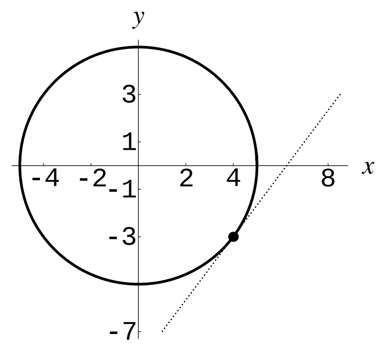
What is the equation of the tangent line to the circle of radius 5 and center $(0, 0)$ at the point P with coordinates $(4, -3)$? If this line has slope m then its equation is clearly $y - (-3) = m(x - 4)$, or

$$y = mx - \{4m + 3\}.$$

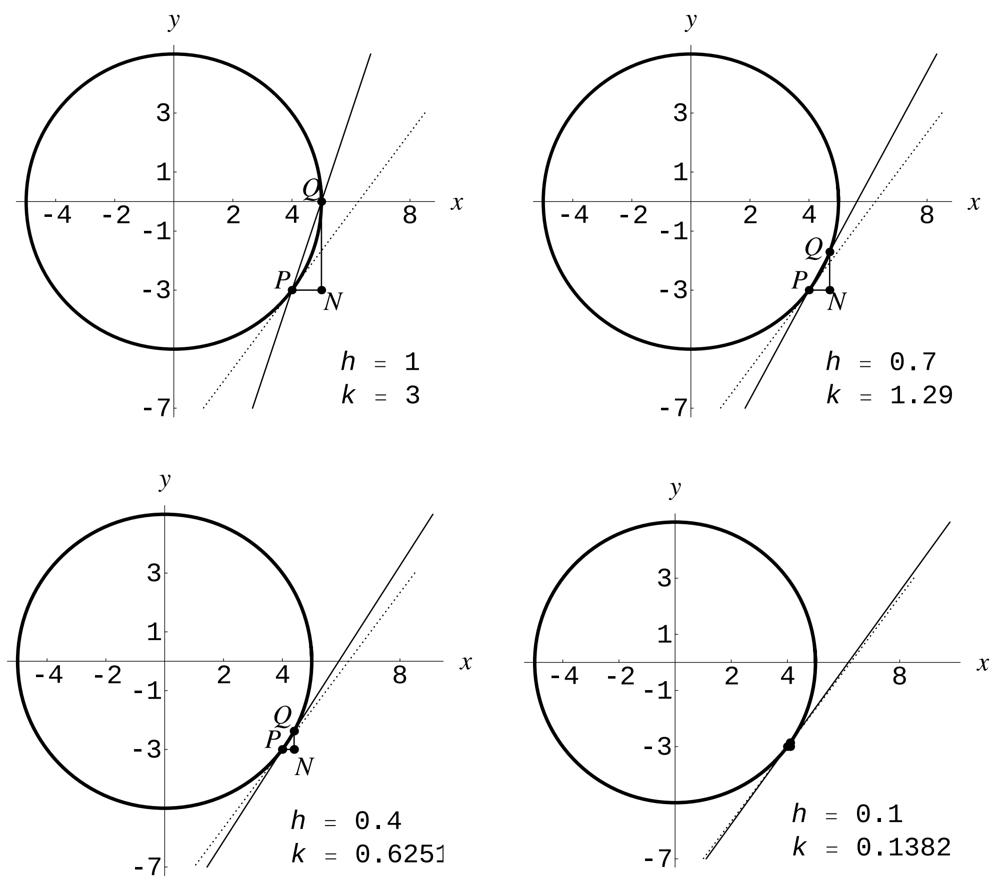
But what is the value of m?



We can approximate m by the slope m_{PQ} of the chord through P and a neighboring point Q that also lies on the circle. Let Q have coordinates

$$(4 + h, -3 + k)$$

where h and k are both small—otherwise, Q would not be a neighboring point. For example, we could suppose that, at the very worst, h can't exceed 1 and k can't exceed 3. Then possible h values are 1, 0.7, 0.4 and 0.1; possible k values—chosen to ensure that $(4+h, -3+k)$ lies on the circle, as shown in the diagrams below—are 3, 1.294, 0.6251 and 0.1382; and the corresponding Q values are (5,0), (4.7, -1.706), (4.4, -2.375) and (4.1, -2.862). Note that, in the fourth diagram, it is already difficult to distinguish P from Q or N.



In the right-angled triangle PNQ, where N has coordinates $(4 + h, -3)$, we have

$$h = PN, \qquad k = NQ$$

$$m_{PQ} = \frac{NQ}{PN} = \frac{k}{h}.$$

Because Q lies on the circle, we must have $(4+h)^2 + (-3+k)^2 = 5^2$, from which

$$k - 3 \;=\; \pm\sqrt{5^2 - (4+h)^2} \;=\; \pm\sqrt{(1-h)(9+h)}.$$

But $k \le 3$ or $k - 3 \le 0$, so that the negative sign must be chosen (and $h \le 1$ or $1 - h \ge 0$, so we don't have to worry about negativity under the square root sign). The upshot is that

$$k - 3 \;=\; -\sqrt{(1-h)(9+h)} \Longrightarrow k \;=\; 3 - \sqrt{(1-h)(9+h)},$$

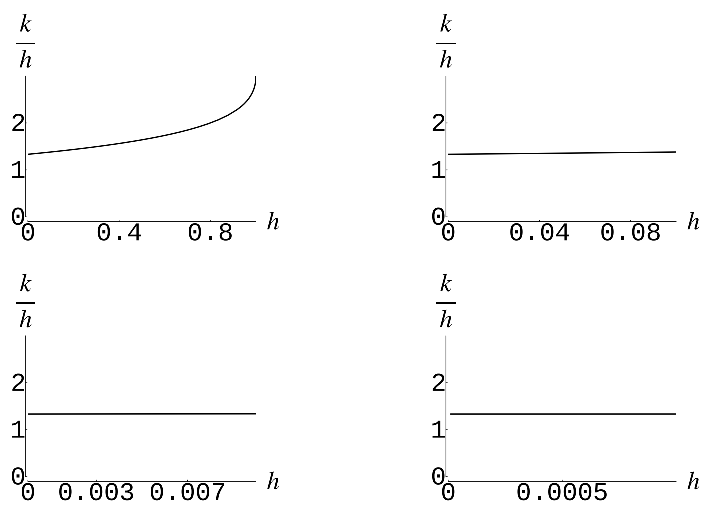
from which

$$\frac{k}{h} \;=\; \frac{3 - \sqrt{(1-h)(9+h)}}{h}.$$

Here is a table of corresponding h and k values:

h	$k = 3 - \sqrt{(1-h)(9+h)}$	$\frac{k}{h} \;=\; \frac{3-\sqrt{(1-h)(9+h)}}{h}$
1	3	3
0.7	1.294	1.849
0.4	0.6251	1.563
0.1	0.1382	1.382
0.01	0.01338	1.338
0.001	0.001334	1.334
0.0001	0.0001333	1.333
0.00001	0.00001333	1.333
……	……	……
10^{-10}	1.333×10^{-10}	1.333

And here are some plots that illustrate precisely the same effect:



We see that as h gets smaller and smaller, k also gets smaller and smaller (and must do so, because otherwise Q would fall off the circle), but in such a way that $\frac{k}{h}$ levels off at 1.333. Because the chord is indistinguishable from the tangent for such small values of h, we guess that its slope is $m = 1.333$.

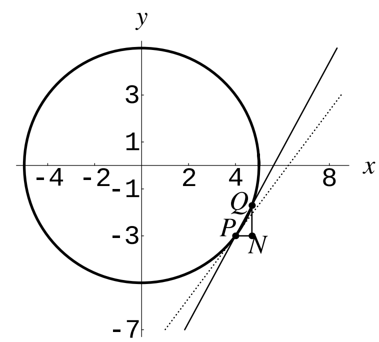
Needless to say, it is easy to confirm from similar triangles that

$$m \;=\; \tfrac{4}{3}.$$

So the equation of the tangent line is $y = \frac{4}{3}x - \frac{25}{3}$ or $4x - 3y = 25$.

In effect we have defined a ratio function by

$$r(h) \;=\; \frac{k}{h} \;=\; \frac{QN}{PN} \;=\; \frac{3-\sqrt{(1-h)(9+h)}}{h}.$$



We implicitly assumed that $h \geq 0$, and we explicitly assumed that $h \leq 1$. Does that mean the domain of r is $[0,1]$? No, not exactly, because $r(0)$ is undefined: $3-\sqrt{(1-0)(9+0)} = 3-3 = 0$, and $\frac{0}{0}$ is undefined. So we have to restrict the domain of r to $(0,1]$ instead:

$$r(h) \;=\; \frac{3-\sqrt{(1-h)(9+h)}}{h}, \qquad 0 < h \leq 1.$$

So h can be any extremely small positive number, as long as it isn't actually zero, and for such small numbers the behavior of r is exemplified by the following table:[1]

h	$r(h)$
10^{-1}	1.381823957491631
10^{-2}	1.337983667155960
10^{-3}	1.333796502184873
10^{-4}	1.333379631687370
10^{-5}	1.333337962983539
10^{-6}	1.333333796296502
10^{-7}	1.333333379629632
10^{-8}	1.333333337962963
10^{-9}	1.333333333796296
10^{-10}	1.333333333379630
10^{-11}	1.333333333337963
10^{-12}	1.333333333333796
10^{-13}	1.333333333333380
10^{-14}	1.333333333333338
10^{-15}	1.333333333333334
10^{-16}	1.333333333333333

It appears that we can make $r(h)$ as close as we please to $\frac{4}{3}$ by making h a positive number as close as is necessary to 0—but not actually zero itself. That is an awful lot to have to write down to record what we have observed; to a considerable extent, mathematics is a shorthand for saying succinctly what take much longer to say in words. So instead of saying, "we can make $r(h)$ as close as we please to $\frac{4}{3}$ by making h a positive number as close as is necessary to 0—but not actually zero itself" we say "the limit of $r(h)$ as h approaches 0 from the right is $\frac{4}{3}$" or—better still—

$$\lim_{h\to 0+} r(h) \;=\; \tfrac{4}{3}.$$

It is much more compact, but means exactly the same (as the words in quotation marks).

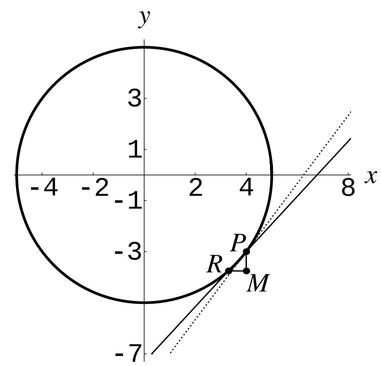
[1]You won't easily be able to reproduce this result on your calculator for subtle reasons that are alluded to in §2.2 of the test, but which are best ignored for present purposes: this is a course on calculus, not numerical analysis.

But we needn't approach $h = 0$ from the right: we can instead approximate m by the slope m_{PR} of the chord through P and neighboring point R with coordinates

$$(4+h, -3+k)$$

such that h and k are both negative. Because R lies on the circle, we must have $(4+h)^2 + (-3+k)^2 = 5^2$ as before and hence $k = 3 - \sqrt{(1-h)(9+h)}$, from which

$$r(h) \;=\; \frac{PM}{RM} \;=\; \frac{-k}{-h} \;=\; \frac{\sqrt{(1-h)(9+h)} - 3}{-h}.$$



If we continue to interpret "h small" as meaning at worst $|h| \le 1$, then we have to restrict the domain of r for negative h to $[-1, 0)$:

$$r(h) \;=\; \frac{\sqrt{(1-h)(9+h)} - 3}{-h}, \qquad -1 \le h < 0.$$

So h can be any extremely small negative number, as long as it isn't actually zero, and for such small numbers the behavior of r is exemplified by the following table:

h	$r(h)$
-10^{-1}	1.288975694324031
-10^{-2}	1.328724153539667
-10^{-3}	1.332870576004604
-10^{-4}	1.333287039094523
-10^{-5}	1.333328703724280
-10^{-6}	1.333332870370576
-10^{-7}	1.333333287037039
-10^{-8}	1.333333328703704
-10^{-9}	1.333333332870370
-10^{-10}	1.333333333287037
-10^{-11}	1.333333333328704
-10^{-12}	1.333333333332870
-10^{-13}	1.333333333333287
-10^{-14}	1.333333333333329
-10^{-15}	1.333333333333333
-10^{-16}	1.333333333333333

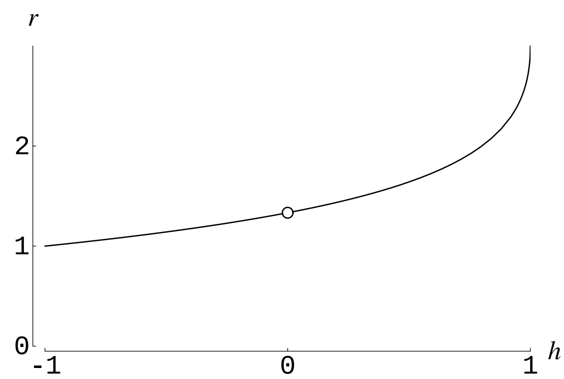
Again, it appears that we can make $r(h)$ as close as we please to $\frac{4}{3}$ by making h a negative number as close as is necessary to 0—but not actually zero itself; and again, we abbreviate this statement by saying that "the limit of $r(h)$ as h approaches 0 from the left is $\frac{4}{3}$" or

$$\lim_{h \to 0-} r(h) \;=\; \tfrac{4}{3}.$$

We can now combine our results for positive and negative h. First, a single formula will do for r:

$$r(h) \;=\; \frac{3 - \sqrt{(1-h)(9+h)}}{h}, \qquad h \in [1, 0) \cup (0, 1].$$

Note that r is undefined at $h = 0$, as indicated in the following sketch of its graph:



Second, in place of $\lim_{h\to 0+} r(h) = \frac{4}{3}$ and $\lim_{h\to 0-} r(h) = \frac{4}{3}$ we can write a single formula

$$\lim_{h\to 0} r(h) \;=\; \tfrac{4}{3}.$$

That is, because the limit from the right and the limit from the left both exist and *are the same*, we can refer to either as simply the limit.

We have already noted that $r(0)$ is undefined. What would be the most natural definition for it? A good answer is to define $r(0)$ in such a way that we can sketch the entire graph of r on the domain $[-1, 1]$ without ever lifting our pen from the page. Then, clearly, we must define $r = \frac{4}{3}$. Whenever $r(0) = \lim_{h\to 0} r(h)$, we say that $r(h)$ is continuous at $h = 0$.

All of these ideas are readily generalized. Whenever we can make $f(x)$ as close as we please to L by making x a number bigger than a that is as close as necessary to a—but not actually a itself—we write

$$\lim_{x\to a+} f(x) \;=\; L.$$

Whenever we can make $f(x)$ as close as we please to L by making x a number smaller than a that is as close as necessary to a—but not actually a itself—we write

$$\lim_{x\to a-} f(x) \;=\; L.$$

Whenever both are true we write

$$\lim_{x\to a} f(x) \;=\; L.$$

Whenever

$$\lim_{x\to a+} f(x) \;=\; f(a)$$

we say that f is continuous from the right at a. Whenever

$$\lim_{x\to a-} f(x) \;=\; f(a)$$

we say that f is continuous from the left at a. And whenever both are true, i.e., whenever

$$\lim_{x\to a} f(x) \;=\; f(a)$$

we simply say that f is continuous at a.