You are now somewhat familiar with the idea that a function can be defined in terms of a limit; for example, it would be possible to define

$$\frac{1}{x^2} \;=\; -\lim_{h\to 0}\frac{1}{h}\left\{\frac{1}{x+h}-\frac{1}{x}\right\}.$$

We wouldn't ordinarily want to do that. However, it can also be shown (though we won't do so here) that the exponential function e^x can be defined in terms of a limit as

$$e^x \;=\; \lim_{n\to\infty}\left(1+\tfrac{x}{n}\right)^n$$

where n is an integer, and this is something we might very well ordinarily want to do. In fact, we want to do it now right now. So let's—instant gratification! Then, in particular, we have

$$e \;=\; e^1 \;=\; \lim_{n\to\infty}\left(1+\tfrac{1}{n}\right)^n$$

A table of values illustrates how $\left(1+\frac{1}{n}\right)^n$ approaches e as $n\to\infty$:

n	$\left(1+\frac{1}{n}\right)^n$	n	$\left(1+\frac{1}{n}\right)^n$
1	2.00	10000	2.71815
10	2.59374	100000	2.71827
100	2.70481	1000000	2.71828
1000	2.71692	1000001	2.71828

(It's terribly slow convergence, but it gets there in the end.)

Now, from the binomial theorem, we have

$$\begin{aligned}(1+a)^n \;=\;& 1+na+\tfrac{n(n-1)}{1.2}a^2+\tfrac{n(n-1)(n-2)}{1.2.3}a^3\\ &+\tfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}a^4+\tfrac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}a^5+\ldots\\ &\text{(plus a lot more terms, in fact, } n-5 \text{ of them).}\end{aligned}$$

If we set $a=x/n$ we get

$$\begin{aligned}\left(1+\frac{x}{n}\right)^n \;=\;& 1+n\left(\tfrac{x}{n}\right)+\tfrac{n(n-1)}{1.2}\left(\tfrac{x}{n}\right)^2+\tfrac{n(n-1)(n-2)}{1.2.3}\left(\tfrac{x}{n}\right)^3\\ &+\tfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}\left(\tfrac{x}{n}\right)^4+\tfrac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}\left(\tfrac{x}{n}\right)^5+\ldots\end{aligned}$$

$$\begin{aligned}=\;& 1+x+\left(1-\tfrac{1}{n}\right)\tfrac{x^2}{1.2}+\left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\tfrac{x^3}{1.2.3}\\ &+\left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\left(1-\tfrac{3}{n}\right)\tfrac{x^4}{1.2.3.4}+\left(1-\tfrac{1}{n}\right)\left(1-\tfrac{2}{n}\right)\left(1-\tfrac{3}{n}\right)\left(1-\tfrac{4}{n}\right)\tfrac{x^5}{1.2.3.4.5}+\ldots\end{aligned}$$

so that, on using $\lim\limits_{n\to\infty}\frac{1}{n}=0$, we have

$$e^x \;=\; \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \;=\; 1+x+\tfrac{x^2}{1.2}+\tfrac{x^3}{1.2.3}+\tfrac{x^4}{1.2.3.4}+\tfrac{x^5}{1.2.3.4.5}+\ldots$$

In particular, we have

$$e^h \;=\; 1+h+\tfrac{h^2}{1.2}+\tfrac{h^3}{1.2.3}+\tfrac{h^4}{1.2.3.4}+\tfrac{h^5}{1.2.3.4.5}+\ldots \;=\; 1+h+hO(h).$$

Now there are two ways in which we can show that e^x is its own derivative. We can either proceed as follows:

$$\begin{aligned}
\frac{d}{dx}\left(e^x\right) &= \lim_{h\to 0} \frac{e^{x+h}-e^x}{h} \\
&= \lim_{h\to 0} \frac{e^x e^h - e^x}{h} \\
&= \lim_{h\to 0} e^x \frac{e^h - 1}{h} \\
&= \lim_{h\to 0} e^x \frac{1+h+hO(h)-1}{h} \\
&= \lim_{h\to 0} e^x \left\{1+O(h)\right\} \\
&= e^x(1+0) \\
&= e^x.
\end{aligned}$$

Or else we can proceed as follows, exploiting linearity*:

$$\begin{aligned}
\frac{d}{dx}\left(e^x\right) &= \frac{d}{dx}\left(1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+\frac{x^5}{1.2.3.4.5}+\ldots\right) \\
&= \frac{d}{dx}(1)+\frac{d}{dx}(x)+\frac{d}{dx}\left(\frac{x^2}{1.2}\right)+\frac{d}{dx}\left(\frac{x^3}{1.2.3}\right)+\frac{d}{dx}\left(\frac{x^4}{1.2.3.4}\right)+\frac{d}{dx}\left(\frac{x^5}{1.2.3.4.5}\right)+\ldots \\
&= \frac{d}{dx}(1)+\frac{d}{dx}(x)+\frac{1}{1.2}\frac{d}{dx}\left(x^2\right)+\frac{1}{1.2.3}\frac{d}{dx}\left(x^3\right)+\frac{1}{1.2.3.4}\frac{d}{dx}\left(x^4\right)+\frac{1}{1.2.3.4.5}\frac{d}{dx}\left(x^5\right)+\ldots \\
&= 0 + 1 + \frac{1}{1.2}\cdot 2x + \frac{1}{1.2.3}\cdot 3x^2 + \frac{1}{1.2.3.4}\cdot 4x^3 + \frac{1}{1.2.3.4.5}\cdot 5x^4 + \ldots \\
&= 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \ldots \\
&= e^x.
\end{aligned}$$

Finally, the product rule:

$$\begin{aligned}
\frac{d}{dx}\left\{f(x)g(x)\right\} &= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h)-f(x)g(x+h)+f(x)g(x+h)-f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{\{f(x+h)-f(x)\}g(x+h)+f(x)\{g(x+h)-g(x)\}}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot g(x+h) + f(x)\cdot\frac{g(x+h)-g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}\cdot g(x+h) + \lim_{h\to 0} f(x)\cdot\frac{g(x+h)-g(x)}{h} \\
&= f'(x) \cdot g(x+0) + f(x) \cdot g'(x) \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned}$$

*And making various other assumptions whose validity we have no choice but to take for granted at this stage in our study of the calculus