{,x}(L,t) + \frac{1}{2}(w_{,x}(L,t))^2\right) = k\int_0^L \sqrt{1 + w_{,x}^2}\,\mathrm{d}x \cong k\left(L + \int_0^L \frac{1}{2}w_{,x}^2\,\mathrm{d}x\right),$$

$$w(0,t) = w_{,xx}(0,t) = 0,$$

$$w(L,t) = w_{,xx}(L,t) = 0. \tag{10}$$

The following series expansion of w satisfies Eq. (10)

$$w(x,t) = \sum_{n=1}^{\infty} \psi_n(t)\sin\left(\frac{n\pi x}{L(t)}\right). \tag{11}$$

It is seen that the argument of the sine functions depends on x and t simultaneously. This important feature allows us to deal with the variation of the rocker length in a straightforward fashion. By taking only the first mode into account, $n = 1$, the following identities are readily verified:

$$w_{,x} = \frac{\pi}{L}\psi\cos\left(\frac{\pi x}{L}\right), \quad w_{,xx} = -\left(\frac{\pi}{L}\right)^2\psi\sin\left(\frac{\pi x}{L}\right),$$

$$w_{,xxxx} = \left(\frac{\pi}{L}\right)^4\psi\sin\left(\frac{\pi x}{L}\right),$$

$$w_{,tt} = \ddot{\psi}\sin\left(\frac{\pi x}{L}\right) - 2\dot{\psi}\frac{\pi\dot{L}}{L^2}x\cos\left(\frac{\pi x}{L}\right) - \left(\frac{\pi\dot{L}}{L^2}\right)^2\psi x^2\sin\left(\frac{\pi x}{L}\right) - \pi\psi\frac{L\ddot{L} - 2\dot{L}^2}{L^3}x\cos\left(\frac{\pi x}{L}\right),$$

$$\psi = \psi_1(t). \tag{12}$$

Inertia forces along the x-axis are small in comparison with those of the y-direction. Therefore, one can assume $u_{,tt} \approx 0$ in Eq. (9a). With regard to this assumption, Eq. (9a) is solved for u, giving

$$u = -\frac{1}{8}\left(\frac{\pi}{L}\right)\psi^2\sin\left(\frac{2\pi x}{L}\right) - \frac{1}{4}\left(\frac{\pi}{L}\right)^2\psi^2 x + kxL + \frac{1}{4}\frac{k\pi^2}{L}x\psi^2. \tag{13}$$

Utilizing Eqs. (11) and (13) in Eq. (9b) and applying the Bubnov–Galerkin global averaging method, yields

$$\ddot{\psi} + f_1(t)\dot{\psi} + f_2(t)\psi + f_3(t)\psi^3 = f_0(t), \tag{14}$$

where

$$f_0(t) = -\frac{2L}{\pi}\ddot{\theta}, \tag{15a}$$

$$f_1(t) = \frac{\dot{L}}{L}, \tag{15b}$$

$$f_2(t) = \frac{k\pi^2 EA}{\rho L} + \frac{\pi^4 EI}{\rho L^4} - \left(\frac{1}{2} + \frac{1}{3}\pi^2\right)\left(\frac{\dot{L}}{L}\right)^2 + \frac{1}{2}\frac{\ddot{L}}{L}, \tag{15c}$$

$$f_3(t) = \frac{\pi^4 kEA}{4\rho L^3}, \tag{15d}$$

$$f_n(t) = f_n\left(t + \frac{2\pi}{\omega}\right), \quad n = 0, 1, 2, 3.$$

We consider a time-dependent transformation for the oscillator (14) through the relation

$$\psi = \xi\sqrt{\frac{L_0}{L}}, \quad L_0 = d - r. \tag{16}$$

By utilizing Eq. (16) in Eq. (14) and performing some algebraic manipulations, we obtain

$$\ddot{\xi} + P(t)\,\xi + Q(t)\,\xi^3 = F(t), \tag{17}$$

where

$$P(t) = -\frac{1}{4}f_1^2(t) - \frac{1}{2}\dot{f}_1(t) + f_2(t),$$

$$Q(t) = f_3(t)\frac{L_0}{L}, \qquad F(t) = f_0(t)\sqrt{\frac{L}{L_0}}.$$

Eq. (17) belongs to the class of Eq. (2). The Hamiltonian function associated with Eq. (17) is given by

$$H = \tfrac{1}{2}\Xi^2 + \tfrac{1}{2}P(t)\,\xi^2 + \tfrac{1}{4}Q(t)\,\xi^4 - F(t)\,\xi, \tag{18}$$

with $\Xi = \dot{\xi}$. This non-autonomous system admits no classical integrals of motion. In the next sections we study Eq. (17) for periodic solutions and examine the geometrical features of the system near resonances.

4. Existence of periodic solutions

The solution of Eq. (17) can be represented by the following integral equation:

$$\xi = \int_0^T \Gamma(t, s)\{F(s) - Q(s)\,\xi^3(s)\}\,\mathrm{d}s, \quad T = 2\pi/\omega, \tag{19}$$

where $\Gamma(t, s)$ is the Green's function for the operator Φ defined as

$$\Phi\xi = \ddot{\xi} + P(t)\,\xi. \tag{20}$$

Green's function can be constructed as follows. Let

$$q(t, s) = -\frac{1}{2T}(t - s)^2 - \frac{1}{2}(t - s); \quad 0 \leqslant t \leqslant s \leqslant T,$$

$$q(t, s) = -\frac{1}{2T}(t - s)^2 + \frac{1}{2}(t - s); \quad 0 \leqslant s \leqslant t \leqslant T. \tag{21}$$

$q(t, s)$ is a piecewise continuous periodic polynomial of second degree, i.e.

$$q^{(i)}(0, s) = q^{(i)}(T, s), \quad i = 0,1,$$

$$\dot{q}(t, t^-) - \dot{q}(t, t^+) = 1. \tag{22}$$

With this $q(t, s)$, we define Green's function as

$$\Gamma(t, s) = q(t, s) + \int_0^T q(t, \tau)R(\tau, s)\,\mathrm{d}\tau. \tag{23}$$

where $R(\tau, s)$ is the resolvant of $\Phi q(t, s)$, i.e.

$$R(t, s) = -\Phi q(t, s) - \int_0^T \Phi q(t, \tau)R(\tau, s)\,\mathrm{d}\tau. \tag{24}$$

In order to estimate Γ, we have

$$\int_0^T |R(t, s)|\,\mathrm{d}s \leqslant \int_0^T |\Phi q(t, s)|\,\mathrm{d}s + \left(\int_0^T |\Phi q(t, \tau)|\,\mathrm{d}\tau\right)\left(\int_0^T |R(\tau, s)|\,\mathrm{d}s\right). \tag{25}$$

Correspondingly

$$\int_0^T |R(t, s)|\,\mathrm{d}s \leqslant \frac{\int_0^T |\Phi q(t, s)|\mathrm{d}s}{1 - \int_0^T |\Phi q(t, s)|\,\mathrm{d}s}. \tag{26}$$

From Eqs. (23) and (26), it can be shown that

$$|\Gamma| \leqslant \max_{t,s\in[0,T]\times[0,T]} |q(t, s)|\frac{1}{1 - \int_0^T |\Phi q(t, s)|\,\mathrm{d}s}. \tag{27}$$

In our case $q \geqslant 0$. Now we calculate $\int_0^T |\Phi q(t, s)|\,\mathrm{d}s$. According to Eqs. (20) and (21), we obtain

$$\Phi q(t, s) = -\frac{1}{T} + P(t)\left[-\frac{1}{2T}(t - s)^2 - \frac{1}{2}(t - s)\right] \quad \text{if } 0 \leqslant t \leqslant s \leqslant T,$$

$$= -\frac{1}{T} + P(t)\left[-\frac{1}{2T}(t - s)^2 + \frac{1}{2}(t - s)\right] \quad \text{if } 0 \leqslant s \leqslant t \leqslant T. \tag{28}$$

Define $M_P = \max_{t\in[0,T]} P(t)$ and $m_P = \min_{t\in[0,T]} P(t)$. Hence, from Eq. (28) we conclude

$$\int_0^T |\Phi q(t,s)|\,\mathrm{d}s = 1 - P(t)\frac{T^2}{12} \leqslant 1 - m_P\frac{T^2}{12} \quad \text{if } M_P T^2 \leqslant 2,$$

$$\int_0^T |\Phi q(t,s)|\,\mathrm{d}s = -1 + P(t)\frac{T^2}{12} \geqslant -1 + M_P\frac{T^2}{12} \quad \text{if } M_P T^2 \geqslant 2. \tag{29}$$

Substitution of Eq. (29) in Eq. (27) leads to

$$|\Gamma| \leqslant \frac{T}{2}\frac{1}{1-1+m_P(T^2/12)} = \frac{6}{m_P T} \quad \text{if } M_P T^2 \leqslant 2,$$

$$|\Gamma| \leqslant \frac{T}{2}\frac{1}{1-(-1+M_P(T^2/12))} = \frac{6T}{24 - M_P T^2} \quad \text{if } M_P T^2 \geqslant 2 \quad \text{and} \quad 24 > M_P T^2. \tag{30}$$

Thus, we proved that Γ exists for $0 < M_P T^2 < 24$. To this end, we use Banach's fixed point theorem to prove the existence of periodic solutions. Let

$$B = \{\xi(t) | \xi(t) \in C[0,T], \|\xi(t)\| = \sup_{t\in[0,T]} |\xi(t)| = M_\xi\}, \tag{31}$$

be a Banach space. Define the map $F: B \to B$ as

$$F\xi = \int_0^T \Gamma(t,s)\{F(s) - Q(s)\xi^3(s)\}\,\mathrm{d}s. \tag{32}$$

By operating F on the sample functions ξ_1 and ξ_2 in B, and subtracting the results, we obtain

$$|F\xi_1 - F\xi_2| \leqslant \int_0^T |\Gamma(t,s)| \times |Q(s)| \times |\xi_1^3(s) - \xi_2^3(s)|\,\mathrm{d}s. \tag{33}$$

Accordingly,

$$|F\xi_1 - F\xi_2| \leqslant \|\xi_1 - \xi_2\| \int_0^T |\Gamma(t,s)| \times |Q(s)| \times |\xi_1^2(s) + \xi_2^2(s) + \xi_1(s)\xi_2(s)|\,\mathrm{d}s. \tag{34}$$

From Eqs. (31) and (34) we have

$$\|F\xi_1 - F\xi_2\| \leqslant \alpha \|\xi_1 - \xi_2\|, \tag{35}$$

$$\alpha = \frac{18|Q(t)|M_\xi^2}{m_P T} \quad \text{if } M_P T^2 < 2,$$
$$= \frac{18T|Q(t)|M_\xi^2}{24 - M_P T^2} \quad \text{if } M_P T^2 \geqslant 2 \text{ and } 24 > M_P T^2. \tag{36}$$

The mapping defined by F is contraction if $\alpha < 1$. By this condition, F has a fixed point denoting T-periodic solution of Eq. (17).

5. Resonances and chaos

T-periodic solutions can be made explicit by studying the Hamiltonian system (18) near resonances. The parameter δ may be assumed to be small in practical problems, and therefore, one could expand $P(t)$, $Q(t)$ and $F(t)$ in Taylor series with respect to δ. Expanding these functions up to third-order terms in δ, yields

$$P(t) = \Omega^2 + P_1(t),$$

$$P_1(t) = A_1\cos\omega t + A_2\cos 2\omega t + A_3\cos 3\omega t + O(\delta^4), \tag{37a}$$

$$Q(t) = B_0 + B_1\cos\omega t + B_2\cos 2\omega t + B_3\cos 3\omega t + O(\delta^4), \tag{37b}$$

$$F(t) = C_1\sin\omega t + C_2\sin 2\omega t + C_3\sin 3\omega t + O(\delta^4), \tag{37c}$$

where A_i's, B_i's and C_i's are constants obtainable from the formulae given in Appendix A. Ω is the linear frequency of the oscillation given by

$$\Omega^2 = \frac{\pi^4 EI}{\rho d^4} + \frac{k\pi^2 EA}{\rho d} + \delta^2\left(\frac{4\pi^4 EI}{\rho d^4} + \frac{k\pi^2 EA}{4\rho d} - \frac{1}{8}\omega^2 - \frac{\pi^2\omega^2}{6}\right).$$

We perform a transformation to the action and angle variables (J, φ) defined by

$$\xi = \sqrt{\frac{2J}{\Omega}}\sin\varphi, \qquad \Xi = \sqrt{2\Omega J}\cos\varphi. \tag{38}$$

In the new variables, our Hamiltonian (18) becomes

$$H = \Omega J + P_1(t)\frac{J}{\Omega}\sin^2\varphi + Q(t)\frac{J^2}{\Omega^2}\sin^4\varphi - F(t)\sqrt{\frac{2J}{\Omega}}\sin\varphi. \tag{39}$$

We consider those motions that take place near the 1:1, 1:2 and 2:1 resonances. They are associated with $\omega \approx \Omega$, $\omega \approx \Omega/2$ and $\omega \approx 2\Omega$, respectively. By neglecting the higher order terms, transformation to resonant variables and first-order averaging over fast angles (see [8]), the following averaged Hamiltonians are obtained for the motions near the mentioned resonances:

$$\bar{H}_{1:1} = \left[(\Omega - \omega) - \frac{A_2}{4\Omega}\cos 2\bar{\varphi}\right]\bar{J} + \left(\frac{3B_0}{8\Omega^2} - \frac{B_2}{4\Omega^2}\cos 2\bar{\varphi}\right)\bar{J}^2 - \frac{C_1}{\sqrt{2\Omega}}\sqrt{\bar{J}}\cos\bar{\varphi}, \tag{40a}$$

$$\bar{H}_{1:2} = (\Omega - 2\omega)\bar{J} + \frac{3B_0}{8\Omega^2}\bar{J}^2 - \frac{C_2}{\sqrt{2\Omega}}\sqrt{\bar{J}}\cos\bar{\varphi}, \tag{40b}$$

$$\bar{H}_{2:1} = \left[\left(\Omega - \frac{1}{2}\omega\right) - \frac{A_1}{4\Omega}\cos 2\bar{\varphi}\right]\bar{J} + \left(\frac{3B_0}{8\Omega^2} - \frac{B_1}{4\Omega^2}\cos 2\bar{\varphi} + \frac{B_2}{16\Omega^2}\cos 4\bar{\varphi}\right)\bar{J}^2, \tag{40c}$$

with $\bar{J}$ and $\bar{\varphi}$ as the new action and angle variables after averaging near resonances. Dynamics of the original system (18) is characterized by the phase portraits of Eqs. (40a)–(40c). The equations of motion corresponding to Eqs. (40a)–(40c) are

$$\dot{\bar{\varphi}} = \frac{\partial \bar{H}_{m:n}}{\partial \bar{J}}, \quad \dot{\bar{J}} = -\frac{\partial \bar{H}_{m:n}}{\partial \bar{\varphi}}, \quad m{:}n = 1{:}1,\ 1{:}2,\ 2{:}1. \tag{41}$$

The phase space flows of the reduced system depend on the position and the type of singular points of Eq. (41). Fig. 2 shows the behavior of level curves of the reduced Hamiltonian (40a). There exist at least one and at most three fixed points. Fixed points of the averaged system indicate periodic solutions of the original system. Stable periodic solutions are associated with the center type fixed points and unstable periodic motions correspond to the saddles. Hamiltonian (40b) is the well-known Andoyer's Hamiltonian and has a topology similar to Eq. (40a). Level curves of Hamiltonian (40c) are totally different from those of Eqs. (40a) and (40b). Fig. 3 illustrates possible phase portraits corresponding to Eq. (40c). In this case, the origin of the averaged system is the trivial fixed point. Rather than this point, the system can have at most four stationary points. As evidenced from Figs. 2 and 3, invariant manifolds of the system can only be homoclinic orbits when the motion takes place near the 1:1 and 1:2 resonances. In the neighborhood of the 2:1 resonance, however, both of homoclinic and heteroclinic structures can exist. Chaotic motions occur in the vicinity of saddle points of the averaged system if the homoclinic or heteroclinic orbits destroy. To make this explicit, we have numerically constructed the Poincaré map of the original system (17) (see Fig. 4) when $\Omega \approx \omega$. The sampling time is set to be $2\pi/\omega$. Chaotic region is observed around the unstable periodic point 1c as expected. Most tori around the stable periodic points 1a and 1b are preserved.

6. Conclusion

In this research we deal with a non-conservative Hamiltonian system. That is Duffing's oscillator with time varying coefficients. It is shown that a flexible crank-rocker mechanism is simulated by such equation. Similar equations can arise in the multi-body dynamics of flexible robots having telescopic arms. Regular response of dynamical systems has many applications in the control of engineering systems. Periodic behavior is the most important regular motion, which is studied in this paper using Banach's fixed-point theorem. This

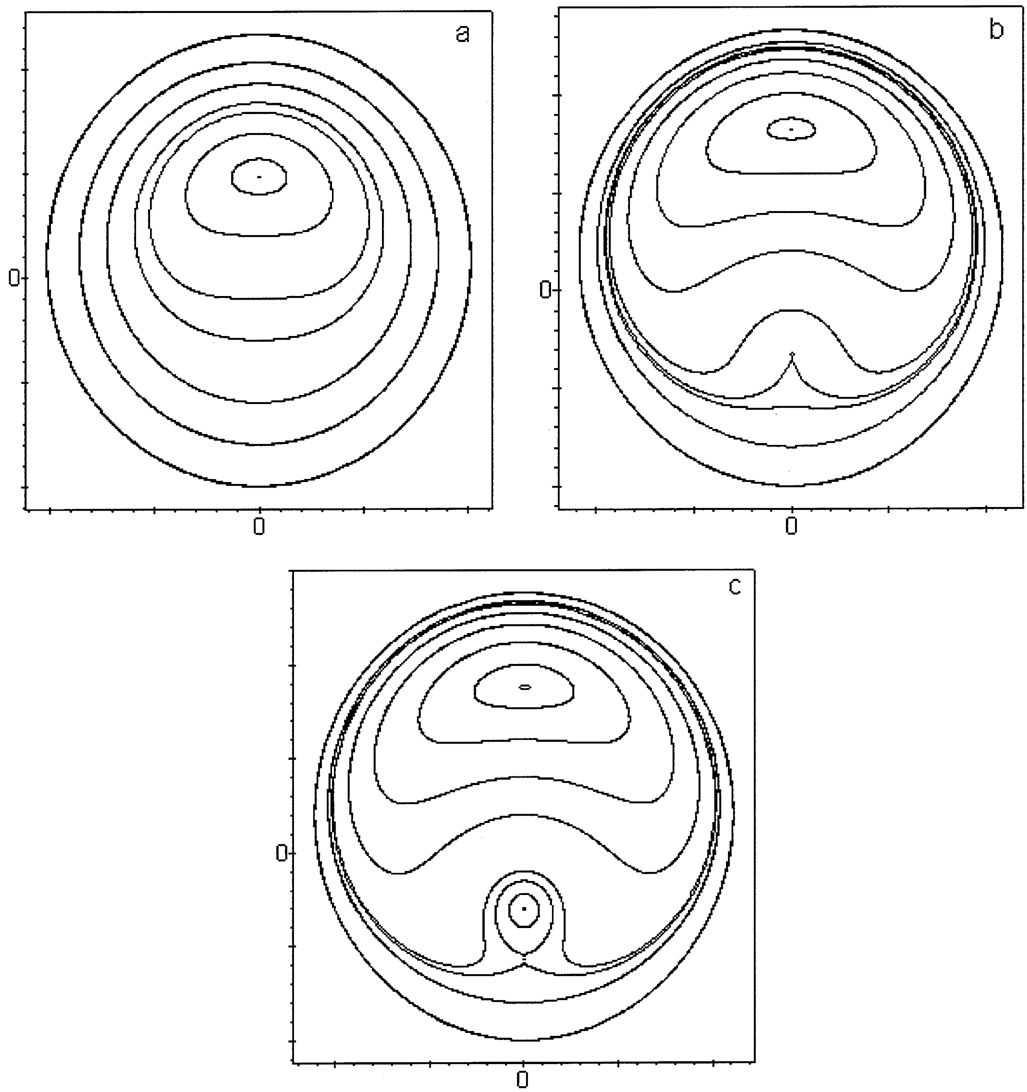

Fig. 2. Possible level curves of the averaged Hamiltonians (Eqs. (40a) and (40b)).

technique is a qualitative approach and provides no idea about the topology of the periodic solutions and their surrounding orbits. To overcome this restriction, we generate the averaged Hamiltonians near certain resonances to reveal the geometrical features of the motion. Findings of Section 5 confirm the predictions of Section 4 only in the case of the 1:1 and 2:1 resonances. This is because of $0 < M_P T^2 < 24$ which is a necessary condition for the existence of Green's function. Outside this range, one could apply other existing mathematical tools for prediction of periodic solutions.

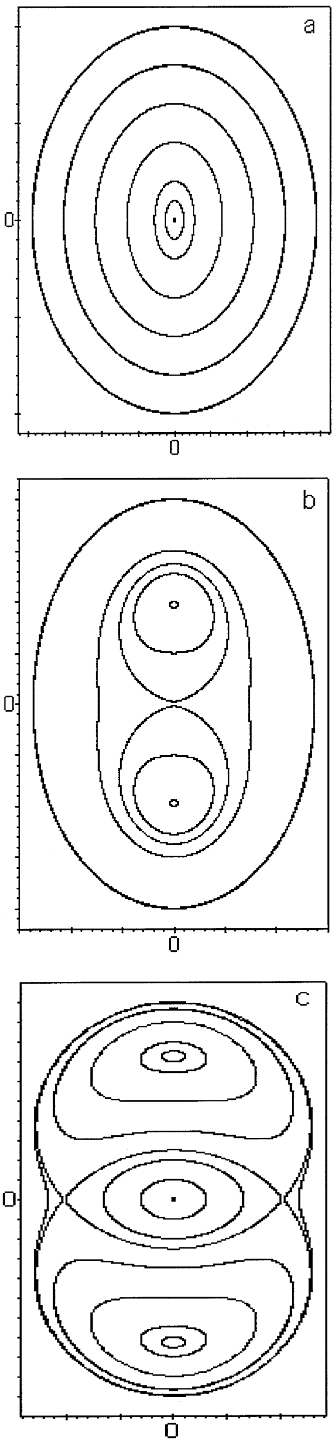

Fig. 3. Possible level curves of the averaged Hamiltonian (Eq. (40c)).

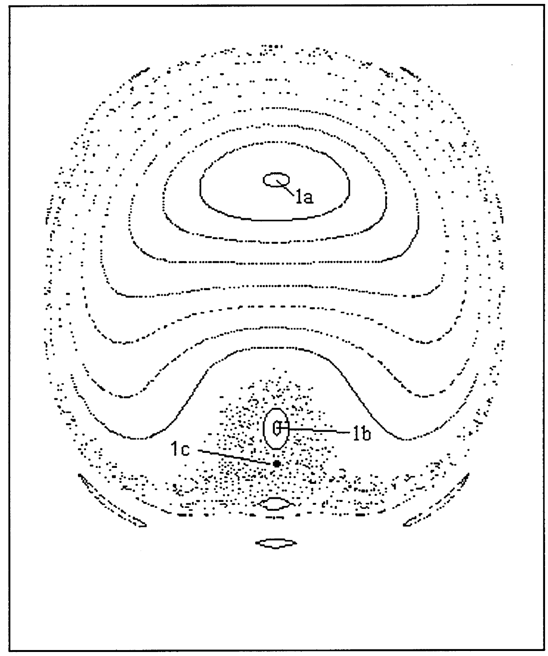

Fig. 4. Numerically constructed Poincaré map of the original system (17) near the 1:1 resonance. The sampling time is set to $2\pi/\omega$.

Appendix A

The constant coefficients in Eq. (37) are determined as follows:

$$A_1 = \delta\left(\frac{4\pi^4 EI}{\rho d^4} + \frac{k\pi^2 EA}{\rho d}\right) + \delta^3\left(-\frac{1}{4}\omega^2 + \frac{12\pi^4 EI}{\rho d^4} + \frac{3k\pi^2 EA}{8\rho d} - \frac{\pi^2\omega^2}{3}\right),$$

$$A_2 = \delta^2\left(\frac{1}{8}\,\omega^2 + \frac{6\pi^4 EI}{\rho d^4} + \frac{3k\pi^2 EA}{4\rho d} + \frac{\pi^2\omega^2}{6}\right),$$

$$A_3 = \delta^3\left(\frac{1}{4}\omega^2 + \frac{8\pi^4 EI}{\rho d^4} + \frac{5k\pi^2 EA}{8\rho d} + \frac{\pi^2\omega^2}{3}\right),$$

$$B_0 = \frac{\pi^4 kEA}{4\rho d^3}(1 - \delta + 4\delta^2 - 4\delta^3),$$

$$B_1 = \frac{\pi^4 kEA}{\rho d^3}(\delta - \delta^2 + 3\delta^3),$$

$$B_2 = \frac{3\pi^4 kEA}{2\rho d^3}(\delta^2 - \delta^3),$$

$$B_3 = \frac{2\pi^4 kEA}{\rho d^3}\delta^3,$$

$$C_1 = \frac{d\omega^2}{16\pi}(32\delta + 16\delta^2 - 15\delta^3),$$

$$C_2 = \frac{5d\omega^2}{4\pi}(2\delta^2 + \delta^3),$$

$$C_3 = \frac{15d\omega^2}{16\pi}\delta^3.$$

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