REGULAR AND CHAOTIC SOLUTIONS OF THE SITNIKOV PROBLEM NEAR THE 3/2 COMMENSURABILITY

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Abstract. Regular solutions at the 3/2 commensurability are investigated for Sitnikov's problem. Utilizing a rotating coordinate system and the averaging method, approximate analytical equations are obtained for the Poincaré sections by means of Jacobian elliptic functions and 3π-periodic solutions are generated explicitly. It is revealed that the system exhibits heteroclinic orbits to saddle points. It is also shown that chaotic region emerging from the destroyed invariant tori, can easily be seen for certain eccentricities. The procedure of the current study provides reliable answers for the long-time behavior of the system near resonances.

Key words: Sitnikov's problem, periodic motion, resonances, chaotic behavior

1. Introduction

Sitnikov's problem is one of the nonintegrable problems in celestial mechanics that represents a rectilinear motion in the restricted problem of three bodies. Two equal primary masses (m_1 and m_2) move on two coplanar elliptic orbits of eccentricity e, $0 \leqslant e < 1$, around their barycentre while a third infinitesimal body moves on a line perpendicular to the motion plane of the first two masses and going through the center of mass. It is customary to normalize the time so that the period of primaries is 2π, the unit of mass so that $m_1 = m_2 = 1/2$ and the unit of length so that the value of the gravitational constant is one.

This nonconservative Hamiltonian system has been studied for regular and stochastic motions through various mathematical methods. Sitnikov (1960) succeeded in proving that oscillatory motions having infinitely many zeros exist for $e > 0$. Alekseev (1968a, 1968b, 1969) then showed that one could apply methods of symbolic dynamics to an oscillator in periodically varying potential field, particularly Sitnikov's system. In an extensive work, Moser (1973) developed a chaos theory in the vicinity of escaping solutions based on the work of Conley (1969). Explicit representations of Moser's theory were developed by Jie Liu and Yi-Sui Sun (1990) through the use of special discrete mappings. Their approach is an approximation of the theory developed by Moser. Certain regular solutions of the problem were generated by Hagel (1992) using a perturbation technique which provides acceptable results in short time intervals. Hagel's approach provides little insight about the periodic motions for nonzero eccentricities. Alfaro and Chiralt (1993) investigated invariant rotational curves of the problem through application of the Birkhoff normal

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form of an elliptic area-preserving mapping. They introduced the eccentric anomaly of the primaries as the independent variable which allowed them to work with high eccentricities. Using elliptic functions, Belbruno *et al.* (1994) gave analytical expressions for the solutions of the circular Sitnikov problem and for the period function of its family of periodic orbits. They also analyzed the linear stability of the family of periodic orbits of Sitnikov's problem and the associated bifurcating families of periodic orbits of the three-dimensional circular restricted three-body problem through numerical scheme.

Since the primaries rotate around their common center of gravity with period 2π, the system will be at 3/2 resonance if it undergoes 3π-periodic motion. The present paper deals with the system evolution near the 3/2 commensurability and provides new analytical insight about the problem.

At first, a rotating coordinate system is utilized to handle the short-period structure of the solutions through the Van der Pol transformation. Then, using the averaging method the original nonautonomous equations of motion are transformed to an autonomous system which has the energy integral. Taking the energy integral into account, the reduced system is integrated in terms of the Jacobian elliptic functions to obtain the long-period portions of the solutions. Analytical results are explicit representations of the Poincaré section that have been compared with numerical solutions showing good agreement between the two approaches. Among the various existing tools for chaos exploration, the method of sections has been used to predict stochastic behavior near hyperbolic points. In the proposed formulation, invariant manifolds are of heteroclinic type that connect hyperbolic fixed points.

2. Equation of Motion

Assuming $r(t)$ as the distance between the primaries, $z(t)$ the distance of the third particle from the center of mass and t representing time, the differential equation of motion will be (Moser, 1973)

$$\ddot{z} + \frac{z}{[r(t)^2 + z^2]^{3/2}} = 0. \tag{1}$$

By expanding $r(t)$ in powers of e (Stumpff, 1965) and applying binomial expansion, Equation (1) is described in a more useful form. For sufficiently small values of e, $0 \leqslant e < 0.2$ and $|z(t)| < |r(t)|$, the following state-space equations are approximately valid (Hagel, 1992):

$$\dot{z} = Z, \tag{2a}$$

$$\dot{Z} = -\omega^2 z - \varepsilon f(z, t), \tag{2b}$$

where

$$f(z,t) = g(t)z + h(t)z^3, \tag{3a}$$

$$g(t) = 24e\cos t + 36e^2\cos 2t + 27e^3\cos t + 53e^3\cos 3t, \tag{3b}$$

$$h(t) = -48 - 240e\cos t, \tag{3c}$$

$$\omega^2 = 8 + 12e^2. \tag{3d}$$

In order to fully benefit from the averaging method, all perturbing terms are collected in the function *f*. Multiplier ε is introduced as a perturbation parameter which eventually will be set to unity.

3. Averaging the Equations of Motion

Any *T*-periodic solution of the Equations (2) may be detected by successive applications of the Van der Pol transformation and the averaging technique. In this way, the original equations of motion, expressed in (z, Z) space, are transformed into a rotating coordinate system with angular velocity $\Omega = 2m\pi/T$ (m is an integer number). The phase portrait of the transformed system corresponds to the Poincaré map of the initial system with a sampling time equal to *T* (Wiggins, 1990) where a fixed point shows a harmonic oscillation with frequency Ω.

Consider a change of variables as

$$\begin{pmatrix} z \\ Z \end{pmatrix} = \begin{bmatrix} \cos\Omega t & -\sin\Omega t \\ -\Omega\sin\Omega t & -\Omega\cos\Omega t \end{bmatrix} \cdot \mathbf{X}, \tag{4}$$

with

$$\mathbf{X} = (x_1, x_2)^T.$$

According to the inverse transformation

$$\mathbf{X} = \begin{bmatrix} \cos\Omega t & -\frac{1}{\Omega}\sin\Omega t \\ -\sin\Omega t & -\frac{1}{\Omega}\cos\Omega t \end{bmatrix} \begin{pmatrix} z \\ Z \end{pmatrix}, \tag{5}$$

and substituting from (4) into (2), equations of motion in **X**-space are obtained

$$\dot{x}_i = \varepsilon F_i(\mathbf{X}, t, \varepsilon), \;\; (i = 1, 2), \tag{6}$$

where

$$\begin{aligned} F_1(\mathbf{X}, t, \varepsilon) &= -\frac{1}{\Omega} F(\mathbf{X}, t, \varepsilon)\sin\Omega t, \\ F_2(\mathbf{X}, t, \varepsilon) &= -\frac{1}{\Omega} F(\mathbf{X}, t, \varepsilon)\cos\Omega t, \\ F(\mathbf{X}, t, \varepsilon) &= \left[\frac{1}{\varepsilon}(\Omega^2 - \omega^2) - (24e\cos t + 36e^2\cos 2t + \right. \\ &\quad \left. + 27e^3\cos t + 53e^3\cos 3t)\right](x_1\cos\Omega t - x_2\sin\Omega t) + \\ &\quad + (48 + 240e\cos t)(x_1\cos\Omega t - x_2\sin\Omega t)^3. \end{aligned} \tag{7}$$

In addition to the resonance criterion at the 3/2 commensurability which implies $\Omega = 2m/3$, the averaging theorem requires $\Omega^2 - \omega^2 = \mathrm{O}(\varepsilon)$. Hence, an appropriate selection is $m = 4 (\Omega = 8/3)$ and therefore $(\Omega^2 - \omega^2)/\varepsilon = \mathrm{O}(1)$. Through the use of the near identity transformation

$$\begin{aligned} x_i &= u_i + \varepsilon W_i(\mathbf{U}, t), \ (i = 1, 2), \\ \mathbf{U} &= (u_1, u_2)^T, \end{aligned} \tag{8}$$

the averaging method (Guckenheimer and Holmes, 1983) gives the new system of differential equations

$$\dot{u}_i = \varepsilon \bar{F}_i(\mathbf{U}) + \varepsilon^2 R_i(\mathbf{U}, t), \tag{9}$$

where the following relations hold

$$\begin{aligned} &\bar{F}_i(\mathbf{U}) = \frac{1}{T}\int_0^T F_i(\mathbf{U}, t, 1)\mathrm{d}t, \qquad T \equiv \frac{2\pi m}{\Omega} = 3\pi, \\ &R_i = \sum_{j=1}^{2}\left(\frac{\partial F_i(\mathbf{U}, t, 1)}{\partial u_j} W_j - \frac{\partial W_i}{\partial u_j} F_j(\mathbf{U}, t, 1) + \frac{\partial W_i}{\partial u_j}\frac{\partial W_j}{\partial t}\right), \\ &W_i = \int_0^t [F_i(\mathbf{U}, \xi, 1) - \bar{F}_i(\mathbf{U})]\mathrm{d}\xi, \quad (i = 1, 2). \end{aligned} \tag{10}$$

The averaged terms in (9) are calculated by the *Maple* symbolic mathematical processor (Heck, 1993) and are given below

$$\begin{aligned} \bar{F}_1(\mathbf{U}) = \frac{1}{3\pi}\Big[&(C_1 e^3 + C_3 e)u_1 - \left(\frac{\pi}{2} + C_4 e^2\right)u_2 - C_2 e u_1^3 + \\ &+ C_5(u_1^2 + u_2^2)u_2 - C_6 e u_1 u_2^2\Big], \end{aligned} \tag{11a}$$

$$\begin{aligned} \bar{F}_2(\mathbf{U}) = \frac{1}{3\pi}\Big[&\left(\frac{\pi}{2} + C_4 e^2\right)u_1 - (C_3 e + C_1 e^3)u_2 + \frac{1}{3}C_6 e u_2^3 - \\ &- C_5(u_1^2 + u_2^2)u_1 + 3C_2 e u_2 u_1^2\Big], \end{aligned} \tag{11b}$$

where constant coefficients C_1 through C_6 are

$$C_1 = \frac{320688}{43225}, \quad C_2 = \frac{545184}{50141}, \quad C_3 = \frac{432}{247},$$
$$C_4 = \frac{27\pi}{4}, \quad C_5 = \frac{81\pi}{4}, \quad C_6 = \frac{995328}{50141}.$$

To investigate the regular behavior, it is adequate to examine the autonomous part of the $\mathrm{O}(\varepsilon)$th order in (9)

$$u_i' = 3\pi \bar{F}_i(\mathbf{U}), \quad (i = 1, 2). \tag{12}$$

The prime denotes differentiation with respect to the slow time, $\tau = (\varepsilon t)/(3\pi)$.

Equation (12) is derivable from a Hamiltonian

$$H = -\frac{1}{2}\left(\frac{\pi}{2} + C_4 e^2\right)(u_1^2 + u_2^2) + (C_1 e^3 + C_3 e)u_1 u_2 + \\ +\frac{1}{4}C_5(u_1^2 + u_2^2)^2 - C_2 e u_1^3 u_2 - \frac{1}{3}C_6 e u_1 u_2^3, \tag{13}$$

and thus

$$u_1' = \frac{\partial H}{\partial u_2}, \quad u_2' = -\frac{\partial H}{\partial u_1}. \tag{14}$$

The aim of next discussion is explicit generation of solutions for the system (14). This is accomplished with the aid of elliptic functions.

4. Analytical Integration by Quadratures

Performing the standard canonical transformation to action angle variables J and φ as

$$u_1 = \sqrt{2J}\sin\varphi, \quad u_2 = \sqrt{2J}\cos\varphi, \tag{15}$$

a new Hamiltonian is obtained

$$H = A(\varphi)J^2 + B(\varphi)J, \tag{16}$$

with

$$A(\varphi) = C_5 - C_7 e \sin 2\varphi - C_8 e \sin 4\varphi, \tag{17a}$$

$$B(\varphi) = (C_1 e^3 + C_3 e)\sin 2\varphi - \left(\frac{\pi}{2} + C_4 e^2\right), \tag{17b}$$

$$C_7 = \frac{1}{3}C_6 + C_2, \quad C_8 = \frac{1}{6}C_6 - \frac{1}{2}C_2. \tag{17c}$$

Thus, Hamilton's equations will be

$$J' = -\frac{\partial H}{\partial \varphi}, \tag{18a}$$

$$\varphi' = \frac{\partial H}{\partial J} = 2A(\varphi)J + B(\varphi). \tag{18b}$$

Since the Hamiltonian is an integral of the motion, Equation (16) can be solved for J as a function of φ, giving

$$J = \frac{1}{2A(\varphi)}\left[-B(\varphi) \pm \sqrt{B(\varphi)^2 + 4HA(\varphi)}\right], \quad J \geqslant 0. \tag{19}$$

From (18) and (19), one achieves

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\tau} = \pm\sqrt{B(\varphi)^2 + 4HA(\varphi)}, \tag{20}$$

consequently

$$\int \frac{\mathrm{d}\varphi}{\sqrt{A_1 + A_2 \sin 2\varphi + A_3 \sin 4\varphi + A_4 \cos 4\varphi}} = \pm\tau, \quad \tau \geqslant 0, \tag{21}$$

where coefficients A_1, A_2, A_3 and A_4 are given by

$$\begin{aligned} A_1 &= \left(\frac{\pi}{2} + C_4 e^2\right)^2 + \frac{1}{2}(C_1 e^3 + C_3 e)^2 + 4HC_5, \\ A_2 &= -4HC_7 e - (\pi + 2C_4 e^2)(C_1 e^3 + C_3 e), \\ A_3 &= -4HC_8 e, \quad A_4 = -\frac{1}{2}(C_1 e^3 + C_3 e)^2. \end{aligned}$$

Using the following identities

$$p = \tan\varphi, \tag{22a}$$

$$\sin 2\varphi = \frac{2p}{1+p^2}, \quad \cos 2\varphi = \frac{1-p^2}{1+p^2}, \tag{22b}$$

Equation (21) leads to

$$\int \frac{\mathrm{d}p}{\sqrt{B_4(p-z_1)(p-z_2)(p-z_3)(p-z_4)}} = \pm\tau, \tag{23}$$

where z_i's ($i = 1, \ldots, 4$) are the roots of the following equation

$$B_4 p^4 + B_3 p^3 + B_2 p^2 + B_1 p + B_0 = 0, \tag{24}$$

with

$$\begin{aligned} &B_0 = A_1 + A_4, \quad B_1 = 2A_2 + 4A_3, \\ &B_2 = 2A_1 - 6A_4, \quad B_3 = 2A_2 - 4A_3, \quad B_4 = A_1 + A_4. \end{aligned}$$

Carrying out the transformation (Davis, 1962)

$$\Lambda^2 = \alpha\frac{p - z_1}{p - z_2}, \quad \alpha = \frac{z_2 - z_4}{z_1 - z_4}, \tag{25}$$

the integral Equation (23) becomes

$$\int \frac{\mathrm{d}\Lambda}{\sqrt{(1-\Lambda^2)(1-k^2\Lambda^2)}} = \lambda\tau, \quad -\infty < \tau < +\infty, \tag{26}$$

$$\lambda = \frac{\sqrt{B_4(z_2 - z_4)(z_1 - z_3)}}{2}, \quad k^2 = \left(\frac{z_2 - z_3}{z_1 - z_3}\right)\left(\frac{z_1 - z_4}{z_2 - z_4}\right).$$

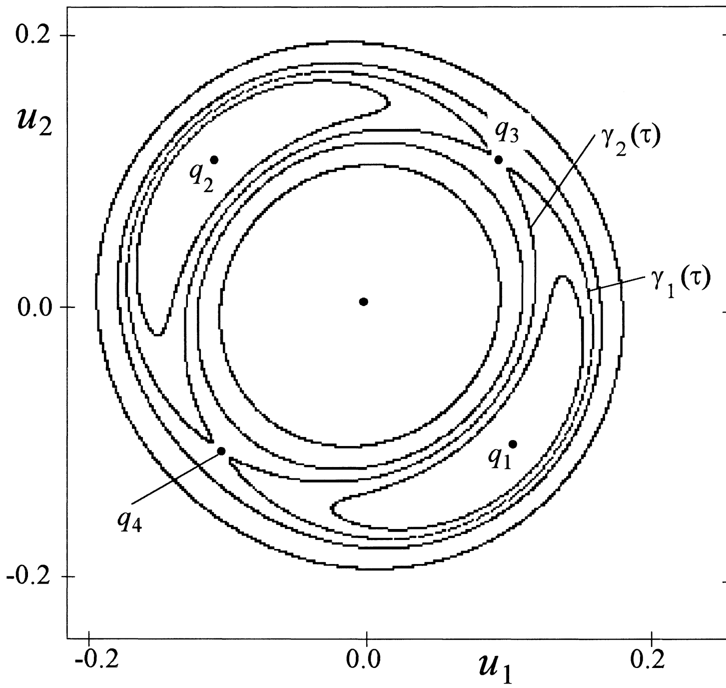

Figure 1. Phase portrait of the averaged system constructed by applying the analytical solutions for $e = 0.03$.

Equation (26) is integrated by means of the Jacobian elliptic function sn to get

$$\Lambda = \mathrm{sn}(\lambda\tau + d, k), \tag{27}$$

where d is a constant parameter and available from the selected initial conditions. On substituting Λ from (27) into (25) and solving for p, Equation (22a) results in

$$\varphi = \arctan\left(\frac{z_2\mathrm{sn}^2(\lambda\tau + d, k) - z_1\alpha}{\mathrm{sn}^2(\lambda\tau + d, k) - \alpha}\right). \tag{28}$$

Applying (28) in (19) completes the integration procedure. Having the explicit forms of the solutions, geometrical features of the averaged system can be extracted.

5. Discussion of the Results

Using the obtained results, the phase portrait of system (18) is plotted in Figure 1 for $e = 0.03$. As this figure shows, there exist four fixed points q_1 through q_4 along with the trivial stationary point at the origin. Fixed points q_1 and q_2 are centers that describe 3π-periodic solutions of the original system. In accordance with (4) and

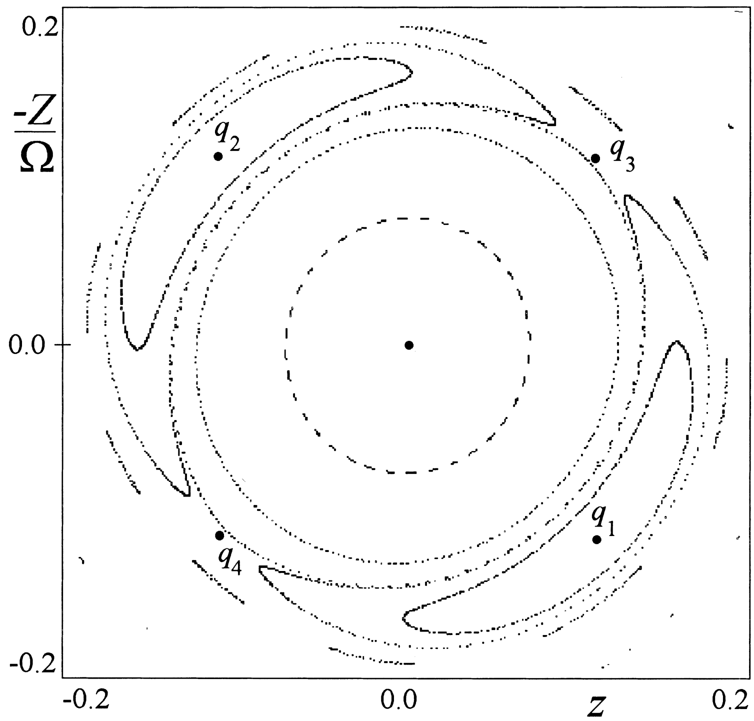

Figure 2. Numerically generated Poincaré map for the Sitnikov problem at 3/2 resonance for $e = 0.03$. Sampling time is set to 3π.

defining J_i and φ_i as the coordinates of $q_i (i = 1, \ldots, 4)$, periodic solutions at the 3/2 commensurability will be

$$z(t) = -\sqrt{2J_n} \sin\left(\frac{8}{3}t - \varphi_n\right), \tag{29a}$$

$$Z(t) = -\Omega\sqrt{2J_n} \cos\left(\frac{8}{3}t - \varphi_n\right), \quad (n = 1, 2). \tag{29b}$$

Analytical expressions for J_i's and φ_i's have been derived and given in Appendix A.

The other two points, q_3 and q_4, are saddles connected by heteroclinic orbits $\gamma_1(\tau)$ and $\gamma_2(\tau)$. These points are unstable periodic solutions. In order to verify the accuracy of the mentioned theory, the Poincaré section of differential Equations (2) is numerically constructed with a sampling time equal to 3π and shown in Figure 2. Results of the numerical integration not only reflect many qualitative characteristics of the system, but also confirm the obtained analytical solutions. Closed curves in the Poincaré map are cross-sections of periodic tubes in three-dimensional (z, Z, t) phase space (Berdichevsky *et al.*, 1994). Numerical experiments reveal that by increasing the eccentricity of the primaries to 0.055, chaotic motion occurs in the vicinity of the hyperbolic points as shown in Figure 3. At first, the appearance of chaotic behavior is due to break down of invariant manifolds of the hyperbolic points q_3 and q_4

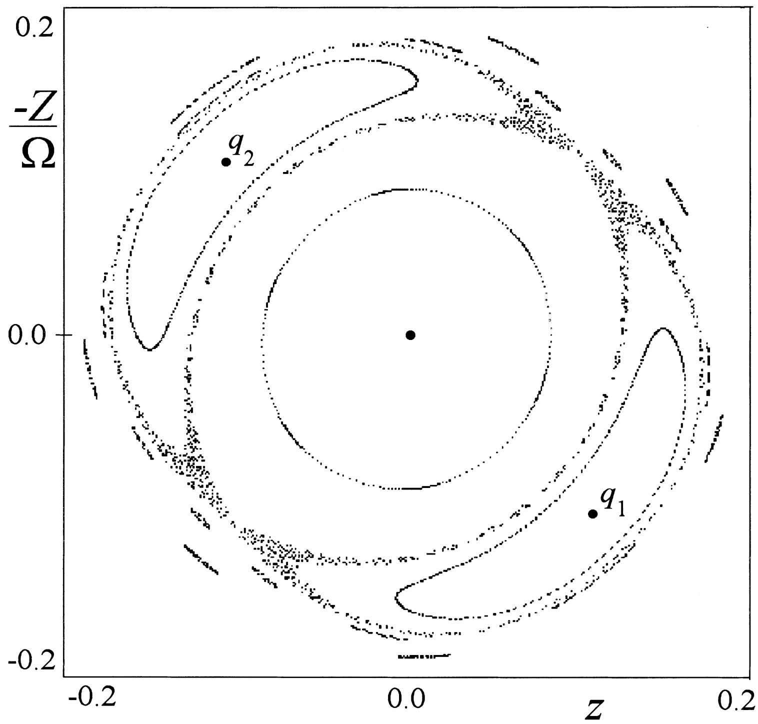

Figure 3. Numerically generated Poincaré map for the Sitnikov problem at 3/2 resonance for $e = 0.055$. Sampling time is set to 3π.

while periodic tubes surround the chaotic zone. However, by further increasing the parameter e to 0.08, surrounding periodic tubes disappear and the main stochastic region joins the escaping one (see Figure 4). As evidenced by Figure 4, transition to chaos is via the period-doubling mechanism. This phenomenon originates from $q_1(q_2)$ which is a 1-cycle periodic point and propagates outward through generation of new n-cycles ($n > 1$).

Prediction of the chaotic motion can also be accomplished through Melnikov's theory. Although the method works successfully when explicit forms of the heteroclinic or homoclinic orbits are known, in dealing with averaged systems it should be used cautiously (Guckenheimer and Holmes, 1983).

6. Conclusion

The formulation adopted in the present research permits for the study of regular and chaotic motions around resonant orbits in Sitnikov's problem. The equations of motion are the same as given by Hagel (1992) and are approximately valid for $0 \leqslant e < 0.2$ and $|z(t)| < |r(t)|$. Therefore, any subsequent results are based on such assumptions. To investigate regular behavior near the 3/2 commensurability,

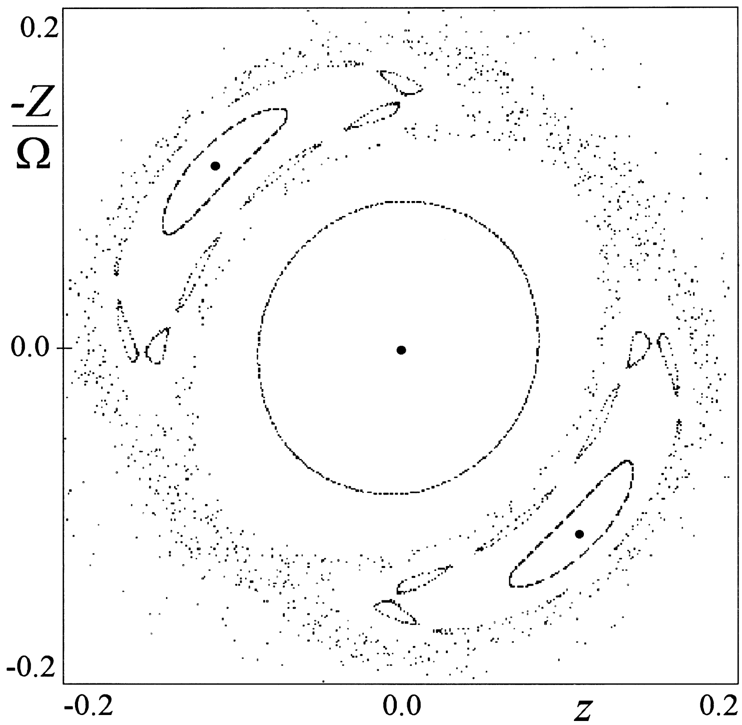

Figure 4. Numerically generated Poincaré map for the Sitnikov problem at 3/2 resonance for $e = 0.08$. Sampling time is set to 3π.

the equations of motion in (z, Z, t) phase space were reduced to a two-dimensional Hamiltonian system using the averaging method. Then, by applying a transformation to action angle variables, the averaged system was integrated in terms of Jacobian elliptic functions. The obtained analytical answers have been verified by numerical construction of the Poincaré maps for the original nonautonomous system. Using the method of sections it was shown that chaotic sea emerging from the destroyed invariant manifolds can easily be seen for certain eccentricities. Figures 1 and 4 indicate that the obtained analytic solutions remain credible for relatively large values of e if the motion starts close to elliptic fixed points. The technique of the current study can be repeated for any Ω satisfying $\Omega^2 - \omega^2 = O(\varepsilon)$. As a plan to future researches, statistical considerations and the cascade of period-doubling may be investigated using modern mathematical methods.

Appendix A

Stationary points of (18) are determined through solving the following relations:

$$\frac{dA(\varphi)}{d\varphi}J^2 + \frac{dB(\varphi)}{d\varphi}J = 0, \tag{30a}$$

$$2A(\varphi)J + B(\varphi) = 0. \tag{30b}$$

The obvious solution of (30a) is $J = 0$. Removing this answer from calculations, one acquires

$$\frac{\mathrm{d}A(\varphi)}{\mathrm{d}\varphi}B(\varphi) = 2\frac{\mathrm{d}B(\varphi)}{\mathrm{d}\varphi}A(\varphi). \tag{31}$$

Utilizing the change of variable $s = \tan\varphi$ and substituting from (17) into (31) yields

$$s^4 + d_3s^3 + d_2s^2 + d_1s + d_0 = 0, \tag{32}$$

or

$$(s - s_1)(s - s_2)(s - s_3)(s - s_4) = 0, \tag{33}$$

where s_i's ($i = 1, \ldots, 4$) are the roots of Equation (32) and

$$\begin{aligned}
d_3 &= (Q_2 + Q_3)/Q_1, \quad d_2 = 12C_8(\pi + 2C_4e^2)/Q_1,\\
d_1 &= (Q_3 - Q_2)/Q_1,\\
d_0 &= Q_4/Q_1, \quad Q_1 = R_1 + R_2, \quad Q_2 = 4C_7e(C_3 + C_1e^2),\\
Q_3 &= -8C_8e(C_3 + C_1e^2), \quad Q_4 = R_1 - R_2,\\
R_1 &= -4C_4C_8e^2 - 2\pi C_8, \quad R_2 = \pi C_7 - 4C_3C_5 + 2C_4C_7e^2 - 4C_1C_5e^2.
\end{aligned}$$

Since Equations (12) are invariant under reflections $u_1 \to -u_1$ and $u_2 \to -u_2$, it is easily understood that $\varphi_2 = \varphi_1 + \pi$ and $\varphi_4 = \varphi_3 + \pi$. Thus, by taking two real roots of (33), one obtains

$$\varphi_1 = \arctan(s_1), \quad \varphi_2 = \varphi_1 + \pi, \tag{34a}$$
$$\varphi_3 = \arctan(s_2), \quad \varphi_4 = \varphi_3 + \pi. \tag{34b}$$

At the end, applying (34) in (30b) gives

$$J_i = \frac{-B(\varphi_i)}{2A(\varphi_i)}, \quad (i = 1, 2, 3, 4). \tag{35}$$

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